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Local Operators, the Black Hole Interior, and the Information Paradox in AdS/CFT

Puri Advanced School on String Theory

Overview

Lecture 1

Local operators

a) AdS/CFT as a correspondence between correlators

b) Solns of the wave eqn about a black hole

c) Modes of GFFs as creation and annihilation operators

d) Implications: thermal occupancy for bulk modes

e) Interior operators: what do we need for a smooth horizon

f) Rindler space in d-dimensions
→ Need for $\tilde{O}$ to get smooth correlators

g) Another motivation: the eternal BH

h). Final formula for mirror operators

1. AdS/CFT is commonly phrased as follows. We have some bulk field $\phi(x, z)$, and dual boundary operators $O(x, z)$. Now, consider the boundary condition:

$$\phi(x, z) \xrightarrow[z \to \epsilon]{} z^{d-\Delta} J(x)$$

in a coordinate system where

$$ds^2 = \frac{dz^2 + dx_i^2}{z^2}$$

asymptotically (near $z=0$).
Then we have

$$\int e^{-S_{bulk}} D\phi \Big|_{bound} = \langle e^{-\int J(x) O(x)} \rangle_{CFT}.$$

2. There is a less abstract way of phrasing the correspondence. We simply take

$$\langle O(x_1) \dots O(x_n) \rangle = Z^n \lim_{z \to 0} z^{-\Delta_1} \dots z^{-\Delta_2} \langle \phi(x_1, z_1) \dots \phi(x_n, z_n) \rangle$$

where the RHS is a bulk correlator and the LHS is a boundary correlator.
The advantage here is that we never need to deform the asymptotic geometry.
The same statement is true in non-trivial states.

(3) This is the statement that boundary correlators are <u>boundary</u> <u>values</u> of bulk Green functions.

<u>Question</u>: Can we reconstruct bulk Green functions at interior points using these boundary values.

or alternately

<u>Question</u>: Can we write down expressions for bulk local operators, in terms of boundary operators?

(4) Clearly we have to use the bulk equations of motion. So, the first thing we need is solutions to the bulk eom. Consider a minimally coupled scalar. We start directly with the black hole in AdS, with geometry:

$$ds^2 = \frac{1}{z^2}\left[-h(z)dt^2 + \frac{dz^2}{h(z)} + dx_i^2\right]$$

where now i runs only over spatial directions

and $$h(z) = 1 - \frac{z^d}{z_0^d}.$$

We have $$\beta = \frac{4\pi z_0}{d}$$

(4) We can find solutions to the wave-eq'n in a straightforward manner.

consider

$$\phi(x,z,t) = e^{i\omega t} e^{i\vec{k}\cdot\vec{x}} \gamma_{\omega,k}(z)$$

Then we get an ODE (2nd order, linear) for $\gamma(z)$.

There are clearly two independent solutions for γ, but recall that we need normalizability at the boundary:

$$\gamma_{\omega,k} \to z^{-\Delta}.$$

near $z=0$.

This eliminates one solution for each ω, k.

There is another regime in which we can solve the eqn. Near $z = z_0$.

Here, introducing the tortoise coordinate z_*, so that

$$\frac{dz}{h(z)} = dz_*$$

we find that the solution above goes like

$$\phi(z_*,x,t) = c(\omega,k)\left(e^{-i\delta_{\omega,k}} e^{i\omega(t-z_*)+ik\cdot\vec{x}} + e^{i\delta_{\omega,k}} e^{i\omega(t+z_*)+ik\cdot\vec{x}}\right).$$

near $z = z_0$.

(5) If we normalize and expand a local bulk operator as:

$$\hat{\phi}(x,z,t) = \int \left(\frac{a_{\omega,k}}{\sqrt{\omega}} \hat{f}_{\omega,k}(x,z,t) + h.c.\right) d\omega \frac{d^{d-1}k}{(2\pi)^d}$$

Note that for ϕ

where

$$\hat{f}_{\omega,k} \xrightarrow[z\to z_0]{} \frac{1}{\sqrt{2}} \left(e^{i\omega(t-z_*)+i\vec{k}\cdot\vec{x}} e^{-i\delta_{\omega,k}} + e^{i\omega(t+z_*)+i\vec{k}\cdot\vec{x}} e^{i\delta_{\omega,k}} \right).$$

then by quantizing $\hat{\phi}$, we need

$$[a_{\omega,k}, a^\dagger_{\omega',k'}] = \delta(\omega-\omega')\,\delta^{d-1}(k-k')$$

Can we identify $a_{\omega,k}$ with some operators in the CFT?

