- So gauging reduces this to a theory with target $\mathbb{PT} \subset \mathbb{P}^{3|4}$. This is (projective) twistor space!
- Can work with homogeneous coordinates: $Z^I(\sigma) = (\lambda_A, \mu^{A'}, \chi^a)$
  Weyl spinors; fermionic coordinate

- These are related to chiral Minkowski super-space $\mathbb{M}$ with coordinates (x,θ) by the incidence relations: $\begin{cases} \mu^{A'} = i x^{AA'} \lambda_A \\ \chi^a = \theta^{aA} \lambda_A \end{cases}$
- These are algebraic relations describing a linear embedding $\mathbb{P}^1 \hookrightarrow \mathbb{PT}$, with (x,θ) the parameters of this embedding
- So $(x,\theta) \in \mathbb{M} \leftrightarrow X \cong \mathbb{P}^1 \subset \mathbb{PT}$. This is the basic twistor correspondence.

- <u>Gravitational Vertex Operators</u>:

- We are interested in gravitational degrees of freedom, so we want vertex operators corresponding to deformations of the complex and Hermitian structure of $\mathbb{PT}$
- In Q-cohomology, this is captured by $F \in H^{0,1}(\mathbb{PT}, T \oplus T^*\mathbb{PT})$.
- On $\mathbb{T}$, this can be represented by a pair $(f^I, g_I) \in \mathcal{O}(1,-1)$, which obey $\partial_I f^I = 0 = Z^I g_I$.
  ensure reps descend to $\mathbb{PT}$!

- Corresponding vertex operators are thus:

$$V_f = \int_\Sigma Y_I f^I \quad , \quad V_g = \int_\Sigma g_I \partial Z^I$$

cplx structure; Hermitian structure

- Correlators in the worldsheet CFT are then calculated by Wick contracting all available Y's with Zs and then integrating over the zero modes of the remaining Zs.
- When $\Sigma_1 \cong \mathbb{P}^1$, the Wick contraction is performed as $\left\langle Y_I(\sigma) Z^J(\sigma') \right\rangle_d = \delta_I^J \frac{D\sigma}{(\sigma\sigma')} \frac{(\xi\sigma')^{d+1}}{(\xi\sigma)^{d+1}}$

  $SL(2,\mathbb{C})$ inner product $\rightarrow (\sigma\sigma')$

  where $\xi \in \mathbb{P}^1$ is arbitrarily chosen, reflecting the ambiguity of inverting $\bar{\partial}|_{\mathbb{P}^1}$ on forms of weight d.
- When acting on a function of Z, Y differentiates, then Wick contracts.
- So we have $\sum_{d=0}^{\infty} \int_{\mathcal{M}_{n,d}} d\mu_d \left\langle V_{f_1} \cdots V_{f_a} V_{g_1} \cdots V_{g_b} \right\rangle_d$, where: $\mathcal{M}_{n,d} = \overline{\mathcal{M}}_{0,n}(\mathbb{PT}, d)$

  $d\mu_d$ is the measure

  a, b are such that this is well-defined at tree-level.

- Claim (Berkovits & Witten): this corresponds to $N=4$ conformal supergravity.

- In a sense, this is to be expected: twistors are the spinors of the superconformal group, and there is no structure in the twistor-string theory which breaks conformal invariance
- Furthermore, one can see that operators corresponding to space-time translations are not diagonalizable $\Leftrightarrow$ violation of unitarity.

- Conformal Gravity:

- Action: $S^{CG}[g] = \frac{1}{\kappa^2} \int_M d\mu \, C^{abcd} C_{abcd}$

  Weyl tensor

• Field Equations: $B_{ab} = 2\left(\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'}\right)\Psi_{ABCD} = 2\left(\nabla^{C'}_{A}\nabla^{D'}_{B} + \Phi^{C'D'}_{AB}\right)\tilde{\Psi}_{A'B'C'D'} = 0$

Bach tensor; trace-free Ricci; ASD Weyl; SD Weyl

• Clearly, these are 4th order equations, which is the source of all the things we don't like about conformal gravity (eg., lack of unitarity)

• However, we have the following fact:

• <u>Fact / Theorem</u> (Maldacena / Anderson): $S^{CG}[dS_4] \propto \Lambda \, S^{EH}[dS_4]$

cosmological constant of dS_4; Einstein-Hilbert action

• This means that we should be able to compute the tree-level S-matrix of Einstein gravity by using Einstein scattering states in conformal gravity in a de Sitter background!

