

3. Driven systems displaying generic scale invariance

driven lattice gas, asymmetric exclusion process:

random walkers on d -dimensional lattice, biased along "||" direction

interactions: hard-core exclusion, site occupation $n_i = 0$ or 1 only

equivalent Ising "spin" representation: $\sigma_i = n_i - \frac{1}{2} = \pm \frac{1}{2}$

coarse-grained continuum description: densities $S(\vec{x}, t) = g(\vec{x}, t) - \frac{1}{2}$

half-filled lattice: $\langle g \rangle = \frac{1}{2}$, $\langle S \rangle = 0$ zero mean magnetization

particle number conservation $\rightarrow$ continuity equation $\frac{\partial S(\vec{x}, t)}{\partial t} + \vec{\nabla} \cdot \vec{j}(\vec{x}, t) = 0$

- transverse spatial sector, $(d-1)$ -dimensional: noisy diffusion current

$$\vec{j}_\perp(\vec{x}, t) = -D \vec{\nabla}_\perp S(\vec{x}, t) + \vec{\eta}_\perp(\vec{x}, t), \quad \langle \vec{\eta}_\perp \rangle = 0$$

$$\langle \eta_i(\vec{x}, t) \eta_j(\vec{x}', t') \rangle = 2\tilde{D} \delta(\vec{x} - \vec{x}') \delta(t - t') \delta_{ij}$$

note: Einstein relation can be restored formally via rescaling

- longitudinal "||" sector: subject to bias $\hat{=}$ effective drive, non-linearity

$$\text{exclusion interaction} \sim g(\vec{x}, t) [1 - g(\vec{x}, t)] = \frac{1}{4} - S(\vec{x}, t)^2$$

$$\rightarrow j_\parallel(\vec{x}, t) = j_0 - c D \nabla_\parallel S(\vec{x}, t) - \frac{Dg}{2} S(\vec{x}, t)^2 + \eta_\parallel(\vec{x}, t), \quad \langle \eta_\parallel \rangle = 0$$

$$\langle \eta_\parallel(\vec{x}, t) \eta_\parallel(\vec{x}', t') \rangle = 2\tilde{c} D \delta(\vec{x} - \vec{x}') \delta(t - t')$$

ratio $w = \frac{\tilde{c}}{c}$ measures deviation from thermal equilibrium ($\ln T = 1$)

$\rightarrow$ DDS Langevin equation: with $\zeta(\vec{x}, t) = -\nabla_\parallel \eta_\parallel(\vec{x}, t) - \vec{\nabla}_\perp \cdot \vec{\eta}_\perp(\vec{x}, t)$

$$\frac{\partial S(\vec{x}, t)}{\partial t} = D(c \nabla_\parallel^2 + \vec{\nabla}_\perp^2) S(\vec{x}, t) + \frac{Dg}{2} \nabla_\parallel S(\vec{x}, t)^2 + \zeta(\vec{x}, t), \quad \langle \zeta \rangle = 0$$

$$\langle \zeta(\vec{x}, t) \zeta(\vec{x}', t') \rangle = -2D(\tilde{c} \nabla_\parallel^2 + \vec{\nabla}_\perp^2) \delta(\vec{x} - \vec{x}') \delta(t - t')$$

drive (and non-linearity) breaks Ising $\mathbb{Z}_2$ symmetry $S \rightarrow -S$
and spatial inversion symmetry

14 Gaussian correlation function: $C_0(q_{||}, \vec{q}_{\perp}, \omega) = \frac{2D(\tilde{c}q_{||}^2 + \vec{q}_{\perp}^2)}{\omega^2 + D^2(cq_{||}^2 + \vec{q}_{\perp}^2)^2}$

"massless" $\rightarrow$ generic scale invariance

static structure factor $C_0(q_{||}, \vec{q}_{\perp}, t=0) = \frac{\tilde{c}q_{||}^2 + \vec{q}_{\perp}^2}{cq_{||}^2 + \vec{q}_{\perp}^2}$

origin $\vec{q} \rightarrow 0$: discontinuity singularity, $q_{||} = q \cos \theta$, $q_{\perp} = q \sin \theta$:

$\lim_{\vec{q} \rightarrow 0} C_0(q_{||}, \vec{q}_{\perp}, t=0) = \frac{\tilde{c} + \tan^2 \theta}{c + \tan^2 \theta}$ depends on angle θ if $w \neq 1$

