Scaling hypotheses and scaling relations. **(10 pts.)**

(a) The *scaling hypothesis* for the *singular* part of the *free energy* near a *second-order (continuous) phase transition* reads

$$f_{\text{sing.}}(\tau, h) = |\tau|^{2-\alpha} \hat{f}\left(h/|\tau|^{\Delta}\right) ,$$

where $\tau = (T - T_c)/T_c$, and $\hat{f}$ denotes a *non-singular scaling function.* At vanishing external field $h = 0$, we have $f_{\text{sing.}}(\tau, 0) = |\tau|^{2-\alpha} \hat{f}(0)$, and consequently the *specific heat* is

$$C_{h=0} = -\frac{T}{T_c^2} \left(\frac{\partial^2 f_{\text{sing.}}}{\partial \tau^2} \right)_{h=0} \approx -\frac{1}{T_c} (2-\alpha)(1-\alpha) |\tau|^{-\alpha} \hat{f}(0) \sim |\tau|^{-\alpha}$$

as $\tau \to 0$ both from above and below.

(b) We obtain the *equation of state* for the *order parameter* (e.g., magnetization) via the partial derivative

$$m = -\left(\frac{\partial f_{\text{sing.}}}{\partial h} \right)_{\tau} = -|\tau|^{2-\alpha} \hat{f}'\left(h/|\tau|^{\Delta}\right) |\tau|^{-\Delta} .$$

At $h = 0$, we find for the *spontaneous magnetization* (*coexistence line*)

$$m = -|\tau|^{2-\alpha-\Delta} \hat{f}'(0) \sim |\tau|^{\beta} , \quad \beta = 2 - \alpha - \Delta .$$

For the *critical isotherm* at $\tau = 0$, on the other hand, the temperature dependence in the scaling function $\hat{f}'$ must cancel the algebraic prefactor, i.e., $\hat{f}'(x) \to x^{(2-\alpha-\Delta)/\Delta}$ as $x \to \infty$. Consequently,

$$\phi \propto h^{(2-\alpha-\Delta)/\Delta} = h^{\beta/\Delta} \sim h^{1/\delta} , \quad \delta = \Delta/\beta ,$$

and $2 - \alpha = \beta + \Delta = \beta(1+\delta)$. Next, we consider the *isothermal susceptibility*

$$\chi_T = \left(\frac{\partial m}{\partial h} \right)_{\tau, h \to 0} = |\tau|^{2-\alpha-2\Delta} \hat{f}''(0) \sim |\tau|^{-\gamma} , \quad \gamma = \alpha + 2(\Delta - 1) .$$

We may now eliminate Δ: $\gamma + 2\beta = \alpha + 2(\Delta - 1) + 2(2 - \alpha - \Delta) = 2 - \alpha$, and with the earlier scaling relation also get $\gamma + \beta = \beta\delta$. This completes the set of *scaling relations* for the *critical exponents* $\alpha, \beta, \gamma, \delta$, and Δ,

$$\Delta = \beta\delta , \quad \alpha + \beta(1+\delta) = 2 = \alpha + 2\beta + \gamma , \quad \gamma = \beta(\delta - 1) ,$$

which leaves us with only *two independent exponents* for the free energy and thus the entire thermodynamics near a continuous phase transition.

(c) Similarly, we exploit the scaling ansatz for the *correlation function*

$$G(\vec{q},\tau) = |\vec{q}|^{-2+\eta}\, \hat{G}(|\vec{q}|\,\xi) \ , \quad \xi \propto |\tau|^{-\nu} \ .$$

First, we employ the *fluctuation-response theorem*,

$$\chi_\tau = \frac{1}{k_{\rm B}T}\, G(\vec{q}=0,\tau) \approx \frac{1}{k_{\rm B}T_c}\, \lim_{\vec{q}\to 0} |\vec{q}|^{-2+\eta}\, \hat{G}(|\vec{q}|\,\xi) \ .$$

As $\vec{q} \to 0$, the scaling function must cancel the singular prefactor, and therefore

$$\chi_\tau \propto |\vec{q}|^{-2+\eta}\, (|\vec{q}|\,\xi)^{2-\eta} \propto |\tau|^{-\nu\,(2-\eta)} \sim |\tau|^{-\gamma} \ , \quad \gamma = \nu\,(2-\eta) \ .$$

