INVARIANT MEASURES AND ERGODICITY FOR INFINITE DIMENSIONAL STOCHASTIC SYSTEMS

P. SUNDAR

1. Introduction

Consider a dynamical system with state space E. If $x \in E$ is the initial state of the system, let usl denote the state of the system at time t by $T_t x$ where $T_t : E \to E$ is a measure preserving transformation. The important equality $T_t(T_s x) = T_{t+s}x$ holds in several systems. Let ϕ be an observable real-valued function on E. In taking observations, the time to take observations is much longer than the natural time scale for molecular interactions. Therefore, we end up with measurements of the average

$$\frac{1}{t}\int_0^t \phi(T_s x)ds$$

instead of $\phi(x)$. Boltzmann hypothesized that if orbits wandered all over the the state space, then the time averages would converge, as $t \to \infty$, to a phase space average. Our aim is to identify the conditions under which the time-averages converge to a phase-space mean.

In 1931, Birkhoff proved an important result known as the ergodic theorem while von Neumann established the mean ergodic theorem. Since then, several improvements, extensions and refinements have been made of these results. The ensuing theory is one of the major topics in ergodic theory. The avid reader can consult books on ergodic theory by Halmos, Sinai, Petersen, and Walters, to name a few.

In this lecture notes, our aim is to provide an elementary introduction to ergodic behavior of solutions of stochastic partial differential equations with particular attention paid to stochastic Navier-Stokes systems. There is an enormous and ever-growing literature on this topic. We content ourselves with presenting a few basic results that could enable one to read other interesting and sophisticated works on ergodicity. The ensuing presentation has benefitted considerably from the works of Da Prato [3], Zabczyk [4], Flandoli [6], Mattingly [7], etc.

First we discuss a few results such as the Krylov-Bogolyubov criterion that ensure existence of invariant measures for time homogeneous Markov processes taking values in a complete separable metric space E. Next, we introduce the notion of ergodic measures, and prove that the theorem of Doob which states that the strong Feller property and irreducibility imply ergodicity of an invariant measure.

Ornstein-Uhlenbeck processes that take values in a real separable Hilbert space are studied in Section 3. First, we show that the mild solution of the linear SPDE is strong Feller and irreducible. Then, we identify an invariant measure and prove that the invariant measure is the unique ergodic measure. The ergodic measure is Gaussian as one would expect. The Stokes equation perturbed by an additive noise is a good example of this class of equations.

In Section 4, we introduce SPDEs with Lipschitz nonlinearities and an additive noise. Under suitable hypotheses, irreducibility and strong Feller properties are established. An invariant measure is identified which implies the ergodicity of the stochastic system. The differentiability properties of solutions and the Bismut-Elworthy formula are proved in order to obtain the strong Feller property. The proof of the existence of an invariant measure is long. Hence we present only the ideas of the proof.

The two-dimensional Navier Stokes equations with an additive noise for incompressible fluids in a bounded domain is discussed in Section 6. The solvability of such equations is well-known (see Menaldi and Sritharan [8]). In the next section, exponential stability of solutions is proved as in [10]. It is used in establishing the existence of a unique ergodic measure in Section 8.

2. **Definitions and Basic Results**

Let (E, d) be a complete, separable metric space with Borel σ-field denoted by $\mathcal{E}$. Given $\{X_t^x : x \in E,\ t \geq 0\}$, a family of time-homogeneous E-valued Markov processes indexed by x, with continuous paths and $X_0^x = x$, define

$$P_x(t, B) = P\{X_t^x \in B\},$$

the *transition* probability measure for all $t \geq 0$, $x \in E$ and $B \in \mathcal{E}$. Then, $P.(t, B)$ is an $\mathcal{E}$-measurable function for all fixed t and B. The Markov property yields the Chapman-Kolmogorov equation:

$$P_x(t+s, B) = \int_E P_x(t, dy) P_y(s, B). \tag{2.1}$$

For any given probability measure ν on $(E, \mathcal{E})$, one can define a probability measure

$$P_\nu(t, B) := \int_E P_x(t, B)\nu(dx).$$

If ξ is a random variable with distribution ν and is independent of $\{X^x : x \in E\}$, then one can construct a continuous time-homogeneous Markov process, denoted X^ξ, such that $X_0^\xi = \xi$, and for any $t > 0$, $P_\nu(t, \cdot)$ is the distribution of X_t^ξ.

One can define a semigroup of linear operators S_t on $B_b(E)$ as follows:

$$S_t\phi(x) = \mathbb{E}\,\phi(X_t^x) = \int_E \phi(y)P_x(t, dy). \tag{2.2}$$

The semigroup, S_t is strongly continuous and is called a Markov semigroup since S_t is a positivity preserving contraction and $S_t 1 = 1$.