$\rightarrow$ singularity disappears in equilibrium

anisotropic scaling: $C(q_{||}, \vec{q}_{\perp}, \omega) = |\vec{q}_{\perp}|^{-2+\eta} \hat{C}\left(\frac{q_{||}}{|\vec{q}_{\perp}|^{1+\Delta}}, \frac{\omega}{|\vec{q}_{\perp}|^z}\right)$

mean-field approximation: $\eta = 0$, $z = 2$, $\Delta = 0$

but non-linearity only in "||" sector: $\eta = 0$, $z = 2$ actually exact

Fourier transform: $C(x_{||}, \vec{x}_{\perp}, t) = t^{-\frac{\lambda_c}{z}} \tilde{C}\left(\frac{x_{||}}{|\vec{x}_{\perp}|^{1+\Delta}}, \frac{t}{|\vec{x}_{\perp}|^{z_{||}}}\right)$

with $z_{||} = \frac{z}{1+\Delta} = \frac{z}{1+\Delta}$, $\lambda_c = d-1+1+\Delta + \underbrace{z-2+\eta}_{=0} = d+\Delta$

emergent Galilean invariance: not present on microscopic level

$x_{||} \rightarrow x'_{||} = x_{||} - Dgv t$, $\vec{x}_{\perp} \rightarrow \vec{x}'_{\perp} = \vec{x}_{\perp}$, $t \rightarrow t' = t$, with arbitrary $v = \text{const.}$

$S(x_{||}, \vec{x}_{\perp}, t) \rightarrow S'(x'_{||}, \vec{x}'_{\perp}, t') = S(x_{||} - Dgv t, \vec{x}_{\perp}, t) - v$

$\rightarrow$ fixes anisotropy scaling exponent:

$v \sim S \sim t^{-\frac{\lambda_c}{z}} \sim t^{-\frac{d+\Delta}{4}} \sim \frac{x_{||}}{t} \sim t^{-1+\frac{1}{z_{||}}} \sim t^{-1+\frac{1+\Delta}{2}}$

$\rightarrow d+\Delta = 2(1-\Delta)$, $\Delta = \frac{2-d}{3}$, $d_c = 2$ upper critical dimension

$z_{||} = \frac{6}{5-d}$ ($= \frac{3}{2}$ for $d=1$), $\lambda_c = \frac{2}{3}(d+1)$ ($= \frac{4}{3}$ for $d=1$)

aging scaling exponents too fixed by conservation law:

$C(\vec{x}=0, t, t' \rightarrow 0) \sim t^{-\frac{\lambda_c}{z}} \left(\frac{t}{t'}\right)^{\theta-1}$, $\theta-1 = 1+\Delta = \frac{5-d}{3}$

15 compare noisy Burgers equation for vorticity-free velocity field $\vec{u}(\vec{x}, t)$:

$$\frac{\partial \vec{u}(\vec{x}, t)}{\partial t} + (Dg) \frac{1}{2} \vec{\nabla} [\vec{u}(\vec{x}, t)^2] = D \vec{\nabla}^2 \vec{u}(\vec{x}, t) + \vec{\zeta}(\vec{x}, t), \quad \langle \vec{\zeta} \rangle = 0$$

$$\langle \zeta_i(\vec{x}, t) \zeta_j(\vec{x}', t') \rangle = -2D \nabla_i \nabla_j \delta(\vec{x} - \vec{x}') \delta(t - t') \quad (\text{again, } h_B T = 1)$$

invariant under Galilei transformation $\vec{u}(\vec{x}, t) \rightarrow \vec{u}'(\vec{x}', t') = \vec{u}(\vec{x} - (Dg)\vec{v}t, t) + \vec{v}$:

$$\frac{\partial \vec{u}'}{\partial t} = \frac{\partial \vec{u}}{\partial t} - (Dg)(\vec{v} \cdot \vec{\nabla}) \vec{u} = D \vec{\nabla}^2 \vec{u} - (Dg) \frac{1}{2} \vec{\nabla} \vec{u}^2 - (Dg)(\vec{v} \cdot \vec{\nabla}) \vec{u} + \vec{\zeta} =$$

integrability: $\nabla_i u_j = \nabla_j u_i$ $= - (Dg) \frac{1}{2} \vec{\nabla} (\vec{u} + \vec{v})^2 = D \vec{\nabla}^2 \vec{u}' - (Dg) \frac{1}{2} \vec{\nabla} \vec{u}'^2 + \vec{\zeta}$

