Quantum Geometry of Correlated Many-Body State

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Outline I

What is an insulator?

Is an insulator a property of low lying excited states or just the ground state property?

Conventional way: the gap between the ground state and the first excited state. Ex: Band Insulator

Mott insulator, Anderson Insulator, or any other kind of insulator(except many body localisation) can be characterized by one definition?

Walter Kohn's view on the insulating state

- The insulating behaviour is a property of the ground state.
- Electrons organise themselves in the ground state as to satisfy a many electrons localisation.
- The insulating state emerges whenever the ground state of the extended system breaks up into sum of function ψ_M $(\Psi(r_1, r_2, \cdot\cdot, r_n) = \sum_M \phi_M(r_1, r_2, \cdot\cdot, r_n))$ which are localized in disconnected region of $\mathcal{R}_M$ of the high dimensional configuration space. Hence electronic localization in an insulator occurs in the configuration space not in the position space
- Kohn aruged that such disconnectedness is in fact a signature of the insulating wave function.

Resta prescription for Many electrons localization

- Resta and Sorrela proposed how to measure the electronic localization length to address the shape of the ground state wavefunction.
- They define the localization tensor $< r_\alpha r_\beta >_c$. It has dimension of square of length; it is an intensive quantity that characterizes the ground state.
- It is related with Quantm geoemtric tensor $\eta_{\alpha\beta}$.

Non-interacting fermions

- Consider N_B tight binding model.

$$H = \sum_{i\alpha, j\beta} t_{ij}^{\alpha\beta} c_{i\alpha}^{\dagger} c_{j\beta} + h.c$$

- In momentum space **k** (where $\mathbf{k}$ takes values in BZ) H is:

$$H = \sum_{\mathbf{k}\alpha\beta} h(\mathbf{k})_{\alpha\beta} c_{\mathbf{k}\alpha}^{\dagger} c_{\mathbf{k}\beta}$$

  The single particle Hamiltonian $h(\mathbf{k})$ in the quasi momentum space **k** is $N_B \times N_B$ matrix.
- Its spectrum is denoted by, $h(\mathbf{k}) u^n(\mathbf{k}) = \epsilon_{\mathbf{n}}(\mathbf{k}) \mathbf{u}^{\mathbf{n}}(\mathbf{k})$
- The single-particle states in the n^{th} band are denoted by $\rho^n(\mathbf{k}) = \mathbf{u}^{\mathbf{n}}(\mathbf{k})(\mathbf{u}^{\mathbf{n}}(\mathbf{k})^{\dagger})$
- Basically, the single particle states, $u^n(\mathbf{k})$ define mapping from the BZ to the projective hilbert space, $\mathbf{k} \to \rho^n(\mathbf{k})$. The projective hilbert space is the space of physical states.

Space of the physical states

- Quantum theory represents the physical states by rays in a Hilbert space.
- A ray in Hilbert space is the following. Consider a vector $|u^n(\mathbf{k})>$. Take it to be normalized $< u^n(\mathbf{k})|u^n\mathbf{k}) >= 1$.
- $|u^n(\mathbf{k})>$ and $e^{i\phi(\mathbf{k})}|u^n(\mathbf{k})>$ represents the same states. So the phase is not detectable. Each state has a direction. These what we call rays.
- Correspondence between normalized vectros in a Hilbert space and the physical states is many to one.

Non-interacting fermions:Bargmann invariant

- Each band of states, we associate a nth Bargmann invariant with every ordrerd sequence of n point in the BZ, $\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2, \cdot\cdot \mathbf{k}_n)$.
  $B^n(\mathbf{k}_1, \mathbf{k}_2, \cdot\cdot \mathbf{k}_n) = Tr(\rho^n(\mathbf{k}_1)\rho^n(\mathbf{k}_2)\rho^n(\mathbf{k}_3)\cdots\rho^n(\mathbf{k}_n))$
- The quantum distances and geometric phases in the BZ come from this map.
- Quantum distance:

$$d^2(\mathbf{k}_1, \mathbf{k}_2) = 1 - B^{(2)}(\mathbf{k}_1, \mathbf{k}_2)$$

- Geometric phase:

$$\Omega(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{B^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{|B^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)|}$$

Quantum geoemtry of Interacting Fermions

- Just we have seen the single particle state is parametrized by quasi momentum $\mathbf{k}$.
- But in the case of many body states there is no such parmetrization. Let us consider a many body state $|\psi >$:

$$|\psi >= \sum_{\mathbf{k}} c_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\cdot\cdot\mathbf{k}_n} |\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_n, \cdot\cdot \mathbf{k}_n >$$

- We can see there is no parametrization as $|\psi(\mathbf{k}) >$ for many body state $|\psi >$.
- Then there is a natural question to ask how to generalize a concept of quantum distance in a physical paramter space such as BZ wave space.

