STOCHASTIC CONTROL PROBLEM AND VISCOSITY SOLUTION

K. SURESH KUMAR

1. Stochastic Control

In this section, we briefly describe the stochastic optimal control problem. The problem is about minimizing a cost, i.e. an 'accumulated' cost associated with a controlled state dynamics which is random over some 'nice' class of controls. We describe the situation when the state dynamics is given by a controlled stochastic differential equation (sde in short) and the cost criterion α-discounted.

1.1. State dynamics.

In this subsection, we describe in some details the state dynamics. Let U be a compact metric space and $V = \mathcal{P}(U)$ denote the space of probability measures with Prohorov topology. Let $\bar{b} : \mathbb{R}^d \times U \to \mathbb{R}^d, \ \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be suitable functions. Set

$$b(x, v) \ = \ \int_U \bar{b}(x, u)v(du), \ x \in \mathbb{R}^d, \ v \in V \, .$$

The state dynamics $X(\cdot)$ is an $\mathbb{R}^d$-valued process described by solution to the sde

$$dX(t) \ = \ b(X(t), v(t))dt \ + \ \sigma(X(t))dW(t) \, . \tag{1.1}$$

Here $W(\cdot)$ is a standard d-dimentional Wiener process, $v(\cdot)$ is a jointly measurable process which is non anticipative, i.e., for each $0 \leq s < t, \ W(t) - W(s)$ is independent of $\{\mathcal{F}_t\}, \ \mathcal{F}_t =$ right continuous and complete augmentation of $\sigma(X_0, v(s), W(s) | s \leq t)$. We frequently interchange $X(t)$ with X_t to avoid exesive use of paranthesis . We call the collection of all such processes to be admissible and is denoted by $\mathcal{U}$. A detailed description of various classes such as feedback controls, Markov controls, stationary Markov controls and prescribed controls can be found in [2].

1.2. Cost Criterion.

We consider the discounted cost criterion with discount rate $\alpha > 0$. Precisely, for each $v(\cdot) \in \mathcal{U}$, if $X(\cdot)$ is a (weak) solution to the sde (1.1), then the discounted cost associated with $(X(\cdot), v(\cdot))$ is given by

$$J(x, v(\cdot)) \ = \ \mathbb{E}_x \Big[\int_0^\infty e^{-\alpha t} r(X_t, v_t) dt \Big].$$

Here $\mathbb{E}_x$ denote the expectation with respect to the stochastic process $X(\cdot)$ with the initial condition $X(0) = x$.

The stochatic control problem is about minimizing the cost $J(x, v(\cdot))$ over all admissible controls.

2. Viscosity solution of Second order elliptic PDES

In this section, we introduce the notion of viscosity solution to second order elliptic pdes.

Consider the general second order nonlinear equation

$$F(x, u(x), Du(x), D^2u(x)) = 0, x \in O \tag{2.1}$$

where $F : O \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to \mathbb{R}$ is a continuous function and O is any open set in $\mathbb{R}^d$. Here $\mathcal{S}(d)$ denotes the space of all $d \times d$ real valued symmetric matrices.

We call F degenerate elliptic if the following condition is satisfied:

$$F(x, r, p, Q) \leq F(x, r, p, Q') \text{ whenever } Q \geq Q'.$$

Here $Q \geq Q'$ means that $Q - Q'$ is nonnegative definite matrix. Further, it is called proper if it is degenerate elliptic and satisfies the monotonicity condition in the r-variable i.e.,

$$F(x, r, p, Q) \leq F(x, s, p, Q')$$

From now on, we always assume our equations to be proper, unless otherwise specified. To motivate the definition of viscosity solutions, we prove the following proposition.

Proposition 2.1. *Let u be a classical solution of the equation (2.1) and $\phi : O \to \mathbb{R}$ be any C^2 function. Then if $u - \phi$ has a locam maximum (local minimum) at a point $x_0 \in O$, then*

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq (\geq) 0.$$

Proof. If $u - \phi$ has a local maximum at x_0, then $Du(x_0) = D\phi(x_0)$ and $D^2u(x_0) - D^2\phi(x_0) \leq 0$. Now the degenerate ellipticity completes the proof. □

The above proposition motivates the following definition:

Definition 2.1. *A continuous function $u : O \to \mathbb{R}$ is called a viscosity subsolution (resp., viscosity supersolution) if*

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \leq (\textit{resp.}, \ \geq) 0$$

whenever x_0 is local maximum (resp., local minimum) of $u - \phi$ for $\phi \in C^2(0)$ with $\phi(x_0) = u(x_0)$.

A continuous function u which is both viscosity sub and super-solution is called a viscosity solution.

Remark 2.1. *In the defintion of viscosity solution, local maximum can be replaced by global maximum and also by strict local or global maximum. Also C^2 functions can be replaced by smooth functions. Also we can assume that the local maximum is zero. Similar remark applies for supersolutions also.*

3. Stability of Viscosity Solutions

In this section, we present the corner stone of the theory of viscosity solutions i.e., the stability result.

Theorem 3.1. (Stability) *Let $F, F_n, n = 1, 2, \cdots$ be proper and assume that $F_n \to F$ uniformly on compact sets. Let u_n be viscosity solution to the equation*

$$F_n(x, u_n(x), Du_n(x), D^2u_n(x)) = 0 \text{ in } O$$

and assume that $u_n \to u$ uniformly on compact sets. Then u is viscosity soltion to (2.1).

Proof. We prove that u is a viscosity subsolution. The proof of supersolution is similar.

Let $u - \phi$ has a strict local maximum at $x_0 \in O$ where $\phi \in C^2(O)$ with $u(x_0) = \phi(x_0)$. Let B be an open ball around x_0 such that

$$0 = u(x_0) - \phi(x_0) = \sup_B (u - \phi) > \sup_{\partial B} (u - \phi).$$

Choose $x_n \in \bar{B}$ such that

$$u_n(x_n) - \phi(x_n) = \sup_B (u_n - \phi).$$

Choose a subsequence, by an abuse of notation, itself, such that $x_n \to \bar{x}$ for some $\bar{x} \in \bar{B}$. Now for any $x \in B$, we have the following:

$$u_n(x) - \phi(x) \leq u_n(x_n) - \phi(x_n) \, , n \geq 1.$$

Thus by letting $n \to \infty$, observing that $u_n \to u$ uniformly in $\bar{B}$, we get

$$u(x) - \phi(x) \leq u(\bar{x}) - \phi(\bar{x}) \text{ for all } x \in B.$$

Hence $\bar{x} = x_0$, since x_0 is the strict local maximum.