Definition 2.1. A probability measure μ on $(E, \mathcal{E})$ is called an **invariant measure** for the transition probability S_t if it satisfies

$$\mu\,(B) = \int_E P_x(t, B)\, d\mu(x).$$

for all $B \in \mathcal{E}$, or equivalently, if for all $\phi \in C_b(E)$, and all $t \geq 0$,

$$\int_E \phi(x)\, d\mu(x) = \int_E (S_t\, \phi)(x)\, d\mu(x). \tag{2.3}$$

In other words, $E_\mu\phi(\xi) = E_\mu\phi(X_t^\xi)$.

Definition 2.2. The Markov semigroup S_t is said to have the **Feller** property if S_t maps $C_b(E)$ to $C_b(E)$. The semigroup possesses the **strong Feller** property if S_t maps $B_b(E)$ to $C_b(E)$.

Proposition 2.3. *Suppose that S_t has the Feller property. Let μ_t denote $P_\nu(t, \cdot)$, the distribution of X_t^ξ. If μ_t converges weakly to μ as $t \to \infty$, then μ is an invariant measure for S_t.*

Proof. With ν being the distribution of the initial variable ξ, we can write, for any $\phi \in C_b(E)$,

$$\int_E \phi(x)d\mu_t(x) = \int_E (S_t\phi)(x)d\nu(x) \to \int_E \phi(x)d\mu(x)$$

as $t \to \infty$. Therefore, for any fixed $s > 0$,

$$\int_E (S_{t+s}\phi)(x)d\nu(x) \to \int_E \phi(x)d\mu(x) \tag{2.4}$$

as $t \to \infty$. On the other hand,

$$\int_E (S_{t+s}\phi)(x)d\nu(x) = \int_E (S_t(S_s\phi))(x)d\nu(x) \to \int_E (S_s\phi)(x)d\mu(x) \tag{2.5}$$

as $t \to \infty$ since $S_s\phi \in C_b(E)$ by the Feller property. From (2.4) and (2.5), it follows that

$$\int_E (S_s\phi)(x)d\mu(x) = \int_E \phi(x)d\mu(x)$$

for all $s > 0$. In other words, μ is an invariant measure. □

Next, a constructive proof for the existence of an invariant measure due to Krylov and Bogolyubov, is given.

Theorem 2.4. *Suppose that S_t has the Feller property. Suppose that for some $x \in E$, there exists a sequence of times $t_n \uparrow \infty$ such that the sequence of Borel probability measures $\{\eta_n\}$ defined by*

$$\eta_n(B) = \frac{1}{t_n}\int_0^{t_n} P_x(s, B)ds \tag{2.6}$$

is tight. Then there exists an invariant measure for S_t.

Proof. By hypothesis (2.6), there exists a subsequence $\{\eta_{n_j}\}$ which converges weakly to a probability measure η. We will show that η is an invariant measure. In what follows, we will call this subsequence as $\{\eta_n\}$ for notational simplicity.

Recall that X_t^x is a Feller process so that its semigroup S_t maps C_b to C_b. Hence, for any bounded continuous function f on E,

$$\int (S_tf)(y)\eta(dy) = \lim_{n\to\infty} \int (S_tf)(y)\eta_n(dy). \tag{2.7}$$

We will consider $\int (S_tf)(y)\eta_n(dy)$ for any fixed n. By the Chapman-Kolmogorov equation, and the Fubini theorem,

$$\begin{aligned}
\int (S_tf)(y)\eta_n(dy) &= \frac{1}{t_n}\int_0^{t_n}\int (S_tf)(y)P_x(s, dy)ds \\
&= \frac{1}{t_n}\int_0^{t_n}\int\int f(z)P_y(t, dz)P_x(s, dy)ds \\
&= \frac{1}{t_n}\int_0^{t_n}\int f(z)P_x(t+s, dz)ds \\
&= \frac{1}{t_n}\int_0^{t_n}(S_{t+s}f)(x)ds.
\end{aligned}$$

The last expression can be written as $\frac{1}{t_n}\int_t^{t_n+t}(S_uf)(x)du$ which can be split as

$$\frac{1}{t_n}\Big[\int_0^{t_n}(S_uf)(x)du+\int_{t_n}^{t_n+t}(S_uf)(x)du-\int_0^{t}(S_uf)(x)du\Big].$$

Using this expression on the right side of Equation (2.7), we obtain

$$\begin{aligned}\int(S_tf)(y)\eta(dy)&=\lim_{n\to\infty}\frac{1}{t_n}\Big[\int_0^{t_n}(S_uf)(x)du\Big]\\&=\lim_{n\to\infty}\int f(y)\eta_n(dy)\\&=\int f(y)\eta(dy)\end{aligned}$$

by the weak convergence of η_n. Thus η is therefore an invariant measure. □

Proposition 2.5. *Let μ be an invariant measure for Equation* (5.1)*. If we set the distribution of the initial distribution to be μ, then the corresponding Markov process is a stationary process.*

The next result is on uniqueness of invariant measures.