Generalization of concept of distance on a physical parameter space for MBS

- Let us first investigate for the mean field state. The mean field state for a multiband system is written as:

$$|n >= \prod_{\mathbf{k},\alpha}(u_{\alpha}^{n}(\mathbf{k})c_{\mathbf{k},\alpha}^{\dagger})|0 >$$

- The density matrix at $\mathbf{k_1}$ and $\mathbf{k_2}$ are $\rho^n(\mathbf{k_1}) = \mathbf{u^n}(\mathbf{k_1})(\mathbf{u^n}(\mathbf{k_1})^\dagger$ and $\rho^n(\mathbf{k_2}) = \mathbf{u^n}(\mathbf{k_2})(\mathbf{u^n}(\mathbf{k_2})^\dagger$
- The second Bargmann invariant:

$$B^{(2)}(\mathbf{k}_1, \mathbf{k}_2) = Tr(\rho^n(\mathbf{k_1})\rho^{\mathbf{n}}(\mathbf{k_2}))$$

Mean-field states

$$B^{(2)}(\mathbf{k_1},\mathbf{k_2}) = (u^{n\dagger}(\mathbf{k_1})u^{n}(\mathbf{k_2})(u^{n\dagger}(\mathbf{k_2})\mathbf{u^n}(\mathbf{k_1}))$$

$$= - < 0|(u^{n\dagger}(\mathbf{k_1})c_{\mathbf{k_1}})(u^{n\dagger}(\mathbf{k_2})c_{\mathbf{k_2}})(u^{n\dagger}(\mathbf{k_2})c^{\dagger}_{\mathbf{k_2}})(u^{n\dagger}(\mathbf{k_1})c^{\dagger}_{\mathbf{k_1}})|0 >$$

$$= - < \mathbf{k_1},\mathbf{k_2}|(u^{n\dagger}(\mathbf{k_2})c^{\dagger}_{\mathbf{k_2}})(u^{n\dagger}(\mathbf{k_1})c^{\dagger}_{\mathbf{k_1}})|0 >$$

$$=< \mathbf{k_1},\mathbf{k_2}|E(\mathbf{k_1},\mathbf{k_2})c^{\dagger}_{\mathbf{k_1}}c^{\dagger}_{\mathbf{k_2}}E(\mathbf{k_1},\mathbf{k_2})|0 >$$

$E(\mathbf{k_1},\mathbf{k_2})$ is an unitary operator, which we may call certain type of exchange operator, such that

$$E(\mathbf{k_1},\mathbf{k_2})|0 >= |0 >$$

$$E(\mathbf{k_1},\mathbf{k_2})|0 >= |\mathbf{k_1},\mathbf{k_2} >$$

$$E(\mathbf{k_1},\mathbf{k_2})c^{\dagger}_{\mathbf{k_1}}c^{\dagger}_{\mathbf{k_2}}E(\mathbf{k_1},\mathbf{k_2}) = -c^{\dagger}_{\mathbf{k_2}}c^{\dagger}_{\mathbf{k_1}}$$

Exchange operator

- $B^{(2)}(\mathbf{k_1},\mathbf{k_2}) =< n|E(\mathbf{k_1},\mathbf{k_2})|n >$
- The Bargmann invarinat $B^{(2)}$ can be written as expectation value of E over many body state $|n >$.
- $E(\mathbf{k_1},\mathbf{k_2})$=e$^{\frac{\pi}{2}\sum_\alpha (c^\dagger_{\mathbf{k_1}\alpha} c_{\mathbf{k_2}\alpha} - h.c)}$

Distance and Geometric phase

The second Bargmann invarant has been used to define a distance between two states $|\psi_1>$ and $|\psi_2>$

$$d(\mathbf{k_1},\mathbf{k_2}) = \sqrt{1 - B^2(\mathbf{k_1},\mathbf{k_2})^{\frac{\lambda}{2}}}$$

- λ is a real number ≥ 1
- maximum distance is 1 when states are orthogonal to each other.
- minimum distance is 0.
- satisfies the traingle inequality
- The phase of Bargmann invariant as a geometric phase is related with 3-vertex Bargmann invariant.