$d=1$, one dimension: $S(\vec{x}, t) = -u(\vec{x}, t)$ (or $g \rightarrow -g$)

$\rightarrow$ DDS Langevin equation (for $w=1$) $\rightarrow \gamma=0, z=z_{||} = \frac{3}{2}$

actually equilibrium dynamics with $P_{st}(u) \sim e^{-\frac{1}{2} \int u(x)^2 dx}$ (kinetic energy)

check equilibrium condition, $\vec{F}_{rev}(\vec{u}) = -\frac{Dg}{2} \vec{\nabla} \vec{u}^2$:

$$\int d^d x \frac{\delta}{\delta \vec{u}(\vec{x})} [\vec{\nabla} \vec{u}(\vec{x})^2] e^{-\frac{1}{2} \int \vec{u}(\vec{x}')^2 d^d x'} =$$

$$= \int \left[\underbrace{2 \vec{\nabla} \cdot \vec{u}(\vec{x}) - \vec{u}(\vec{x}) \cdot \vec{\nabla} \vec{u}(\vec{x})^2}_{\text{boundary term, } = 0} \right] d^d x e^{-\frac{1}{2} \int \vec{u}(\vec{x}')^2 d^d x'}$$

only for $d=1$: $-\int u(x) \frac{du(x)^2}{dx} dx = \int u(x)^2 \frac{du(x)}{dx} dx = \frac{1}{3} \int \frac{du(x)^3}{dx} dx$

also boundary term

$\rightarrow$ "hidden" fluctuation-dissipation theorem for DDS, Burgers equation, $d=1$

variant: add attractive nearest-neighbor spin/particle interactions

$\rightarrow$ driven Ising lattice gas, Katz-Lebowitz-Spohn (KLS) model

$\rightarrow d > d_{qc} = 1$: continuous non-equilibrium phase transition at T_c

$T > T_c$: generic scale invariance, DDS scaling laws

$T < T_c$: ordered stripes along drive direction, slow coarsening

$T \rightarrow T_c$: driven model B, critical KLS model
upper critical dimension $d_c = 5$

16 continuum Langevin description:

$$\frac{\partial S(\vec{x}, t)}{\partial t} = D \left[c \nabla_{\parallel}^2 + \vec{\nabla}_{\perp}^2 (\tau - \vec{\nabla}_{\perp}^2) \right] S(\vec{x}, t) + \frac{Dg}{2} \nabla_{\parallel} S(\vec{x}, t)^2 + \frac{Du}{6} \vec{\nabla}_{\perp}^2 S(\vec{x}, t)^3 + \zeta(\vec{x}, t)$$

turns out irrelevant in RG sense ($d_c' = 3 < d_c = 5$)

Gaussian correlation function:

$$C_0(q_{\parallel}, \vec{q}_{\perp}, \omega) = \frac{2D(\tilde{c} q_{\parallel}^2 + \vec{q}_{\perp}^2)}{\omega^2 + D^2 [c q_{\parallel}^2 + \vec{q}_{\perp}^2 (\tau + \vec{q}_{\perp}^2)]^2}$$

anisotropic critical scaling: $C(\tau, q_{\parallel}, \vec{q}_{\perp}, \omega) = |\vec{q}_{\perp}|^{-z-2+\gamma} \hat{C}_{\pm} \left(\frac{\tau}{|\vec{q}_{\perp}|^{1+\Delta}}, \frac{q_{\parallel}}{|\vec{q}_{\perp}|^{1+\Delta}}, \frac{\omega}{|\vec{q}_{\perp}|^z} \right)$

→ mean-field critical exponents: $\nu = \frac{1}{2}, \gamma = 0, z = 4, \Delta = 1$

fluctuations critical only in transverse sector ($\tilde{c}$ in noise irrelevant),

but non-linearity pertains only to "||" sector → $\nu = \frac{1}{2}, \gamma = 0, z = 4$ exact