(6) One hint comes from the large-N behaviour of these operators. We know that interactions in the bulk are controled by $\frac{1}{N}$. So, at leading order we have:

$$\langle a_{\omega_1,k_1} \ldots a_{\omega_n,k_n} \rangle$$

$$= \langle a_{\omega_1} a_{\omega_2} \rangle \langle a_{\omega_3} a_{\omega_4} \rangle \ldots \langle a_{\omega_{n-1}} a_{\omega_n} \rangle$$

$$+ \text{permutations}$$

$$\propto n_{\omega_1} \delta(\omega_1+\omega_2)\, n_{\omega_3} \delta(\omega_3+\omega_4) \ldots$$

$$+ \text{permutations}$$

7 In fact the boundary duals of these fields have exactly the same property at large N.
They are generalized free-fields.

$$\langle O(x_1,t_1) \ldots O(x_n,t_n)\rangle$$
$$= \langle O(x_1,t_1)\, O(x_2,t_2)\rangle \langle O(x_3,t_3)\, O(x_4,t_4)\rangle \ldots + \text{permutations}$$

By Fourier transforming this we find an expression for the commutator

$$[O_{\omega,k}, O_{\omega',k'}] = 2\pi\,\delta(\omega+\omega')(2\pi)^{d-1}\delta(k+k')\, G(\omega,k).$$

So we identify:

$$a_{\omega,k} \longleftrightarrow \frac{O_{\omega,k}}{\sqrt{G(\omega,k)}}.$$

8 We get an immediate consistency check and a bonus. To see this, we must understand some properties of CFT correlators at temp β.
The CFT correlator we are looking at is:

$$\langle\Psi|\, O(t,x)\, O(0,0)\,|\Psi\rangle$$

where $|\Psi\rangle$ is dual to the B.H

(9) At leading order

$$\langle\psi| O(x,t)\, O(0,0) |\psi\rangle = \mathrm{Tr}\left(e^{-\beta H} O(x,t)\, O(0,0)\right)$$

Notice that

$$\mathrm{Tr}\left(e^{-\beta H} O(x,t-i\beta)\, O(0,0)\right).$$

$$= \mathrm{Tr}\left(e^{-\beta H} e^{\beta H} O(x,t) e^{-\beta H} O(0,0)\right)$$

$$= \mathrm{Tr}\left(e^{-\beta H} O(0,0)\, O(x,t)\right).$$

Now, ignoring the x-coordinate:

$$\langle O(x,t)\, O(0,0)\rangle.$$

$$= \int d\omega\, d\omega' \langle O_{\omega,k}\, O_{\omega',k'}\rangle\, e^{i\omega t} e^{ik\cdot x}\, \delta(\omega+\omega')\, \delta(k+k')$$

$$= \int \langle P_\omega\rangle e^{i\omega t}\, d\omega.$$

(ignoring x)

$$\langle O(0)\, O(t)\rangle = \int \langle P_{-\omega}\rangle e^{i\omega t}$$

So,

$$\langle P_\omega\rangle = e^{\beta\omega}\langle P_{-\omega}\rangle$$

From the relation above.

Also

$$\langle [O(x,t), O(0,0)]\rangle = \int (P_\omega - P_{-\omega}) e^{i\omega t}$$

$$P_\omega - P_{-\omega} = (e^{\beta\omega}-1) P_{-\omega} = G_\omega$$

Hence

$$\langle a_\omega\, a_{\omega'}\rangle = \langle n_\omega + 1\rangle\, \delta(\omega+\omega').$$

with

$$\langle n_\omega \rangle = \frac{P_\omega}{G_\omega} = \frac{e^{\beta\omega}}{e^{\beta\omega}-1} - 1 = \frac{1}{e^{\beta\omega}-1}$$

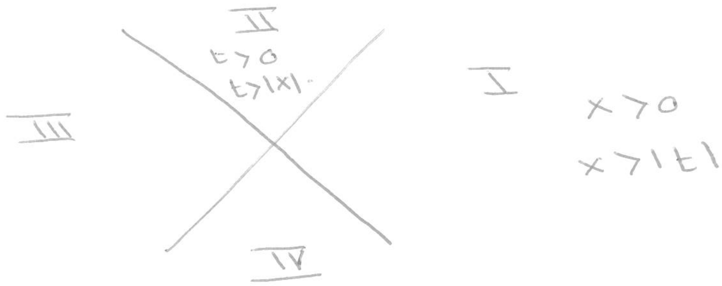
$$= \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}$$

So, we see that the bulk modes are <u>thermally</u> <u>occuppied</u>!

(10) Let me make a comment about the continuous spectrum we see here. Here we are working on R^3, but the continous spectrum also extends to S^3. This is an artifact of the large-N limit and arises because the black hole consists of many very finely spaced states, with spacing e^{-N^2}.

11) Now, let us try and construct local operators inside the black-hole. The key point is that this requires us to <u>double the degrees</u> of freedom.

12) Let me remind you what happens in Rindler space.



