Gaussian relaxational models in the ordered phase (12 points)

Model A/B ($a = 0, 2$) coupled Langevin equations: $\alpha = 1, \dots, n$

$$\frac{\partial S^\alpha(\vec{x},t)}{\partial t} = -D(i\vec{\nabla})^a \left[\left(\tau - \vec{\nabla}^2 + \frac{u}{6}\sum_{\beta=1}^{n} S^\beta(\vec{x},t)^2\right) S^\alpha(\vec{x},t) - h^\alpha(\vec{x},t)\right] + \zeta^\alpha(\vec{x},t)$$

$$\langle \zeta^\alpha(\vec{x},t)\rangle = 0\,, \quad \langle \zeta^\alpha(\vec{x},t)\zeta^\beta(\vec{x}',t')\rangle = 2Dk_BT\,(i\vec{\nabla}_x)^a\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')\,\delta^{\alpha\beta}$$

$T < T_c$ ($\tau < 0$): parametrize $S^\alpha = \pi^\alpha$, $\alpha = 1, \dots, n-1$

$$S^n = \sqrt{\frac{6|\tau|}{u}} + \sigma$$

in terms of the fluctuating fields π^α, σ:

$$\frac{\partial \pi^\alpha(\vec{x},t)}{\partial t} = -D(i\vec{\nabla})^a \Big[\Big(\cancel{-|\tau|} - \vec{\nabla}^2 + \frac{u}{6}\Big(\cancel{\frac{6|\tau|}{u}} + 2\sqrt{\frac{6|\tau|}{u}}\,\sigma(\vec{x},t) + \sigma(\vec{x},t)^2 + \sum_{\beta=1}^{n-1}\pi^\beta(\vec{x},t)^2\Big)\pi^\alpha(\vec{x},t) - h^\alpha(\vec{x},t)\Big] + \zeta^\alpha(\vec{x},t)$$

linearize: $\approx D(i\vec{\nabla})^a\left[\vec{\nabla}^2\pi^\alpha(\vec{x},t) + h^\alpha(\vec{x},t)\right] + \zeta^\alpha(\vec{x},t)$

$$\frac{\partial \sigma(\vec{x},t)}{\partial t} = -D(i\vec{\nabla})^a \Big[\Big(\cancel{-|\tau|} - \vec{\nabla}^2 + \frac{u}{6}\Big(\cancel{\frac{6|\tau|}{u}} + 2\sqrt{\frac{6|\tau|}{u}}\,\sigma(\vec{x},t) + \sigma(\vec{x},t)^2 + \sum_{\beta=1}^{n-1}\pi^\beta(\vec{x},t)^2\Big)\Big(\sqrt{\frac{6|\tau|}{u}} + \sigma(\vec{x},t)\Big) - h^n(\vec{x},t)\Big] + \zeta^n(\vec{x},t)$$

linearize: $\approx -D(i\vec{\nabla})^a\left[\left(2|\tau| - \vec{\nabla}^2\right)\sigma(\vec{x},t) - h^n(\vec{x},t)\right] + \zeta^n(\vec{x},t)$

Fourier transform:
$$\left(-i\omega + Dq^{2+a}\right)\pi^\alpha(\vec{q},\omega) = Dq^a h^\alpha(\vec{q},\omega) + \zeta^\alpha(\vec{q},\omega)$$
$$\left[-i\omega + Dq^a\left(2|\tau| + q^2\right)\right]\sigma(\vec{q},\omega) = Dq^a h^n(\vec{q},\omega) + \zeta^n(\vec{q},\omega)$$

with $\langle \zeta^\alpha(\vec{q},\omega)\rangle = 0$, $\langle \zeta^\alpha(\vec{q},\omega)\zeta^\beta(\vec{q}',\omega')\rangle = 2Dk_BT q^a (2\pi)^{d+1}\delta(\vec{q}+\vec{q}')\delta(\omega+\omega')\delta^{\alpha\beta}$

→ dynamic response functions in Gaussian approximation:

$$\chi_0^{\alpha\beta}(\vec{q},\omega) = \frac{\partial\langle\pi^\alpha(\vec{q},\omega)\rangle}{\partial h^\beta(\vec{q},\omega)} = \chi_0^T(\vec{q},\omega)\,\delta^{\alpha\beta}\,,$$

$$\chi_0^T(\vec{q},\omega) = \frac{1}{-\frac{i\omega}{Dq^a} + q^2}$$ transverse response function

"massless": $n-1$ Goldstone modes

$$\chi_0^{nn}(\vec{q},\omega) = \frac{\partial\langle\sigma(\vec{q},\omega)\rangle}{\partial h^n(\vec{q},\omega)} = \chi_0^L(\vec{q},\omega)\,,$$

$$\chi_0^L(\vec{q},\omega) = \frac{1}{-\frac{i\omega}{Dq^a} + 2|\tau| + q^2}$$ longitudinal response function

and $\chi_0^T(\vec{q},t) = D q^a \, e^{-Dq^{2+a}t} \, \theta(t)$ always "critical"

$$\chi_0^L(\vec{q},t) = D q^a \, e^{-Dq^a(2|\tau|+q^2)t} \, \theta(t)$$

dynamic correlation functions in Gaussian approximation:

$$C_0^T(\vec{q},\omega) = \frac{2Dk_BTq^a}{\omega^2+(Dq^{2+a})^2} \, , \quad C_0^L(\vec{q},\omega) = \frac{2Dk_BTq^a}{\omega^2+[Dq^a(2|\tau|+q^2)]^2}$$

$$C_0^T(\vec{q},t) = \frac{k_BT}{q^2} \, e^{-Dq^{2+a}|t|}$$

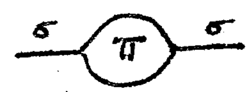
$$C_0^L(\vec{q},t) = \frac{k_BT}{2|\tau|+q^2} \, e^{-Dq^a(2|\tau|+q^2)|t|}$$

note static limits: $\chi_0^T(\vec{q}) = \chi_0^T(\vec{q},\omega=0) = \frac{1}{q^2}$ "massless"