Set $\phi_n(x) = \phi(x) + (u_n(x_n) - \phi(x_n))$, then $\phi_n(x_n) = u_n(x_n)$ and $u_n - \phi_n$ has a local maximum at x_n. Hence using the subsolution property for u_n, we have

$$F(x_n, \phi(x_n), D\phi(x_n), D^2\phi(x_n)) \leq 0.$$

Now letting $n \to \infty$, we obtain

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

This completes the theorem. □

4. Sub and Super differentials

In this section, we give an alternate definition of viscosity solution which will be useful in certain cases.

Definition 4.1. *Let $u : O \to \mathbb{R}$ be a continuous function. Fix $x \in O$. Introduce the sets*

$$\begin{aligned}
D^+u(x) &= \{p \in \mathbb{R}^d : \limsup_{y\to x} \frac{u(y) - u(x) - p\cdot(y-x)}{|y-x|} \leq 0\}, \\
D^-u(x) &= \{p \in \mathbb{R}^d : \liminf_{y\to x} \frac{u(y) - u(x) - p\cdot(y-x)}{|y-x|} \geq 0\}, \\
\bar{J}_O^{2,+}u(x) &= \{(p,Q) \in \mathbb{R}^d \times \mathcal{S}(d) : \limsup_{y\to x} \frac{u(y) - u(x) - p\cdot(y-x) - \frac{1}{2}(y-x)^\perp Q(y-x)}{|y-x|^2} \leq 0\}, \\
\bar{J}_O^{2,-}u(x) &= \{(p,Q) \in \mathbb{R}^d \times \mathcal{S}(d) : \liminf_{y\to x} \frac{u(y) - u(x) - p\cdot(y-x) - \frac{1}{2}(y-x)^\perp Q(y-x)}{|y-x|^2} \geq 0\}.
\end{aligned}$$

The sets $D^+u(x), D^-u(x), \bar{J}_O^{2,+}u(x), \bar{J}_O^{2,-}u(x)$ are respectively called super differential, sub differential, superjet and subjet .

Remark 4.1. *If $(p,Q) \in \bar{J}_O^{2,+}u(x)$, it is obvious from the definition that $(p,Q') \in \bar{J}_O^{2,+}u(x)$ for all $Q \geq Q'$. Thus $\bar{J}_O^{2,+}u(x)$ is always either empty or infinite. Also the superjet need not be closed if it is nonempty. Similar remark holds for the superjet. However, sub and superdifferentials can be finite sets and are always closed.*

We now prove a characterization of these sets which gives an equivalent definition of viscosity solution in terms of these sets.

Lemma 4.1. *For $x \in O$,*

$$\begin{aligned}
\bar{J}_O^{2,+}u(x) &= \{(D\phi(x), D^2\phi(x)) : u - \phi \text{ has a local maximum at } x \text{ for some test function } \phi\} \\
\bar{J}_O^{2,-}u(x) &= \{(D\phi(x), D^2\phi(x)) : u - \phi \text{ has a local minimum at } x \text{ for some test function } \phi\}
\end{aligned}$$

Proof. We will prove the characterization for $\bar{J}_O^{2,+}$. The characterization of $\bar{J}_O^{2,-}$ follows from the fact that $\bar{J}_O^{2,+}u = -\bar{J}_O^{2,-}(-u)$.

Fix $x \in O$. Let $\phi \in C^2(O)$ be such that $u(x) = \phi(x)$ and $u - \phi$ has a local maximum at x. Then for some $r_0 > 0$, when ever $\|y - x\| < r_0$, we have

$$\begin{aligned}
u(y) &\leq \phi(y) + u(x) - \phi(x) \\
&= u(x) + \nabla\phi(x)\cdot(y-x) + \frac{1}{2}(y-x)^\perp \nabla^2\phi(x)(y-x) + o(|y-x|^2)
\end{aligned}$$

Therefore

$$\limsup_{y\to x} \frac{u(y) - u(x) - \nabla\phi(x)\cdot(y-x) - \frac{1}{2}(y-x)^\perp \nabla^2\phi(x)(y-x)}{|y-x|^2} \leq 0.$$

i.e. $(\nabla\phi(x), \nabla^2\phi(x)) \in \bar{J}_O^{2,+}u(x)$.

Now we prove the converse. Suppose $(p,Q) \in \bar{J}_O^{2,+}u(x)$. Choose $r_0 > 0$ such that $B_{r_0}(x) \subseteq O$. Define $f : B_{r_0}(x) \to [0,\ \infty)$ as follows.

$$f(y) = \begin{cases} \dfrac{[u(y) - u(x) - p\cdot(y-x) - \frac{1}{2}(y-x)^\perp Q(y-x)]^+}{|y-x|^2} & \text{if } \quad y \neq x \\ 0 & \text{if } \quad y = x \end{cases}$$

Note that from $(p, Q) \in \bar{J}^{2,+}_O u(x)$ it follows that

$$\limsup_{y \to x} \frac{[u(y) - u(x) - p \cdot (y-x) - \frac{1}{2}(y-x)^\perp Q(y-x)]^+}{|y-x|^2} = 0.$$

Hence f is continuous at $y = x$. Thus f is continuous.

Define $h : [0, r_0) \to [0, \infty)$ by

$$h(r) \ = \ \sup\{f(y) | |y-x| \leq r\}.$$

Then clearly h is nondecresing, $h(0) = 0$ and is continuous. (Here continuity follows from the following observation: f attains maximum on compcat sets, h is nondcreasing)

From $y \in B_{r_0}(x),\ y \neq x$, we have

$$h(|y-x|)|y-x|^2 \geq f(y)|y-x|^2 \geq u(y) - u(x) - p \cdot (y-x) - \frac{1}{2}(y-x)^\perp Q(y-x).$$

Hence

$$u(y) \leq u(x) + p \cdot (y-x) + \frac{1}{2}(y-x)^\perp Q(y-x) + h(|y-x|)|y-x|^2.$$

Define

$$\eta(r) \ = \int_r^{\sqrt{3}r} \int_t^{2t} h(s) ds dt,\ 0 \leq r < \frac{r_0}{\sqrt{3}} := r_1.$$

Then $\eta \in C^2([0, r_1)),\ \eta(0) = \eta'(0) = \eta''(0) = 0$. Also

$$\eta(r) \geq \int_r^{\sqrt{3}r} t h(t) dt \geq \ h(r) \int_r^{\sqrt{3}r} dt = r^2 h(r).$$

Now set

$$\phi(y) \ = \ p \cdot (y-x) + \frac{1}{2}(y-x)^\perp Q(y-x) + \eta(|y-x|),\ y \in B_{r_1}(x).$$

The $\phi \in C^2(B_{r_1}(x))$ and $(p, Q) = (\nabla\phi(x), \nabla^2\phi(x))$. Also for $y \in\in B_{r_1}(x)$,

$$\begin{aligned} u(y) - \phi(y) &= u(y) - p \cdot (y-x) - \frac{1}{2}(y-x)^\perp Q(y-x) - \eta(|y-x|) \\ &\leq u(y) - p \cdot (y-x) - \frac{1}{2}(y-x)^\perp Q(y-x) - h(|y-x|)|y-x|^2 \\ &\leq u(x) = u(x) - \phi(x) \end{aligned}$$

i.e. x is a local maximum for $u - \phi$. Hence for any translation of ϕ. This completes the proof. □

Thus we have the following equivalent definition to the viscosity solutions.