Fourier transform: $C(\tau, x_{\parallel}, \vec{x}_{\perp}, t) = t^{-\frac{\lambda_c}{z}} \tilde{C} \left(\tau |x_{\parallel}|^{\frac{1}{\nu_{\parallel}}}, \frac{x_{\parallel}}{|\vec{x}_{\perp}|^{1+\Delta}}, \frac{t}{|x_{\parallel}|^{z_{\parallel}}} \right)$

with $\nu_{\parallel} = \nu(1+\Delta) = \frac{1}{2}(1+\Delta), z_{\parallel} = \frac{z}{1+\Delta} = \frac{4}{1+\Delta}, \lambda_c = d-1+1+\Delta-2+\gamma = d-2+\Delta$

Galilean invariance: $v \sim S \sim t^{-\frac{\lambda_c}{2z}} \sim t^{-\frac{d-2+\Delta}{8}} \sim \frac{x_{\parallel}}{t} \sim t^{-1+\frac{1}{z_{\parallel}}} \sim t^{-1+\frac{1+\Delta}{4}}$

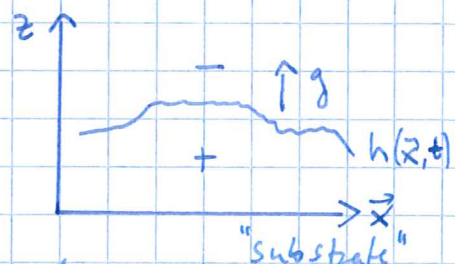
→ $d-2+\Delta = 2(3-\Delta), \Delta = \frac{8-d}{3}, \nu_{\parallel} = \frac{11-d}{6}, z_{\parallel} = \frac{12}{11-d}, \lambda_c = \frac{2}{3}(d+1)$

aging scaling: $C(\vec{x}=0, t, t' \rightarrow 0) \sim t^{-\frac{\lambda_c}{z}} \left(\frac{t}{t'} \right)^{\theta-1}, \theta-1 = \Delta = \frac{8-d}{3}$

driven interface, e.g., in Ising ferromagnet:

field h_z moves interface along z direction

interface height function: $z = h(\vec{x}, t)$



interfacial energy: $H[h] = (\sigma) \int \sqrt{1 + [\vec{\nabla} h(\vec{x}, t)]^2} d^d x$ surface tension σ

small fluctuations: $\approx \text{const.} + \frac{\sigma}{2} \int [\vec{\nabla} h(\vec{x}, t)]^2 d^d x$

relaxation rate too proportional to surface element:

$$\frac{\partial h(\vec{x}, t)}{\partial t} = D \sqrt{1 + [\vec{\nabla} h(\vec{x}, t)]^2} \left(- \frac{\delta H[h]}{\delta h(\vec{x}, t)} + g \right) + \zeta(\vec{x}, t)$$

$\approx \vec{\nabla}^2 h(\vec{x}, t)$

$|\vec{\nabla} h|^2 \ll 1: \approx D \vec{\nabla}^2 h(\vec{x}, t) + Dg + \frac{Dg}{2} [\vec{\nabla} h(\vec{x}, t)]^2 + \zeta(\vec{x}, t)$

17 shift to comoving frame: $S(\vec{x}, t) = h(\vec{x}, t) - Dg t$

→ Kardar-Parisi-Zhang (KPZ) equation:

$$\frac{\partial S(\vec{x}, t)}{\partial t} = D \nabla^2 S(\vec{x}, t) + \frac{Dg}{2} [\nabla S(\vec{x}, t)]^2 + \zeta(\vec{x}, t), \quad \langle \zeta \rangle = 0$$

$$\langle \zeta(\vec{x}, t) \zeta(\vec{x}', t') \rangle = 2D \delta(\vec{x} - \vec{x}') \delta(t - t') \quad (\text{after rescaling})$$

also for surface growth, random deposition, curvature-driven:

$$\frac{\partial h(\vec{x}, t)}{\partial t} = v(\nabla h) = v(0) + \vec{v} \cdot \nabla h + \frac{Dg}{2} (\nabla h)^2 + \dots$$

shift $S(\vec{x}, t) = h(\vec{x} - \vec{v}t, t) - v(0)t$ to comoving frame

add diffusive relaxation: surface smoothening, white noise $\rightarrow$ KPZ

linear model:
$$\frac{\partial S(\vec{x}, t)}{\partial t} = D \nabla^2 S(\vec{x}, t) + \zeta(\vec{x}, t)$$