Given (d+1) dim Minkowski space

$$ds^2 = -dt^2 + dx^2 + \Sigma dy_i^2$$

we can define Rindler coordinates by

Region I: $x + t = e^{(x_R + t_R)}$

$x - t = e^{(x_R - t_R)}$

Region II:

$x + t = e^{(x_R + t_R)}$

$t - x = e^{(t_R - x_R)}$

Now we can expand the field both in region I and region II

(3) The full solutions can be found in appendix B of the "infalling" paper but the key point is that in region I, near the boundary, we find

$$\phi = \int_{\omega>0} \frac{d\omega\, d^{d-1}k}{(2\pi)^d} \frac{1}{\sqrt{2\omega}} a_\omega \left(e^{i\omega(t_R + x_R)} + e^{i\delta} e^{i\omega(t_R - x_R)} \right) + h.c.$$

in region II, near the boundary, we find:

$$\phi = \int_{\omega>0} \frac{d\omega\, d^{d-1}k}{(2\pi)^d} \cdot \frac{1}{\sqrt{2\omega}} \left(A_\omega e^{i\omega(t_R + x_R)} + B_\omega e^{i\omega(x_R - t_R)} \right) + h.c.$$

Now by looking at the 2-point function across the horizon and demanding the right short-distance behaviour, we find (Candelas, vacuum polarization in Schwarzschild spacetime, 1979).

that

a) $A_\omega = a_\omega$ [we see this by continuity of the modes $e^{i\omega(t_R + x_R)}$.

b). the state has the property that:

$$B_\omega |\Omega\rangle = e^{-\pi\omega} a_\omega^\dagger |\Omega\rangle.$$

So, $\langle a_\omega B_\omega \rangle \neq 0$

14) This is not at all surprising if we think of the origin of the modes B_ω

Just like A_ω are the left-movers from region I, the B_ω are the right-movers from region $\underline{III}$: $B_\omega \sim \tilde{a}_\omega$

recall that the Minkowski vacuum looks like:

$$|\Omega_{Mink}\rangle = e^{\int d\omega\, e^{-\pi\omega} a^+_\omega \tilde{a}^+_\omega} |\Omega^R_{I}, \Omega^R_{III}\rangle$$

and so,

$$\tilde{a}_\omega |\Omega_{Mink}\rangle = e^{-\pi\omega} a^+_\omega |\Omega_{Mink}\rangle$$

$$\tilde{a}^+_\omega |\Omega_{Mink}\rangle = e^{\pi\omega} a_\omega |\Omega_{Mink}\rangle$$

15) Now, let us return to the black hole

The horizon is semi-classically supposed to look like Rindler space. This means that we <u>require</u> some new operators $\tilde{O}_{\omega,R}$ and $\tilde{O}^+_{\omega,R}$ which satisfy

$$\tilde{O}_{\omega,R} |\Psi\rangle = e^{-\beta\omega/2} (O_{\omega,R})^+ |\Psi\rangle$$

$$\tilde{O}^+_{\omega,R} |\Psi\rangle = e^{\beta\omega/2} (O_{\omega,R}) |\Psi\rangle$$

16) Here is another way to understand this. At late-times, the collapsing geometry should approximate the eternal BH

Collapsing Geom

eternal AdS-B.H

The eternal B.H. is dual to the thermofield doubled state:

$$|\psi\rangle_{tfd} = \sum_{E} e^{-\beta E/2} |E\rangle_{I} |E\rangle_{II}$$

and we can think of the $\tilde{O}$ modes as coming from the second CFT.

17) To summarize; we require operators in the CFT that satisfy

$$\langle\psi| O_{\omega_1} \tilde{O}_{\omega_1'} \tilde{O}_{\omega_2'} \cdots \tilde{O}_{\omega_\ell'} O_{\omega_2} \cdots O_{\omega_\ell} |\psi\rangle$$

$$= e^{-\beta(\omega_1' + \ldots \omega_\ell')/2} \langle\psi| O_{\omega_1} O_{\omega_2} \cdots O_{\omega_\ell} (O_{\omega_\ell'})^\dagger \cdots (O_{\omega_2'})^\dagger (O_{\omega_1'})^\dagger |\psi\rangle$$

in position space, if we add also that $|\psi\rangle$ is close to thermal

$$\langle\psi| O(t_1) O(t_2) \cdots \tilde{O}(t_1') \cdots \tilde{O}(t_\ell') \cdots O(t_\ell) |\psi\rangle$$

$$= \frac{1}{Z_\beta} \operatorname{tr}\big(e^{-\beta H} O(t_1) O(t_2) \cdots O(t_\ell) O(t_\ell' + \tfrac{i\beta}{2}) \cdots O(t_2' + \tfrac{i\beta}{2}) O(t_1' + \tfrac{i\beta}{2})\big).$$

So the $\tilde{O}$ commute with O but when we convert them to ordinary ops, their <u>relative ordering</u> is reversed.

Both the <u>central equations</u> above just come from demanding that $|\psi\rangle$ <u>looks like</u> the thermofield entangled state.