Definition 4.2. *A continuous function u is said to be viscosity subsolution if*

$$F(x, u(x), p, Q) \leq 0 \text{ for all } (p, Q) \in \bar{J}^{2,+}_O u(x)$$

and u is said to be viscosity supersolution if

$$F(x, u(x), p, Q) \geq 0 \text{ for all } (p, Q) \in \bar{J}^{2,-}_O u(x).$$

5. Some Properties of Sub and Super differentials

In this section, we review some properties of sub and super differentials.

Theorem 5.1. (i) *$D^+u(x)$ and $D^-u(x)$ are closed and convex.*
(ii) *If both $D^+u(x)$ and $D^-u(x)$ are non empty, then u is differentiable at x and $D^+u(x) = D^-u(x)$, a singleton.*
(iii) *$\{x \in O : D^+u(x) \neq \emptyset\}$ and $\{x \in O : D^-u(x) \neq \emptyset\}$ are dense in O.*

We now present the anologous properties for the sub and superjets.

Theorem 5.2. (i) *$\bar{J}^{2,+}_O u(x)$ and $\bar{J}^{2,-}_O u(x)$ are convex.*
(ii) *If $\bar{J}^{2,+}_O u(x)$ and $\bar{J}^{2,-}_O u(x)$ are non empty, then u is differentiable at x and $\bar{J}^{2,+}_O u(x) = \bar{J}^{2,-}_O u(x)$. Further if $(p, Q), (p, Q') \in \bar{J}^{2,+}_O u(x)$ then $Q = Q'$ and hence they are singletons.*
(iii) *$\{x \in O : \bar{J}^{2,+}_O u(x) \neq \emptyset\}$ and $\{x \in O : \bar{J}^{2,-}_O u(x) \neq \emptyset\}$ are dense in O.*

Proof. We will prove only (ii) as the proofs of (i) and (iii) are essentially same to their counterparts in previous theorem.

Let ϕ_1 and ϕ_2 be C^2 functions such that

$$\phi_1(y) \leq u(y) \leq \phi_2(y)$$

for all $y \in O$ such that

$$\phi_1(x) = u(x) = \phi_2(x).$$

Then x is a local maximum of $\phi_1 - \phi_2$. Hence,

$$\nabla \phi_1(x) = \nabla \phi_2(x) \text{ and } \nabla^2 \phi_1(x) \leq \nabla^2 \phi_2(x).$$

Combining this with the Remark 4.1, we get that

$$\bar{J}_O^{2,+} u(x) = \bar{J}_O^{2,-} u(x)$$

provided they are nonempty. Let $(p, Q), (p, Q') \in \bar{J}_O^{2,+} u(x)$. Then

$$\begin{aligned} \frac{(y-x)^{\perp}(Q-Q')(y-x)}{|y-x|^2} &= \frac{u(y) - u(x) - p \cdot (y-x) - \frac{1}{2}(y-x)^{\perp} Q (y-x)}{|y-x|^2} \\ &\quad - \frac{u(y) - u(x) - p \cdot (y-x) - \frac{1}{2}(y-x)^{\perp} Q' (y-x)}{|y-x|^2}. \end{aligned}$$

Let λ be any eigen value and z be the corresponding eigen vector of $Q - Q'$. Choose $y = x + \frac{1}{n} z$ in the above equality and let $n \to \infty$ to obtain

$$\begin{aligned} \lambda &= \lim_{y \to x} \frac{u(y) - u(x) - p \cdot (y-x) - \frac{1}{2}(y-x)^{\perp} Q (y-x)}{|y-x|^2} \\ &\quad - \lim_{y \to x} \frac{u(y) - u(x) - p \cdot (y-x) - \frac{1}{2}(y-x)^{\perp} Q' (y-x)}{|y-x|^2} \\ &= 0. \end{aligned}$$

Here we have used the fact that (p, Q) and (p, Q') are in both sub and superjets. Thus the eigen values of $Q - Q'$ are zero. Since $Q - Q'$ is symmetric and real valued $Q = Q'$. Thus $\bar{J}_O^{2,+} u(x) = \bar{J}_O^{2,-} u(x)$ is singlton if they are nonempty. This completes the proof of part (ii). □

6. Vanishing Viscosity Method

In this section, we introduce the vanishing viscosity method which is the basis for the name "viscosity solution". This also gives an application to the stability result of viscosity solution.

Consder the Hamilton-Jacobi equation

$$u + H(Du) = f, \; x \in \mathbb{R}^N \tag{6.1}$$

where $H : \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz continuous function. We assume that $f : \mathbb{R}^N \to \mathbb{R}$ is bounded and Lipschitz continuous function.

Consider the second order equation

$$-\epsilon \Delta u_\epsilon + u_\epsilon + H(Du_\epsilon) = f \tag{6.2}$$

Due to the uniform ellipticity, there is a unique classical solution of (6.2) with the property

$$\|u_\epsilon\|_\infty \leq \|f\|_\infty. \tag{6.3}$$

Further if v_ϵ denotes the classical solution of (6.2) where f is replaced by any other bounded Lipschitz continuous function g, we have from the maximum principle, the following estimate

$$\|u_\epsilon - v_\epsilon\|_\infty \leq \|f - g\|_\infty. \tag{6.4}$$

Now for any fixed $h \in \mathbb{R}^N$, let g be defined by $g(x) = f(x + h)$. Then $v_\epsilon(x) = u_\epsilon(x + h)$ for all $x \in \mathbb{R}^N$. Now using (6.3) and (6.4), we obtain the following estimates

$$\|u_\epsilon\|_\infty \leq \|f\|_\infty \text{ and } \|u_\epsilon(\cdot) - u_\epsilon(\cdot + h)\|_\infty \leq C|h|$$

where C is the Lipschitz constant of f. Using the above inequalities, we conclude that $\{u_\epsilon\}$ is equibounded and equiLipschitz continuous. Now applying Ascoli-Arzela's theorem, $u_\epsilon \to u$ locally uniformly along a subsequenece, which we denote again by u_ϵ by an abuse of notation. The stability of viscosity solutions yields that u is a viscosity solution of (6.1).

7. Viscosity solutions for HJB equations

In this section, we address existence and uniqueness of solution to the Hamilton-Jacobi-Bellman equation arising from the discounted cost stochastic optimal control problem.