Edwards-Wilkinson equation: noisy diffusion equation

scaling exponents: critical Gaussian model A, $\eta = 0, z = 2$

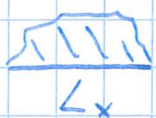
effectively equilibrium, $P_{st}[S] \sim e^{-\frac{1}{2} \int [\nabla S(\vec{x})]^2 d^d x}$ (interfacial energy)

general scaling form for correlation function: $C(\vec{x}, t) = |\vec{x}|^{2\chi} \hat{C}\left(\frac{t}{|\vec{x}|^2}\right)$

with roughness (wandering) exponent $\chi = \frac{z-d-\eta}{2}$

$d < 2$: $\chi > 0$, surface "rough" up to length scale $L_x(t) \sim t^{1/2}$

consider growing interface "bulge": $L_x(t) \sim t^{1/2}$, $L_z \sim L_x^\chi$

$L_z \uparrow$  $\frac{dL_x}{dt} \sim \frac{L_z}{L_x} \sim L_x^{\chi-1} \rightarrow L_x(t) \sim t^{\frac{1}{2-\chi}} \rightarrow \text{scaling relation } z + \chi = 2$

$\tilde{u}(\vec{x}, t) = -\nabla S(\vec{x}, t)$: KPZ equation $\leftrightarrow$ noisy Burgers equation

Galilean invariance $\leftrightarrow$ local tilt symmetry: $S(\vec{x}, t) \rightarrow S'(\vec{x}', t') = S(\vec{x} - Dg \vec{v} t, t) - \vec{v} \cdot \vec{x}$
 $\rightarrow v \sim \frac{S}{x} \sim x^{\chi-1} \sim \frac{x}{t} \sim x^{1-z}$ recovers $z + \chi = 2$

mean-field exponents $\eta = 0, z = 2, \chi = 0$ reached at $d_c = 2$

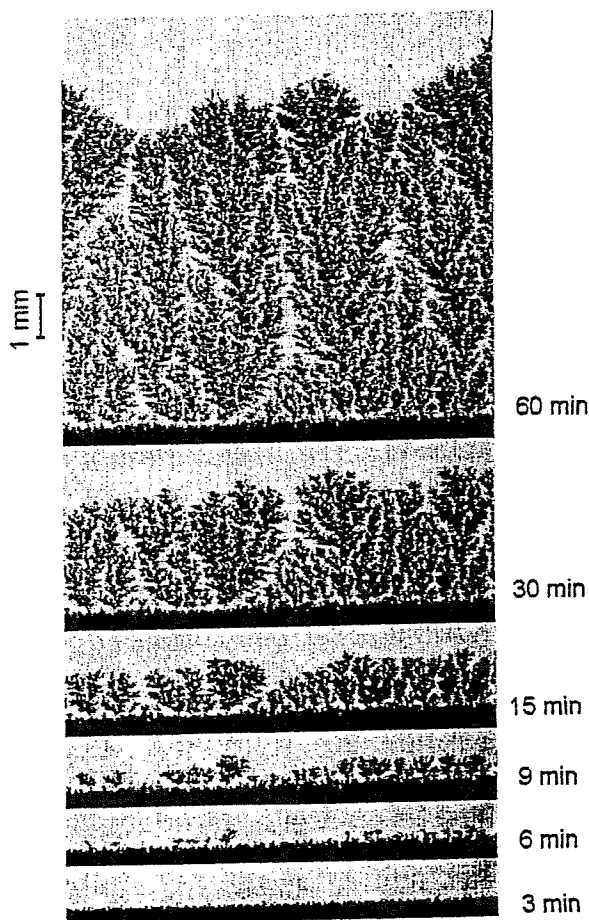


FIG. 2. *In situ* lateral micrographs showing the interface evolution from $t=0$ to $t=60$ min for Ag electrodeposition at $j = 1 \text{ mA cm}^{-2}$ in $5 \times 10^{-3} \text{ M Ag}_2\text{SO}_4 + 10^{-2} \text{ M H}_2\text{SO}_4 + 0.5 \text{ M Na}_2\text{SO}_4$.