Quantum distance for strongly interacting fermions

- How to define the quantum distance for the correlated systems?
- We define induced distance between two points on the BZ as distance between $|\Psi >$ and $E(k_1, k_2)|\Psi >$

$$d^2(k_1, k_2) = 1 - | < \Psi|E(k_1, k_2)|\Psi > |^2$$

where

$$E(k_1, k_2) = e^{\frac{\pi}{2}(c^\dagger_{k_1} c_{k_2} - h.c)}$$

$$|\Psi >= \sum_{\{k\}} C_{k_1,k_2,......k_L} |k_1 \otimes k_2 \otimes ...... \otimes k_L >$$

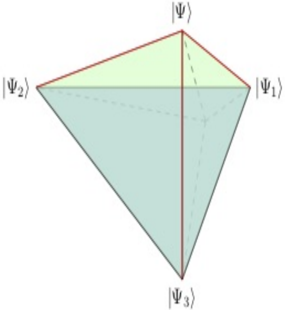
Quantum distance

- $d(k,k) = 0$, $d(k_1,k_2) = d(k_2,k_1)$
- Triangle inequality $d(k_1,k_2) + d(k_2,k_3) \geq d(k_3,k_1)$ is also satisfied. Can be proven by Ptolemy inequality.
- $|1 >= E(k_2,k_3)|GS >$, $|2 >= E(K_3,k_1)|GS >$, $|3 >= E(k_1,K_2)|GS >$
- It can be shown that $< 1|2 >=< 2|3 >=< 3|1 >$. This condition helps to prove triangle inequality.

Ptolemay inquality

Four normalized Hilbert space vectors $|\Psi>$, $|1>$,$|2>$, and $|3>$ can constructed. Six distances d_{ij} between any four distinct points $i, j = 1, 2, 3, 4$ satisfy the following Ptolemy inequality

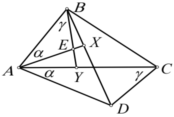
$$d_{ij}d_{kl} + d_{ik}d_{jl} \geq d_{il}d_{jl}$$



Theorem (Ptolemy's Inequality) Let *ABCD* be a quadrilateral. We have

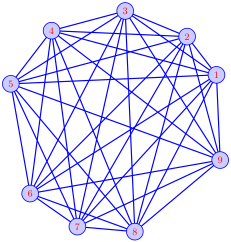
$$AB \times CD + DA \times BC \geq AC \times BD$$

with equality if and only if *ABCD* is a cyclic quadrilateral.



Ptolemaic distance space

- All these distances induced on the pair of physical paramter space is a metric space. It is also called Ptolemaic metric or we can call it distance space.
- We have distance space. The question is what to do with it. What information we can extract for the physical system?
- How to characterize distance space mathematically?
- we can view this distance space as a network or a graph:



Geometric Phase

- Our exchange operator generated states $|GS >= |\Psi_1 >, |\Psi_2 >= E(12)|GS >, , |\Psi_3 >= E(13)|GS >, ...$
- The third order Bargmann Invariant: $B^3(|\Psi_1 >, |\Psi_2 >, |\Psi_3 >) = Tr(\rho(\Psi_1)\rho(\Psi_2)\rho_(\Psi_3))$

- This satisfies additive properties of phases.
- Total geometric phase can be computed.
- The question is how to express the third order Bargmann invriant as $B^{(3)}(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3})$?
- Can it be written as expectation value of some operator $C(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3})$ where C is:

$$C(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) = E(\mathbf{k_1}, \mathbf{k_2})E(\mathbf{k_2}, \mathbf{k_3})$$

- This form works for mean-field state but not the correlated states.

Application:Quantum distance in the t-V modle

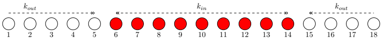
- The hamiltonian is

$$H = \sum_{i}^{L} (-t(c_i^\dagger c_{i+1} + h.c) + V n_i n_j)$$

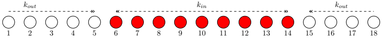
- for V=0

$$|FS >= \prod_{k_n \leq k_f} C_{k_n}^\dagger |0 >$$

- $c_{k_n} = \sum_i c_i e^{-ik_n i}$ and $k_n = \frac{2\pi n}{L}$



Di stance Matrix at $V = 0, \infty$



At $V = 0$:

$$D = \left[\begin{array}{c|cc} & k_{in} & k_{out} \\ \hline k_{in} & 0 & I \\ k_{out} & I & 0 \end{array}\right]$$

Distance Matrix at $V = \infty$ (CDW state):

$$D_{ij} = \begin{bmatrix} 0 & i = j \\ 1 & i = j + \frac{L}{2} \\ \sqrt{\frac{3}{4}} & i \neq j, i \neq j + \frac{L}{2} \end{bmatrix}$$

At an arbitary V:

$$D = \begin{bmatrix} \Delta & \Delta_e \\ \Delta_e & \Delta \end{bmatrix}$$

Distance Matrix

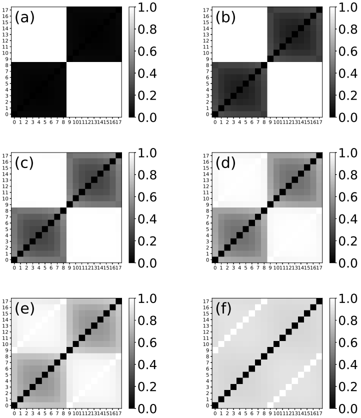


Figure: (a)-(f) Distance matrices obtained from numerical computation for interaction strengths $V = 0.1$ (a), $V = 1$ (b), $V = 2$ (c), $V = 3$ (d), $V = 4$ (e) and $V = 12$ (f).