Let $V = \mathcal{P}(U)$, U a compact metric space. Here V is endowed with the Prohorov topology, i.e., the topology of weak convergence of probability measures. Let $\bar{b} : \mathbb{R}^d \times U \to \mathbb{R}^d, \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $\bar{r} : \mathbb{R}^d \times U \to \mathbb{R}$ be bounded continuous functions. Set

$$b(x, v) \ = \int_U \bar{b}(x, u)v(du), \ r(x, v) = \int_U r(x, u)v(du).$$

We make the following assumptions throughout this section.
(A1) (i) The functions $\bar{b}, \bar{r}$ are Lipschitz continuous in the first argument uniformly over the second, i.e. there exists $K > 0$ such that

$$|\bar{b}(x, u) - \bar{b}(y, u)| \leq K\|x - y\|, \ |\bar{r}(x, u) - \bar{r}(y, u)| \leq K\|x - y\|.$$

(ii) The function σ is Lipschitz continuous.
(A2) The function $a := \sigma\sigma^{\perp}$ satisfies the ellipticity condition, i.e.

$$a(x) \geq 0.$$

The HJB equation is given by

$$\alpha\phi(x) \ = \ \inf_{v\in V}[L\phi(x, v) + r(x, v)], \ x \in \mathbb{R}^d, \tag{7.1}$$

where

$$L\phi(x, v) \ = \ \sum_{i=1}^{d} b_i(x, v)\frac{\partial\phi}{\partial x_i}(x) \ + \ \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2\phi}{\partial x_i\partial x_j}(x) \tag{7.2}$$

for $f \in C^2(\mathbb{R}^d)$.

Consider the value function of the discounted cost stochastic control problem:

$$V(x) \ = \ \inf_{v(\cdot)} \mathbb{E}_x\Big[\int_0^\infty e^{-\alpha t}r(X(t), v(t))dt\Big], \tag{7.3}$$

where $X(\cdot)$ is a solution to the stochastic differential equation (sde)

$$dX(t) \ = \ b(X(t), v(t))dt \ + \ \sigma(X(t))dW(t) \tag{7.4}$$

and infimum is over all V-valued admissible control process $v(\cdot)$.

First we state the following Dynamic programming principle form Prof. Ghosh's lectures.

Theorem 7.1. *Assume (A1) and (A2). The value function satisfies*

$$V(x) \ = \ \inf_{v(\cdot)} \mathbb{E}_x\Big[\int_0^\tau e^{\alpha t}r(X_t, v_t)dt + e^{\alpha\tau}V(X_\tau)\Big]$$

for all bounded stopping times τ. Here infimum is over all feed back controls.

Theorem 7.2. *Assume (A2) and (A2). If $V \in C(\mathbb{R}^d)$. Then V is a viscosity solution to (7.1).*

Proof. We follow the arguments in [1].

Fix $x_0 \in \mathbb{R}^d$. Let $\psi \in C_b^2(\mathbb{R}^d)$ such that $V - \psi$ has a local minimum at x_0 and $V(x_0) = \psi(x_0)$. Let $\epsilon > 0$ be such that $V(x) \geq \psi(x)$ for $|x - x_0| \leq \epsilon$. Then using DPP, we have for each $\delta > 0$

$$V(x_0) \ = \ \inf_{v(\cdot)} \mathbb{E}_{x_0}\Big[\int_0^{\tau\wedge\delta} e^{-\alpha t}r(X_t, v_t)dt + e^{-\alpha(\tau\wedge\delta)}V(X_\tau)\Big], \tag{7.5}$$

where τ is the exit time of the process $X(\cdot)$ from the ball $B_\epsilon(x_0) = \{x \in \mathbb{R}^d||x - x_0| < \epsilon\}$.

Note that

$$P\{\tau \leq \delta\} \leq K_1\delta^2, \tag{7.6}$$

for some constant which depends only on the bounds of b and σ. See appendix for the proof.

$$\begin{aligned}\psi(x_0) \;=\; V(x_0) &\;=\; \inf_{v(\cdot)} \mathbb{E}_{x_0}\Big[\int_0^{\tau\wedge\delta} e^{-\alpha t} r(X_t, v_t)dt + e^{-\alpha(\tau\wedge\delta)} V(X_{\tau\wedge\delta})\Big] \\ &\;\geq\; \inf_{v(\cdot)} \mathbb{E}_{x_0}\Big[\int_0^{\tau\wedge\delta} e^{-\alpha t} r(X_t, v_t)dt + e^{-\alpha(\tau\wedge\delta)} \psi(X_{\tau\wedge\delta})\Big].\end{aligned} \tag{7.7}$$

Now using Itŏ-Dynkin's formula to $e^{-\alpha t}\psi(X_t)$, we get

$$\mathbb{E}_{x_0}[e^{-\alpha(\tau\wedge\delta)}\psi(X_\tau)] \;=\; \psi(x_0) + \mathbb{E}_{x_0}\Big[\int_0^{\tau\wedge\delta} e^{-\alpha s}[L\psi(X_s, v_s) - \alpha\psi(X_s)]ds\Big]. \tag{7.8}$$

From (7.7) and (7.8), we get

$$\inf_{v(\cdot)} \mathbb{E}_{x_0}\Big[\int_0^{\tau\wedge\delta} e^{-\alpha s}[L\psi(X_s, v_s) + r(X_s, v_s) - \alpha\psi(X_s)]ds\Big] \;\leq\; 0\,. \tag{7.9}$$

For each $v(\cdot)$

$$\begin{aligned}&\mathbb{E}_{x_0}\Big[\int_0^{\tau\wedge\delta} e^{-\alpha s}[L\psi(X_s, v_s) + r(X_s, v_s) - \alpha\psi(X_s)]I\{\tau \leq \delta\}ds\Big] \\ &\leq\; K_2\mathbb{E}_{x_0}\Big[\int_0^{\tau} e^{-\alpha s} I\{\tau \leq \delta\}ds\Big] \\ &\leq\; \frac{K_2}{\alpha}\mathbb{E}_{x_0}[(1 - e^{-\alpha\tau})I\{\tau \leq \delta\}] \\ &\leq\; \frac{K_1K_2}{\alpha}\delta^2.\end{aligned} \tag{7.10}$$

Now in view of (7.6), (7.10), by dividing (7.9) by δ and passing $\delta \to 0$, we get $V(\cdot)$ is a viscosity super solution. A similar argument that V is a viscosity sub solution. This completes the proof. □