• Consequences; as $\Lambda \to 0$: - The $\mathcal{O}(\Lambda^0)$ part of such a CG amplitude must vanish

- The $\mathcal{O}(\Lambda)$ part will correspond to the $\Lambda = 0$ Einstein amplitude!

• So how do we implement this in twistor-string theory?

• Need to introduce a structure which breaks conformal invariance, and then use it to get Einstein vertex operators,

• The "infinity twistor" breaks conformal invariance by singling out a line $I \subset \mathbb{PT}$ which corresponds to infinity in conformally compactified $\mathbb{M}$.

- Infinity twistor: $I^{IJ} = \begin{pmatrix} \Lambda \varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}$, $I_{IJ} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \Lambda \varepsilon_{A'B'} \end{pmatrix}$

$$I^{IJ} I_{JK} = \Lambda \delta^I_K$$

- Then our vertex operators can be modified as:

$$V_f \to V_h = \int_\Sigma Y_I I^{IJ} \partial_J h \ , \quad h \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2))$$

$$V_g \to V_{\tilde{h}} = \int_\Sigma I_{IJ} Z^J \tilde{h}\, \partial Z^I \ , \quad \tilde{h} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-2))$$

- Via the Penrose transform, it's easy to see that these encode the ± 2 helicity multiplets of $N=4$ supergravity.

- Now, the flat space n-point MHV amplitude should correspond to:

$$\mathcal{M}^0_{n,0}(1,\dots,n) = \lim_{\Lambda \to 0} \frac{1}{\Lambda} \int_{\mathcal{M}_{n,1}} d\mu_1 \left\langle V_{h_1} \cdots V_{h_{n-2}} V_{\tilde{h}_{n-1}} V_{\tilde{h}_n} \right\rangle_1$$

- Let us check that the $\mathcal{O}(\Lambda^0)$ part of the correlator vanishes.
- After integrating out the moduli and using standard momentum eigenstates, it's not too hard to see that $I^{IJ} \partial_{iJ} h_i \langle Y_{iI} h_j \rangle_1 \Big|_{\Lambda=0} = \frac{[ij]}{\langle ij \rangle} \frac{\langle \xi j \rangle^2}{\langle \xi i \rangle^2}$.
- So a single Y_i will contribute $\sum_{j \neq i} \frac{[ij]}{\langle ij \rangle} \frac{\langle \xi j \rangle^2}{\langle \xi i \rangle^2} \equiv \phi^i_i$, and if all contractions (i.e., loops, disconnected) then the $\mathcal{O}(\Lambda^0)$ contribution should be: $\prod_{i=1}^{n-2} \phi^i_i$

- But this doesn't generically vanish!?!

- It turns out that if we impose a maximally connected formalism on the Feynman graphs of the worldsheet CFT, we seem to get the correct answer.
- For instance, if we do the same computation in this formalism, we find $\det \tilde{\Phi}_{n-1n}$, where $\tilde{\Phi}$ is a matrix with entries:

$$\tilde{\Phi}_{ij} = \begin{cases} \dfrac{[ij]}{\langle ij\rangle} & \text{if } i \neq j \\[2ex] -\displaystyle\sum_{j \neq i} \dfrac{[ij]}{\langle ij\rangle} \dfrac{\langle \xi j\rangle^2}{\langle \xi i\rangle^2} & \text{if } j = i \end{cases}$$

and $\tilde{\Phi}_{n-1n}$ is $\tilde{\Phi}$ with the $(n-1)^{st}$ and n^{th} rows and columns crossed out.
- This follows from the Matrix-Tree theorem in graph theory.
- But $\sum_{j=1}^{n} \tilde{\Phi}_{ij} \lambda_j^{A_1} \lambda_j^{A_2} = 0$ by momentum conservation, so $\tilde{\Phi}$ has co-rank 3.
- Then $\det \tilde{\Phi}_{n-1n} = 0$, as required for conformal gravity!

- Things to do: - Try to get the whole tree-level S-matrix from this approach (i.e., derive the Cachazo-Skinner formula)
  - Understand *why* we have to impose these rules on the string theory.
  - Can conformal gravity methods be extended to loops?