Distance Matrix

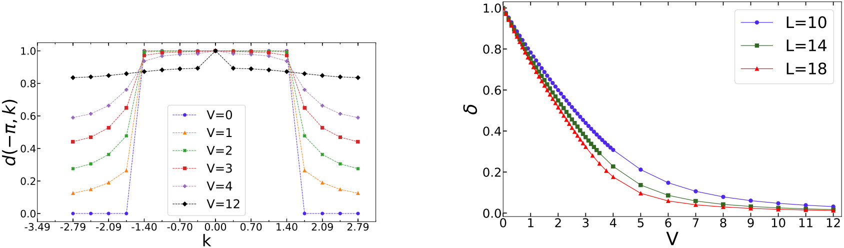
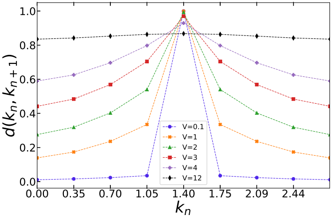


Figure: Distance $d(-\pi, k)$ between $k = -\pi$ and the other k modes in the Brillouin zone (BZ) for different values of the interaction strength V.

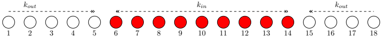
$\delta = d(-\pi, -\pi/2) - d(-\pi, -\pi/2 - 2\pi/L)$, gives a measure of the discontinuity across the Fermi points. It is studied as a function of interaction strength V for different system sizes.

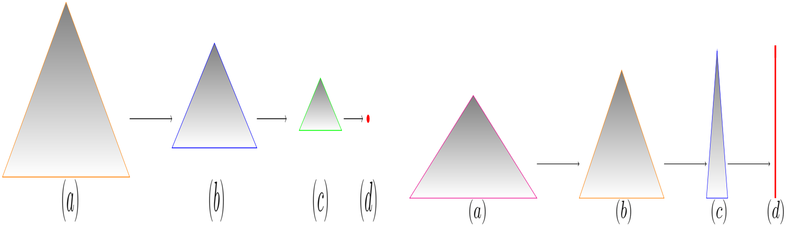
Properties of NN distances



- V=0: all zero except fermi point (delta function sigularity at k_f)
- intermediate V: this singularity remains but smmoothen out.
- $V = \infty$: all NN distances are equal.

Behaviour of triangles

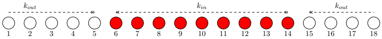




Particle triangles:

Particle-Hole triangles:

Understanding Distances



Define

a radius R in terms sum of NN dist :

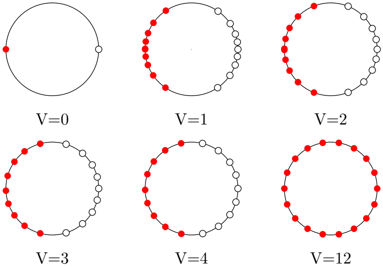
$2\pi R = \sum_i d(k_i, k_{i+1})$

NN distance represnted by an angle:

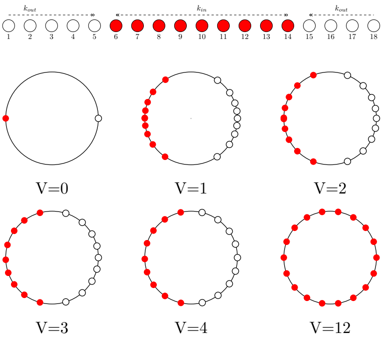
$\theta_{i,i+1} = \frac{d(k_i,k_{i+1})}{R}$

Each quasi momentum by an angle:

$\theta_k = \sum_j^k \theta_{j,j+1}$



Unit Circle representation



- $V = 0$ all the points collaps into $\theta = 0, \pi$.
- Small V: they spread out but points in the feri sea and those outside its are well separated
- In the corss over regions ($2 \leq V \leq 3$): the separation starts closing.
- In the insilating state: the sepration is indistinguishable.
- $V = \infty$: they are eqyally spaced.

Summary

- We presented an approach how to construct distance space for the many body state.
- We have studied properties of distances and tried to extract from it possible cross over transition.
- The Quantum distance probes the shape of the ground state wave function
- Our expresssion of the Quantum distance is very general and can be applied in one band Hubbard model. Other approaches fail to study geometry for such cases.
- We can apply in spin systems/bose system
- We are unable to parametrize the thrid order Bargmann invariant in terms of BZ vectors.