We use the following result. For a proof, see

Theorem 7.3. *(Ishii's lemma) Let O be an open bounded subset of $\mathbb{R}^d$ and u, v be usc and lsc functions in O and ψ be given by*

$$\psi(x, y) \;=\; u(x) - v(y) - \frac{\lambda}{2}|x - y|^2, \;\; x, y \in O.$$

If ψ has a local maximum at $(\hat{x}, \hat{y}) \in O \times O$, then there exists $Q_1, Q_2 \in \mathcal{S}(d)$ such that
(i) $(\lambda(\hat{x} - \hat{y}), Q_1) \in \bar{J}_O^{2,+}u(\hat{x})$,
(ii) $(\lambda(\hat{x} - \hat{y}), Q_2) \in \bar{J}_O^{2,-}v(\hat{y})$,
(iii)

$$\begin{pmatrix} Q_1 & 0 \\ 0 & -Q_2 \end{pmatrix} \leq\; 3\lambda \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

Theorem 7.4. *Let $u_i \in C_b(\mathbb{R}^d),\ i = 1, 2$ be viscosity solutions to (7.1). Then $u_1 = u_2$.*

Proof. It suffices to show that $u_1 \leq u_2$. If not, there exists $x_0 \in \mathbb{R}^d$ such that $u_1(x_0) - u_2(x_0) > 0$.
Set

$$\psi(x) \;=\; \sqrt{1 + |x|^2} + c,$$

where $c > 0$ will be chosen later. Since $u_1(x_0) - u_2(x_0) > 0$, there exists $\delta > 0$ such that

$$\sup_{x\in\mathbb{R}^d} [u_1(x) - u_2(x) - 2\delta\psi(x)] = \rho > 0\,. \tag{7.11}$$

Using $u_i \in C_b(\mathbb{R}^d),\ i = 1, 2$, it follows that

$$\lim_{\|x\|\to\infty} [u_1(x) - u_2(x) - 2\delta\psi(x)] \;=\; -\infty.$$

Hence supremum is attained in (7.11). Let $\bar{x} \in \mathbb{R}^d$ be such that

$$\rho = u_1(\bar{x}) - u_2(\bar{x}) - 2\delta\psi(\bar{x}). \tag{7.12}$$

For each $\lambda > 0$, define

$$\Psi_\lambda(x,y) \;=\; u_1(x) - u_2(y) - \frac{\lambda}{2}|x-y|^2 - \delta(\psi(x)+\psi(y)),\;\; x,y \in \mathbb{R}^d. \tag{7.13}$$

Note that for each $\lambda > 0$

$$\lim_{|x|+|y|\to\infty} \Psi_\lambda(x,y) = -\infty.$$

Hence there exists $(x_\lambda, y_\lambda) \in \mathbb{R}^d \times \mathbb{R}^d$ such that

$$\Psi_\lambda(x_\lambda,y_\lambda) \;=\; \sup_{\mathbb{R}^d\times\mathbb{R}^d} \Psi_\lambda(x,y) \geq \; \Psi_\lambda(\bar{x},\bar{x}) \;=\; \rho > 0\,. \tag{7.14}$$

Note that there exists $R > 0$(independent of λ) such that $\Psi_\lambda(x,y) \leq 0$ for all (x,y) satisfying $|x|+|y| \geq R$. Hence (x_λ, y_λ) is bounded sequence.

From (7.14), it follows that

$$\frac{\lambda}{2}|x_\lambda - y_\lambda|^2 < u_1(x_\lambda) - u_2(y_\lambda) - \delta(\psi(x_\lambda)+\psi(y_\lambda)) < \|u_1\|_\infty + \|u_2\|_\infty. \tag{7.15}$$

From (7.15), it follows that along a sequence $\lambda \to \infty$

$$x_\lambda, y_\lambda \to \tilde{x} \text{ for some } \tilde{x} \in \mathbb{R}^d. \tag{7.16}$$

Also

$$\Psi_\lambda(x_\lambda,y_\lambda) \geq \Psi_\lambda(x_\lambda,x_\lambda)$$

implies that

$$\frac{\lambda}{2}|x_\lambda - y_\lambda|^2 \leq \; u_2(y_\lambda) - u_2(x_\lambda) + \delta(\psi(y_\lambda) - \psi(x_\lambda).$$

Hence from (7.16), along the sequence $\lambda \to \infty$,

$$\frac{\lambda}{2}|x_\lambda - y_\lambda|^2 \to 0\,. \tag{7.17}$$

From now onwards we will restrict ourself to this sequence and without any loss of generality, we write it as $\lambda \to \infty$.

Now let $\lambda \to \infty$ in (7.14), it follows [in view of (7.17)] that

$$u_1(\tilde{x}) - u_2(\tilde{x}) - 2\delta\psi(\tilde{x}) \geq \rho > 0. \tag{7.18}$$

Set

$$u = u_1 - \delta\psi,\;\; \bar{u} = u_2 + \delta\psi.$$

Then

$$\Psi_\lambda(x,y) \;=\; u(x) - \bar{u}(y) - \frac{\lambda}{2}|x-y|^2.$$

Applying Ishii's lemma, there exists $Q_1, Q_2 \in \mathcal{S}(d)$ such that

$$(\lambda(x_\lambda - y_\lambda), Q_1) \in \bar{J}^{2,+}u(x_\lambda),\;\; (\lambda(x_\lambda - y_\lambda), Q_2) \in \bar{J}^{2,}\bar{u}(y_\lambda),$$

where

$$\begin{pmatrix} Q_1 & 0 \\ 0 & -Q_2 \end{pmatrix} \;\leq\; 3\lambda \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Note that

$$\bar{J}^{2,+}u(x) \;=\; \bar{J}^{2,+}(u_1 - \delta\psi)(x) \;=\; \{(q,Q) - \delta(\nabla\psi(x), \nabla^2\psi(x)) \,|\, (q,Q) \in \bar{J}^{2,+}u_1(x)\}.$$

Hence

$$(\lambda\,(x_\lambda - y_\lambda), Q_1) + \delta(\nabla\psi(x_\lambda), \nabla^2\psi(x_\lambda)) \in \bar{J}^{2,+}u_1(x_\lambda).$$

Similary,

$$(\lambda\,(x_\lambda - y_\lambda), Q_2) + \delta(\nabla\psi(y_\lambda), \nabla^2\psi(y_\lambda)) \in \bar{J}^{2,-}u_2(y_\lambda).$$

Since u_1 is a viscosity (sub)solution to (7.1), we get

$$\begin{aligned} \alpha u_1(x_\lambda) \;\;&\leq\;\; \inf_{v\in V}[b(x_\lambda,v)\cdot[\lambda\,(x_\lambda - y_\lambda) + \delta\nabla\psi(x_\lambda)] + r(x_\lambda,v)] \\ &\quad +\frac{1}{2}tr(\sigma\sigma^\perp(x_\lambda)(Q_1 + \delta\nabla^2\psi(x_\lambda)). \end{aligned} \tag{7.19}$$

