Bangalore School on Statistical Physics VIII
Course 2: CRITICAL DYNAMICS
Problem set 1

1. *Scaling hypotheses and scaling relations.*
   (a) The scaling hypothesis for the singular part of the free energy near a second-order (continuous) phase transition reads

$$f_{\text{sing.}}(\tau, h) = |\tau|^{2-\alpha} \, \hat{f}\Big(h/|\tau|^{\Delta}\Big) \ ,$$

   where $\tau = (T - T_c)/T_c$ and $\hat{f}$ denotes a non-singular scaling function. Show that the specific heat $C_{h=0} \sim |\tau|^{-\alpha}$ as $|\tau| \to 0$.
   (b) Determine the critical exponents β, δ, and γ, and derive the scaling relations $\Delta = \beta\,\delta$, $\alpha + \beta\,(1+\delta) = 2 = \alpha + 2\,\beta + \gamma$, $\gamma = \beta\,(\delta - 1)$.
   Hint: Consider the behavior of the derivative $\hat{f}'(x)$ as $x \to \infty$.
   (c) Similarly, use the scaling ansatz for the correlation function

$$G(\vec{q}, \tau) = |\vec{q}|^{-2+\eta} \, \hat{G}(|\vec{q}|\,\xi) \ , \quad \xi \sim |\tau|^{-\nu} \ ,$$

   and show that $\gamma = \nu\,(2-\eta)$, $\beta = \nu\,(d - 2 + \eta)/2$.
   Hint: Remember that $\chi_T = G(\vec{q} = 0, T)/k_{\rm B}T$. Away from the critical point, what is the limit of $G(|\vec{x}| \to \infty, \tau)$ in d dimensions ?

2. *Critical order parameter decay and dynamic correlations.*
   (a) Show that the dynamic scaling ansatz for the order parameter

$$\langle S(\tau, t)\rangle = |\tau|^{\beta} \hat{S}\Big(t/t_c(\tau)\Big)$$

   with the characteristic relaxation time scale $t_c(\tau) \sim \xi(\tau)^z \sim |\tau|^{-z\nu}$ implies the long-time algebraic decay $\langle S(0,t)\rangle \sim t^{-\beta/z\nu}$ at $T = T_c$.
   (b) Confirm that the dynamic scaling form for the response function

$$\chi(\tau, \vec{q}, \omega) = |\vec{q}|^{-2+\eta} \, \hat{\chi}_{\pm}\Big(\vec{q}\,\xi(\tau), \omega\,\xi(\tau)^z\Big)$$

   leads to the scaling forms for the dynamic correlation function

$$C(\tau, \vec{q}, \omega) = |\vec{q}|^{-z-2+\eta} \, \hat{C}_{\pm}\Big(\vec{q}\,\xi(\tau), \omega\,\xi(\tau)^z\Big) \, ,$$
$$C(\tau, \vec{x}, t) = |\vec{x}|^{-d+2-\eta} \, \widetilde{C}_{\pm}\Big(\vec{x}/\xi(\tau), t/\xi(\tau)^z\Big) \ .$$