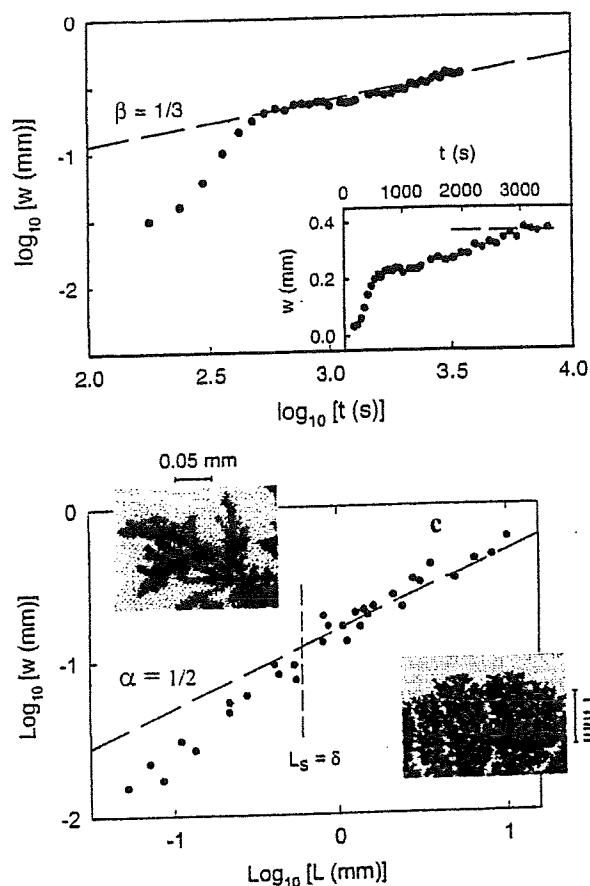


FIG. 3. (a) $\langle h \rangle$ vs t plot resulting from the analysis of profile shown in Fig. 2. The linear portion for $t > t_s$ is drawn. (b) $\text{Log}_{10} w$ vs $\text{Log}_{10} t$ plot resulting from the analysis of profiles shown in Fig. 2. The slope $\beta = \frac{1}{3}$ has been drawn as a visual guide. The inset shows the w vs t plot. (c) $\text{Log}_{10} w$ vs $\text{Log}_{10} L$ plot for $t = 60$ min. The slope $\alpha = \frac{1}{2}$ has been drawn as a visual guide. Patterns for different scale lengths are shown.

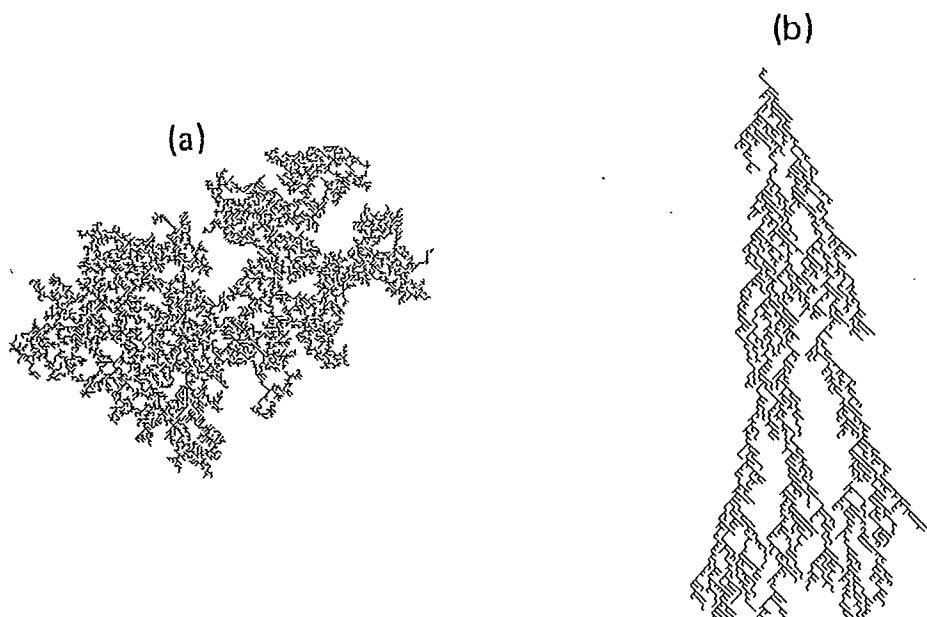


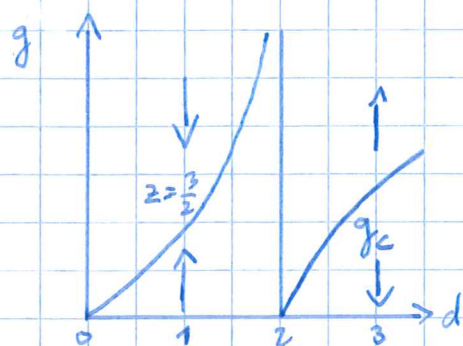
FIG. 1. Isotropic (a) and directed (b) percolation clusters.