$\chi_0^L(\vec{q}) = \frac{1}{2|\tau|+q^2}$ "massive"

$$C_0^T(\vec{q}) = C_0^T(\vec{q},t=0) = \frac{k_BT}{q^2} \, , \quad C_0^L(\vec{q}) = \frac{k_BT}{2|\tau|+q^2}$$

coupling of longitudinal fluctuations to Goldstone modes induces coexistence singularities:



$$\chi^L(\vec{q},\omega) \sim \frac{1}{\left|-\frac{i\omega}{Dq^a}+q^2+h^n\right|^{\frac{d-4}{2}}}$$

Reversible mode-coupling terms and equilibrium condition

(8 points)

equilibrium condition for reversible stationary probability current:

divergence-free in space of random variables:

$$0 \stackrel{!}{=} \int d^d x \sum_\alpha \frac{\delta}{\delta S^\alpha(\vec{x})} \left(F^\alpha_{rev}[S^\alpha](\vec{x})\, e^{-H[S^\alpha(\vec{x})]/k_B T} \right) =$$

$$= \int d^d x \sum_\alpha \left[\frac{\delta F^\alpha_{rev}[S^\alpha](\vec{x})}{\delta S^\alpha(\vec{x})} - \frac{F^\alpha_{rev}[S^\alpha](\vec{x})}{k_B T} \frac{\delta H[S^\alpha]}{\delta S^\alpha(\vec{x})} \right] e^{-H[S^\alpha]/k_B T}$$

ansatz:

$$F^\alpha_{rev}[S^\alpha](\vec{x}) = -\int d^d x' \sum_{\alpha'} \left[Q^{\alpha\alpha'}(\vec{x},\vec{x}') \frac{\delta H[S^\alpha]}{\delta S^{\alpha'}(\vec{x}')} - k_B T \frac{\delta Q^{\alpha\alpha'}(\vec{x},\vec{x}')}{\delta S^{\alpha'}(\vec{x}')} \right]$$

with antisymmetric $Q^{\alpha\alpha'}(\vec{x},\vec{x}') = -Q^{\alpha'\alpha}(\vec{x}',\vec{x})$

$$\rightarrow \sum_\alpha \frac{\delta F^\alpha_{rev}[S^\alpha](\vec{x})}{\delta S^\alpha(\vec{x})} = -\int d^d x' \sum_{\alpha,\alpha'} \left[\frac{\delta Q^{\alpha\alpha'}(\vec{x},\vec{x}')}{\delta S^\alpha(\vec{x})} \frac{\delta H[S^\alpha]}{\delta S^{\alpha'}(\vec{x}')} \right.$$

$$\left. + \underbrace{Q^{\alpha\alpha'}(\vec{x},\vec{x}') \frac{\delta^2 H[S^\alpha]}{\delta S^\alpha(\vec{x}) \delta S^{\alpha'}(\vec{x}')} - k_B T \frac{\delta^2 Q^{\alpha\alpha'}(\vec{x},\vec{x}')}{\delta S^\alpha(\vec{x}) \delta S^{\alpha'}(\vec{x}')}}_{\text{vanish because of antisymmetry in } \alpha \leftrightarrow \alpha',\ \vec{x} \leftrightarrow \vec{x}'} \right]$$

after $\int d^d x$ integration: vanish because of antisymmetry in $\alpha \leftrightarrow \alpha'$, $\vec{x} \leftrightarrow \vec{x}'$

this leaves in equilibrium condition:

$$-\int d^d x \int d^d x' \sum_{\alpha,\alpha'} \left[\frac{\delta Q^{\alpha\alpha'}(\vec{x},\vec{x}')}{\delta S^\alpha(\vec{x})} \frac{\delta H[S^\alpha]}{\delta S^{\alpha'}(\vec{x}')} - \frac{Q^{\alpha\alpha'}(\vec{x},\vec{x}')}{k_B T} \underbrace{\frac{\delta H[S^\alpha]}{\delta S^\alpha(\vec{x})} \frac{\delta H[S^\alpha]}{\delta S^{\alpha'}(\vec{x}')}}_{=0 \text{ from antisymmetry}} \right.$$

$$\left. + \frac{\delta Q^{\alpha\alpha'}(\vec{x},\vec{x}')}{\delta S^{\alpha'}(\vec{x}')} \frac{\delta H[S^\alpha]}{\delta S^\alpha(\vec{x})} \right] e^{-H[S^\alpha]/k_B T} = 0 \checkmark$$

combine to 0 after $\alpha \leftrightarrow \alpha'$, $\vec{x} \leftrightarrow \vec{x}'$ in second term