Similary, since u_2 is a viscosity (super)solution to (7.1), we get

$$\begin{aligned}\alpha u_2(y_\lambda) &\geq \inf_{v\in V}[b(y_\lambda, v)\cdot[\lambda(x_\lambda - y_\lambda) - \delta\nabla\psi(y_\lambda)] + r(y_\lambda, v)] \\ &\quad + \frac{1}{2}tr(\sigma\sigma^\perp(y_\lambda)(Q_2 - \delta\nabla^2\psi(y_\lambda)).\end{aligned} \tag{7.20}$$

From (7.19) and (7.20), we get

$$\begin{aligned}\alpha u_1(x_\lambda) - \alpha u_2(y_\lambda) &\leq \inf_{v\in V}[b(x_\lambda, v)\cdot[\lambda(x_\lambda - y_\lambda) + \delta\nabla\psi(x_\lambda)] + r(x_\lambda, v)] \\ &\quad - \inf_{v\in V}[b(y_\lambda, v)\cdot[\lambda(x_\lambda - y_\lambda) - \delta\nabla\psi(y_\lambda)] + r(y_\lambda, v)] \\ &\quad + \frac{1}{2}tr(\sigma\sigma^\perp(x_\lambda)Q_1 - \sigma\sigma^\perp(y_\lambda)Q_2) \\ &\quad + \frac{\delta}{2}tr(\sigma\sigma^\perp(x_\lambda)\nabla^2\psi(x_\lambda) + \sigma\sigma^\perp(y_\lambda)\nabla^2\psi(y_\lambda))\end{aligned}$$

Hence we get

$$\begin{aligned}\alpha u_1(x_\lambda) - \alpha u_2(y_\lambda) &\leq \sup_{v\in V}|(b(x_\lambda, v) - b(y_\lambda, v))\cdot\lambda(x_\lambda - y_\lambda)| \\ &\quad + \delta\sup_{v\in V}|b(x_\lambda, v)\cdot\nabla\psi(x_\lambda) + b(y_\lambda, v)\cdot\nabla\psi(y_\lambda)| \\ &\quad + \sup_{v\in V}|r(x_\lambda, v) - r(y_\lambda, v)| \\ &\quad + \frac{1}{2}tr(\sigma\sigma^\perp(x_\lambda)Q_1 - \sigma\sigma^\perp(y_\lambda)Q_2) \\ &\quad + \frac{\delta}{2}tr(\sigma\sigma^\perp(x_\lambda)\nabla^2\psi(x_\lambda) + \sigma\sigma^\perp(y_\lambda)\nabla^2\psi(y_\lambda)).\end{aligned} \tag{7.21}$$

Consider

$$\begin{aligned}\lim_{\lambda\to\infty}\sup_{v\in V}|(b(x_\lambda, v) - b(y_\lambda, v))\cdot\lambda(x_\lambda - y_\lambda)| &\leq \lim_{\lambda\to\infty}\mathrm{Lip}(b)\lambda|x_\lambda - y_\lambda|^2 \\ &= 0.\end{aligned} \tag{7.22}$$

Some comments in order. Recall that limit $\lambda\to\infty$ is taken along the sequence mentioned earlier and Lip(b) denote the Lipschitz constant of b. Similarly

$$\lim_{\lambda\to\infty}\sup_{v\in V}|r(x_\lambda, v) - r(y_\lambda, v)| = 0. \tag{7.23}$$

Consider

$$\begin{aligned}&\limsup_{\lambda\to\infty}\sup_{v\in V}|b(x_\lambda, v)\cdot\nabla\psi(x_\lambda) + b(y_\lambda, v)\cdot\nabla\psi(y_\lambda)| \\ &\leq \lim_{\lambda\to\infty}\|b\|_\infty(|\nabla\psi(x_\lambda)| + |\nabla\psi(y_\lambda)|) \leq 2\|b\|_\infty|\nabla\psi(\tilde{x})|.\end{aligned} \tag{7.24}$$

Also

$$\lim_{\lambda\to\infty}tr(\sigma\sigma^\perp(x_\lambda)\nabla^2\psi(x_\lambda) + \sigma\sigma^\perp(y_\lambda)\nabla^2\psi(y_\lambda)) = 2tr(\sigma\sigma^\perp(\tilde{x})\nabla^2\psi(\tilde{x})). \tag{7.25}$$

Consider

$$\begin{aligned}tr(\sigma\sigma^\perp(x_\lambda)Q_1 - \sigma\sigma^\perp(y_\lambda)Q_2) &= tr\Big(\begin{bmatrix}\sigma\sigma^\perp(x_\lambda) & \sigma(x_\lambda)\sigma^\perp(y_\lambda) \\ \sigma(y_\lambda)\sigma^\perp(x_\lambda) & \sigma\sigma^\perp(y_\lambda)\end{bmatrix}\begin{bmatrix}Q_1 & 0 \\ 0 & -Q_2\end{bmatrix}\Big) \\ &\leq tr\Big(\begin{bmatrix}\sigma\sigma^\perp(x_\lambda) & \sigma(x_\lambda)\sigma^\perp(y_\lambda) \\ \\ \sigma(y_\lambda)\sigma^\perp(x_\lambda) & \sigma\sigma^\perp(y_\lambda)\end{bmatrix}\begin{bmatrix}I & -I \\ -I & I\end{bmatrix}\Big) \\ &= 3\lambda\, tr(\sigma\sigma^\perp(x_\lambda) - \sigma(x_\lambda)\sigma^\perp(y_\lambda) - \sigma(y_\lambda)\sigma^\perp(x_\lambda) + \sigma\sigma^\perp(y_\lambda)) \\ &= 3\lambda tr((\sigma(x_\lambda) - \sigma(y_\lambda))(\sigma(x_\lambda) - \sigma(y_\lambda))^\perp) \\ &= 3\lambda\|\sigma(x_\lambda) - \sigma(y_\lambda)\|^2 \\ &\leq 3\lambda\mathrm{Lip}(\sigma)|x_\lambda - y_\lambda|^2.\end{aligned}$$

Hence by letting $\lambda\to\infty$ in (7.21), using (7.17), (7.22), (7.23), (7.24) and (7.25) we get

$$\alpha(u_1(\tilde{x}) - u_2(\tilde{x})) \leq 2\delta\|b\|_\infty|\nabla\psi(\tilde{x})| + \delta tr(\sigma\sigma^\perp(\tilde{x})\nabla^2\psi(\tilde{x})).$$

Now choose $c > 0$ such that

$$2\delta\|b\|_\infty|\nabla\psi(\tilde{x})| + \delta tr(\sigma\sigma^\perp(\tilde{x})\nabla^2\psi(\tilde{x})) \leq \ 2\delta\alpha\psi(\tilde{x}).$$

Such a choice is possible. Hence we get

$$u_1(\tilde{x}) - u_2(\tilde{x}) \leq 2\delta\psi(\tilde{x}). \tag{7.26}$$

This contradicts (7.18). This completes the proof.

To use the above comparison principle one need to know whether the value function is continuous or not.