Applying the same reasoning for the spatial correlations as $|\vec{x}| \to \infty$, we see that

$$|\vec{x}| \to \infty : \ G(\vec{x},\tau) = \frac{1}{|\vec{x}|^{d-2+\eta}}\, \widetilde{G}(|\vec{x}|/\xi) \to \xi^{-(d-2+\eta)} \sim |\tau|^{\nu\,(d-2+\eta)} \ .$$

On the other hand, away from the critical point

$$|\vec{x}| \to \infty : \ G(\vec{x},\tau) = \langle m(\vec{x})\, m(0)\rangle - \langle m\rangle^2 \to -\langle m\rangle^2 \sim |\tau|^{2\beta} \ ,$$

yielding the identity (*hyperscaling relation*)

$$\beta = \frac{\nu}{2}\,(d-2+\eta) \ .$$

Thus, the newly introduced critical exponents of the correlation function are in fact *not* independent and can be expressed in terms of the two thermodynamic exponents.

Critical order parameter decay and dynamic correlations (8 points)

(a) Dynamic scaling ansatz for order parameter:

$$\phi^n(\tau,t) = |\tau|^\beta \hat{\phi}(t/t_c(\tau)), \quad t_c(\tau) \sim \xi(\tau)^z \sim |\tau|^{-z\nu}$$

$$\hat{\phi}(x\to\infty) = \text{const.}, \quad x = \frac{t}{t_c(\tau)} \to \phi^n(\tau, t\to\infty) \sim |\tau|^\beta$$

$x = \frac{t}{t_c(\tau)} \to 0$: $|\tau|$ dependence must cancel as $|\tau| \to 0$

$\to$ requires $\hat{\phi}(x) \sim x^{-\frac{\beta}{z\nu}}$:

$$\phi^n(0,t) \sim |\tau|^\beta \left(t|\tau|^{z\nu}\right)^{-\frac{\beta}{z\nu}} = t^{-\frac{\beta}{z\nu}}$$

$\to$ algebraic order parameter decay at $T = T_c$

(b) Dynamic scaling form for response function: $\xi(\tau) \sim |\tau|^{-\nu}$

$$\chi(\tau,\vec{q},\omega) = |\vec{q}|^{-2+\eta} \hat{\chi}_\pm(\vec{q}\,\xi(\tau), \omega\,\xi(\tau)^z) \quad \text{for } T \gtrless T_c$$

$\to$ dynamic correlation function via fluctuation-dissipation theorem:

$$C(\tau,\vec{q},\omega) = \frac{2k_B T}{\omega} \operatorname{Im} \chi(\tau,\vec{q},\omega) \sim \frac{(q\xi)^z}{\omega(q\xi)^z} q^{-2+\eta} \operatorname{Im} \hat{\chi}_\pm(\vec{q}\xi, \omega\xi^z)$$

$$= |\vec{q}|^{-z-2+\eta} \hat{C}_\pm(\vec{q}\,\xi(\tau), \omega\,\xi(\tau)^z)$$

where $\hat{C}_\pm(\vec{k},\nu) \sim \frac{|\vec{k}|^z}{\nu} \operatorname{Im} \hat{\chi}_\pm(\vec{k},\nu)$, $\vec{k} = \vec{q}\xi$, $\nu = \omega\xi^z$

inverse Fourier transform:

$$C(\tau,\vec{x},t) = \int \frac{d^dq}{(2\pi)^d} \int \frac{d\omega}{2\pi} C(\tau,\vec{q},\omega) e^{i(\vec{q}\cdot\vec{x}-\omega t)}$$

$$= \xi^{z+2-\eta} \xi^{-d-z} \int \frac{d^d(q\xi)}{(2\pi)^d} \int \frac{d(\omega\xi^z)}{2\pi} (|\vec{q}|\xi)^{-z-2+\eta} \hat{C}_\pm(\vec{q}\xi, \omega\xi^z) \cdot e^{i(\vec{q}\xi\frac{\vec{x}}{\xi} - \omega\xi^z t/\xi^z)}$$

$$= |\vec{x}|^{-d+2-\eta} \tilde{C}_\pm\left(\frac{\vec{x}}{\xi(\tau)}, \frac{t}{\xi(\tau)^z}\right)$$

where $\tilde{C}_\pm(\vec{y},x) = |\vec{y}|^{d-2+\eta} \int \frac{d^dk}{(2\pi)^d} \int \frac{d\nu}{2\pi} |\vec{k}|^{-z-2+\eta} \hat{C}_\pm(\vec{k},\nu) \cdot e^{i(\vec{k}\cdot\vec{y}-\nu x)}$

$$\vec{y} = \frac{\vec{x}}{\xi}, \quad x = \frac{t}{\xi^z}$$