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$d=1$: "hidden" FDT, $P_{st}[S]$ as for EW model, $\gamma=0$, $\chi=\frac{1}{2}$, $z=\frac{3}{2}$

→ scaling exponents known exactly, and much about scaling functions as well

$d > d_c = 2$: lower critical dimension for roughening transitions



$g < g_c$: smooth EW surface, $g \rightarrow 0$, $z = z$

$g > g_c$: strong-coupling rough phase, $\chi > 0$

$g = g_c$: non-equilibrium roughening phase transition

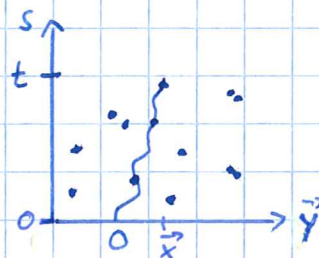
mapping to directed line statistical mechanics, in disordered medium:

line tension ε , random potential $V(\vec{x}, t)$

partition function for single directed line:

($\hat{=}$ dilute magnetic flux lines in type-II superconductors
subject to point-like pinning centers)

$$Z(\vec{x}, t) = \int_{\vec{y}(0)=0}^{\vec{y}(t)=\vec{x}} \mathcal{D}[\vec{y}(s)] \exp \left(-\frac{1}{k_B T} \int_0^t \left[\frac{\varepsilon}{2} \left(\frac{d\vec{y}(s)}{ds} \right)^2 + V(\vec{y}(s), s) \right] ds \right)$$



note: $\hat{=}$ imaginary-time Feynman path integral for quantum particle
with mass ε , $\frac{i}{\hbar} \rightarrow -\frac{1}{k_B T}$ (Wick rotation)

equivalent "Schrödinger" partial differential equation: (linear)

$$k_B T \frac{\partial Z(\vec{x}, t)}{\partial t} = \frac{(k_B T)^2}{2\varepsilon} \nabla^2 Z(\vec{x}, t) + V(\vec{x}, t) Z(\vec{x}, t)$$

diffusion, $D = \frac{k_B T}{2\varepsilon}$

multiplicative noise
with Gaussian-distributed
disorder potential V

Cole-Hopf transformation:

$$S(\vec{x}, t) = \frac{k_B T}{Dg\varepsilon} \ln Z(\vec{x}, t), \quad \text{or free energy } F(\vec{x}, t) = -Dg\varepsilon S(\vec{x}, t)$$

→ KPZ non-linear differential equation with additive noise

$$\partial_t S(\vec{x}, t) = \frac{1}{Dg\varepsilon} V(\vec{x}, t)$$

19 $g < g_c$: smooth interface $\hat{=}$ straight line, line tension dominates over disorder

$g > g_c$: strong-coupling rough phase $\hat{=}$ disorder-dominated vortex glass phase

Kim-Kosterlitz argument: marginal at g_c

$$\rightarrow \chi_c = 0, z_c = 2, \eta_c = 2 - d, \nu_c = \frac{1}{d-2} \quad \text{exact roughening transition exponents}$$

"conserved" KPZ variants: smoother surface relaxation

$$\frac{\partial S(\vec{x}, t)}{\partial t} = -D \vec{\nabla}^2 \left(\vec{\nabla}^2 S(\vec{x}, t) + \frac{g}{2} [\vec{\nabla} S(\vec{x}, t)]^2 \right) + \zeta(\vec{x}, t), \quad \langle \zeta \rangle = 0$$

$$\langle \zeta(\vec{x}, t) \zeta(\vec{x}', t') \rangle = 2D (i\vec{\nabla})^a \delta(\vec{x} - \vec{x}') \delta(t - t')$$

$a = 0$: Wolf-Villain model $\rightarrow$ molecular beam epitaxy

$a = 2$: Sun-Gao-Grant model (fully conserved)

neither displays roughening transition, just generic scale invariance

upper critical dimensions: $d_c = 4 - a$

$$\text{exact scaling relations: } z = 4 - \eta, \quad \chi = \frac{4 - a - d - \eta}{2}$$

$$\epsilon (= d_c - d) \text{ expansion: } \eta = \chi = \frac{\epsilon}{3} + O(\epsilon^2)$$