Set

$$\alpha_0 \ = \ \sup_{x\neq y, v\in V}\Big[\frac{(b(x,v)-b(y,v)\cdot(x-y)}{|x-y|^2} + \frac{1}{2|x-y|^2}\sum_i(\sigma_{ii}(x)-\sigma_{ii}(y))^2\Big].$$

Then using (A1), clearly $\alpha_0 < \infty$.

Theorem 7.5. *The value function is Lipschitz continuous if $\alpha \geq \alpha_0$ and Holder continuous with exponent $\frac{\alpha}{\alpha_0}$ if $\alpha < \alpha_0$.*

Proof. Observe that the value function $V(\cdot)$ is given by

$$V(x) \ = \ \inf_{v(\cdot)}\mathbb{E}_x\Big[\int_0^\infty e^{-\alpha t}r(X_t, v_t)dt\Big]$$

where infimum is over all admissible prescribed controls.

Reason why one can do this is the following. For an admissible feedback pair $(X(\cdot), v(., X_{[0,.]})$ on some complete probability space $(\Omega, \mathcal{F}, P)$ with a Wiener process $W(\cdot)$ in it, set $\tilde{v}_t = v(t, X[0,t]), t \geq 0$. Then corresponding to the presribed control $\tilde{v}(\cdot)$ and the prescribed Wiener process $W(\cdot)$, $X(\cdot)$ is a solution. i.e. corresponding to any feedback admissible pair, there is a corresponding prescribed control with same payoff.

For a prescribed control $v(\cdot)$, let $X_x(\cdot), X_y(\cdot)$ denote the solution to (1.1) for the initial conditions x, y respectively.

Then using Itŏ's fromula, we get for $x \neq y$,

$$\begin{aligned}|X_x(t) - X_y(t)|^2 \ &= \ |x-y|^2 + \int_0^t 2\Big[(b(X_x(s), v(s)) - b(X_y(s), v(s)))\cdot(X_x(s) - X_y(s)) \\ &\quad + \sum_i(\sigma_{ii}(X_x(s)) - \sigma_{ii}(X_y(s)))^2\Big]ds \ + \ \text{a zero mean martingale.}\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}|X_x(t) - X_y(t)|^2 \ &= \ |x-y|^2 + \int_0^t 2\mathbb{E}\Big[(b(X_x(s), v(s)) - b(X_y(s), v(s)))\cdot(X_x(s) - X_y(s)) \\ &\quad + \sum_i(\sigma_{ii}(X_x(s)) - \sigma_{ii}(X_y(s)))^2\Big]ds \\ &= \ |x-y|^2 + \int_0^t 2\mathbb{E}\Big[(b(X_x(s), v(s)) - b(X_y(s), v(s)))\cdot(X_x(s) - X_y(s)) \\ &\quad + \sum_i(\sigma_{ii}(X_x(s)) - \sigma_{ii}(X_y(s)))^2 \times \frac{|X_x(s) - X_y(s)|^2}{|X_x(s) - X_y(s)|^2} \\ &\quad \times I\{X_x(s) \neq X_y(s)\}\Big]ds \\ &\leq \ |x-y|^2 + 2\alpha_0\int_0^t 2\mathbb{E}|X_x(s) - X_y(s)|^2 ds.\end{aligned}$$

Now using Gronwall's inequality, we get

$$\mathbb{E}|X_x(t) - X_y(t)|^2 \ \leq \ |x-y|^2 e^{2\alpha_0 t}, \ t \geq 0. \tag{7.27}$$

Using DPP, we get for eache $T > 0$,

$$V(x) \ = \ \inf_{v(\cdot)}\mathbb{E}\Big[\int_0^T e^{-\alpha t}r(X_x(t), v(t))dt + e^{-\alpha T}V(X_x(T))\Big].$$

Here again the infimum is over all prescribed admissible controls. Hence

$$|V(x) - V(y)| \leq \sup_{v(\cdot)} \mathbb{E}\Big[\int_0^T e^{-\alpha t}|r(X_x(t), v(t)) - r(X_y(t), v(t))|dt\Big] + \frac{\|r\|_\infty}{\alpha} e^{-\alpha T}.$$

Therefore

$$\begin{aligned} |V(x) - V(y)| &\leq \operatorname{Lip}(r)\int_0^T e^{-\alpha t}\mathbb{E}|X_x(t) - X_y(t)|dt + \frac{\|r\|_\infty}{\alpha}e^{-\alpha T} \\ &\leq C\Big[|x-y|e^{(\alpha_0-\alpha)T} + e^{-\alpha T}\Big],\ T \geq 0, \end{aligned} \tag{7.28}$$

where $C > 0$ is a constant independent of T.

Hence when $\alpha_0 < \alpha$, the by letting $T \to \infty$ in (7.28), it follows that V is Lipschitz continuous.

When $\alpha < \alpha_0$. From (7.28), we have

$$|V(x) - V(y)| \leq C \inf_{T>0}\Big[|x-y|e^{(\alpha_0-\alpha)T} + e^{-\alpha T}\Big]. \tag{7.29}$$

$$\inf_{T>0}\Big[|x-y|e^{(\alpha_0-\alpha)T} + e^{-\alpha T}\Big] = |x-y|e^{(\alpha_0-\alpha)T^*} + e^{-\alpha T^*},$$

where

$$T^* = \frac{1}{\alpha_0} ln\Big(\frac{\alpha}{(\alpha_0-\alpha)|x-y|}\Big).$$

Hence

$$\inf_{T>0}\Big[|x-y|e^{(\alpha_0-\alpha)T} + e^{-\alpha T}\Big] = \frac{\alpha_0}{\alpha_0-\alpha}\Big(\frac{\alpha_0}{\alpha_0-\alpha}\Big)^{\frac{\alpha}{\alpha_0}}|x-y|^{\frac{\alpha}{\alpha_0}}. \tag{7.30}$$

From (7.29) and (7.30), the Holder continuity of V follows.

For $\alpha = \alpha_0$, a similar calculation implies the Lipschitz continuity of V. □

8. Note on optimal control

In this section, we briefly talk about the existence of α-discounted optimal control. We provide an approach using the small noise limit (or the vanishing viscosity).

Consider the ϵ-perturbed HJB equation of (7.1)

$$\alpha\phi_\epsilon(x) = \inf_{v\in V}[L_\epsilon\phi_\epsilon(x,v) + r(x,v)],\ x \in \mathbb{R}^d, \tag{8.1}$$

where

$$L_\epsilon f = Lf + \frac{\epsilon}{2}\Delta f.$$

Note that (8.1) corresponds to the HJB equation of the α-discounted stochastic control problem with satet dynamics

$$dX_\epsilon(t) = b(X_\epsilon(t), v(t))dt + \sigma_\epsilon(X_\epsilon(t))dW(t), \tag{8.2}$$

where σ^ϵ is such that $\sigma_\epsilon\sigma_\epsilon^\perp = \sigma\sigma^\perp + \epsilon I$. i.e. The process in (8.2) is a small noise perturbation of (1.1).

We state the following theorem for the small noise perturbed stocahstic control problem.

Theorem 8.1. *Assume (A1) and (A2). The HJB equation (8.1) has a unique solution in $C^2(\mathbb{R}^d)\cap C_b(\mathbb{R}^d)$. Moreover any minimizing selector $v^\epsilon(\cdot)$ is an optimal stationary Markov control to the ϵ-perturbed stochastic control problem.*

Let $v^\epsilon(\cdot)$ be a stationary Markov control as in the above theorem. Let $(X_\epsilon(\cdot)$ be the unique solution to the sde (8.2) corresponding to $v^\epsilon(\cdot)$ and initial condition $x \in \mathbb{R}^d$. Then it is easy to see that $X^\epsilon(\cdot), \bar{v}^\epsilon(\cdot)),\ \bar{v}^\epsilon(t) = v^\epsilon(X^\epsilon(t))$ is tight in $\mathcal{P}(C[0,\infty);\mathbb{R}^D) \times \mathcal{P}(\hat{\mathcal{U}})$ where $\hat{\mathcal{U}}$ is the set of all measurable maps from $[0,\ \infty) \to V$.

Then any limit $(X(\cdot), \bar{v}(\cdot))$ can been show to be the solution of the Martingale problem corresponding to (1.1), see [[2], page p.55] and hence $X(\cdot)$ weak solution to (1.1) corresponding to $\bar{v}(\cdot)$ and initial condition x.

Now one can show that

$$\lim_{\epsilon\to 0}\phi_\epsilon(x) = \phi(x)$$

and

$$\phi(x) \ = \ \mathbb{E}_x\Big[\int_0^\infty r(X_t, \bar{v}_t)dt\Big].$$

i.e. $\bar{v}(\cdot)$ is an optimal admissible control.

9. APPENDIX

We prove the following estimate.

Lemma 9.1. *Assume (A1) and (A2). For $v(\cdot)$ an admissible control, let $X(\cdot)$ be a solution to (1.1) corresponding to $v(\cdot)$ and initial condition x_0. Let τ be the exit time from the ball $B_{x_0}(\epsilon)$. Then*

$$P\{\tau \leq \delta\} \leq K_1\delta^2$$

for some $K_1 > 0$ which is independent of δ.

Proof. The process $X(\cdot)$ is given by

$$X(t) - x_0 \ = \ \int_0^t b(X_s, v_s)ds + \int_0^t \sigma(X_s)dW_s\,, \ t \geq 0.$$

Hence

$$|X(t) - x_0|^4 \ \leq \ K\Big[\Big|\int_0^t b(X_s, v_s)ds\Big|^4 + \Big|\int_0^t \sigma(X_s)dW_s\Big|^4\Big].$$

Therefore

$$E\Big[\sup_{0\leq s\leq t}|X(t) - x_0|^4\Big] \ \leq \ K\Big\{E\Big[\sup_{0\leq s\leq t}\Big|\int_0^t b(X_s, v_s)ds\Big|^4\Big] + \Big[E\Big[\sup_{0\leq s\leq t}\Big|\int_0^t \sigma(X_s)dW_s\Big|^4\Big]\Big\}. \tag{9.1}$$

Consider

$$E\Big[\sup_{0\leq s\leq t}\Big|\int_0^t b(X_s, v_s)ds\Big|^4\Big] \ \leq \ \|b\|_\infty t^4, \tag{9.2}$$

where $\|b\|_\infty = \sup_{v\in V, x\in\mathbb{R}^d}|b(x,v)|$. Also

$$\Big[E\Big[\sup_{0\leq s\leq t}\Big|\int_0^t \sigma(X_s)dW_s\Big|^4\Big] \ \leq \ CE\Big[\int_0^t tr(\sigma\sigma^\perp(X_s)ds\Big]^2 \leq \|tr(\sigma\sigma^\perp\|_\infty Ct^2. \tag{9.3}$$

The first inqeuality follows from Burkholder-Davis-Gundsy inequality, see [[2], p.33]. Combining (9.1), (9.2) and (9.3), we get

$$E\Big[\sup_{0\leq s\leq t}|X(t) - x_0|^4\Big] \leq \hat{K}(t^2 + t^4) \leq 2\hat{K}t^2,\ t \leq 1\,. \tag{9.4}$$

Now for $\delta < 1$,

$$\begin{aligned} P\{\tau \leq \delta\} \ &= \ P\Big\{\sup_{0\leq s\leq t}|X(t) - x_0| \geq \epsilon\Big\} \\ &\leq \ \frac{1}{\epsilon^4}E\Big[\sup_{0\leq s\leq t}|X(t) - x_0|^4\Big] \\ &\leq \ \frac{2\hat{K}}{\epsilon^4}\delta^2 = K_1\delta^2. \end{aligned}$$

The first inequality is by an application of Chebychev's lemma. □

Remark 9.1. *The above with $\sigma = 0$ has a very simple proof. When $\sigma = 0$, we have the ode*

$$dx(t) = b(x(t), v_t)dt, x(0) = x_0.$$

Suppose $x(\cdot)$ is a solution to the above ode. A word about the solution to the ode. Using small noise limit, one can show the existence of the solution in the weak solution frame work of sde which in turn is a solution in Fillipov sense.

Now here τ is a positive number, i.e. τ is a degenerate random variable(dirac mass) with its mass concentrated at $\inf\{t \geq 0 | |x(t) - x_0| \geq \epsilon\}$. Let us denote $\inf\{t \geq 0||x(t) - x_0| \geq \epsilon\}$ by t^*.

Then

$$P\{\tau \leq \delta\} \; = \; \begin{cases} 0 & \text{if} \quad \delta < t^* \\ 1 & \text{if} \quad \delta \geq t^* \end{cases}$$

Hence

$$P\{\tau \leq \delta\} \leq \frac{1}{(t^*)^2}\delta^2.$$

This proves the lemma.

This remark is written due to the question raised by Prof. Adimurthi.

References

[1] Lions P. L. Optimal control of diffusion processes and HJB equations. Part 2: Viscosity solutions and uniqueness, CPDE, 8(11), pp.1229 - 1276, 1983.

[2] Arapostathis, A., Borkar, V.S. and Ghosh, M.K., *Ergodic control of diffusion processes*, Encyclopedia of Mathematics and its applications 143, Cambridge University Press, 2012.

[3] Karatzas, I. and Shreve, S. E., Brownian Motion and stochastic calculus, GTM 113, Second Edition, 1991, Springer, NewYork.

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076.

E-mail address: suresh@math.iitb.ac.in.