Proposition 2.6. *Suppose that S_t has the Feller property. Let μ_t^x denote $P_x(t,\cdot)$, the distribution of X_t^x. If μ_t^x converges weakly to μ as $t\to\infty$ for all $x\in E$, then μ is the unique invariant measure for S_t.*

Proof. For any $\phi\in C_b(E)$,

$$S_t\phi(x)=\int_E\phi(y)d\mu_t^x(y)\to\int_E\phi(y)d\mu(y).$$

Therefore, for any invariant measure ν, we have

$$\begin{aligned}\int_E\phi(x)d\nu(x)&=\int_E S_t\phi(x)d\nu(x)\ \text{ for all }\ t\\&=\lim_{t\to\infty}\int_E S_t\phi(x)d\nu(x)\\&=\int_E\int_E\phi(y)d\mu(y)d\nu(x)\end{aligned}$$

by the bounded convergence theorem. Thus for all $\phi\in C_b(E)$, we have

$$\int_E\phi(x)d\nu(x)=\int_E\phi(y)d\mu(y).$$

Hence μ is the unique invariant measure. □

Definition 2.7. Let μ be an invariant measure for S_t. A set $A \in \mathcal{E}$ is called an **invariant set** with respect to the semigroup S_t if for all $t \geq 0$,

$$S_t 1_A = 1_A \quad \mu\text{a.s.}$$

The measure μ is called **ergodic** if $\mu(A)$ is either 0 or 1 for any invariant set A.

The following result is standard and stated without proof. One can find a proof in the book by Da Prato and Zabczyk.

Proposition 2.8. *If μ is an ergodic measure for the Markov family $\{X_t^x : x \in E\}$, then for any $\phi \in L^2(E, \mu)$,*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t S_s \phi ds = \int \phi d\mu \tag{2.8}$$

where the limit is taken in $L^2(E, \mu)$.

Proposition 2.9. *If μ and ν are two ergodic measures with respect to S_t, and if $\mu \neq \nu$, then μ and ν are singular.*

Proof. Let $A \in \mathcal{E}$ such that $\mu(A) \neq \nu(A)$. By the above proposition, there exists $t_n \uparrow \infty$ such that

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} (S_t 1_A)(x) dt = \mu(A) \;\; \mu \;\text{ almost all } \; x$$

and

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} (S_t 1_A)(x) dt = \nu(A) \;\; \nu \;\text{ almost all } \; x$$

Let the two almost sure sets be denoted B and C respectively. Then, $B \cap C = \emptyset$ and $\mu(B) = \nu(C) = 1$ which completes the proof. □

Theorem 2.10. *If μ is the unique invariant measure for the family $\{X^x : x \in E\}$, then μ is ergodic.*

Proof. Suppose that μ is not ergodic. Then, there exists a set $A \in E$ such that $0 < \mu(A) < 1$ and

$$S_t 1_A = 1_A \;\; \mu \;\text{ a.s.} \tag{2.9}$$

Define another measure ν on $\mathcal{E}$ by

$$\nu(B) = \frac{\mu(A \cap B)}{\mu(A)}.$$

We will show that ν is an invariant measure. Consider

$$\begin{aligned}\int_E P_x(t,B)d\nu(x) &= \frac{1}{\mu(A)}\int_A P_x(t,B)d\mu(x)\\ &= \frac{1}{\mu(A)}\{\int_A P_x(t,A\cap B)d\mu(x)+\int_A P_x(t,A^c\cap B)d\mu(x)\}.\end{aligned} \tag{2.10}$$

From $S_t 1 = 1$ and (2.9). it follows that

$$P_x(t,A\cap B)\le P_x(t,A)=0 \;\; \mu \;\text{ a.s. on }\; A^c$$

and

$$P_x(t,A^c\cap B)\le P_x(t,A^c)=0 \;\; \mu \;\text{ a.s. on }\; A$$

Hence, equation (2.10) can be written as

$$\int_E P_x(t,B)d\nu(x)=\frac{1}{\mu(A)}\int_E P_x(t,A\cap B)d\mu(x).$$

Using invariance of μ on the right side,

$$\int_E P_x(t,B)d\nu(x)=\frac{\mu(A\cap B)}{\mu(A)}=\nu(B).$$

Thus ν is also an invariant measure, a contradiction to the uniqueness of invariant measures. □

Definition 2.11. A Markov family X^x is called **irreducible** if for all $t>0$, any non-empty open set G in E, and any $x\in E$, we have

$$P\{X_t^x\in G\}>0.$$

That is, $P_x(t,G)>0$.

Proposition 2.12. *If the semigroup S_t is irreducible and strongly Feller, then the family of probability measures $\{P_x(t,\cdot) : x\in E\}$ are mutually equivalent for each $t>0$.*

Proof. Pick any $t,s>0$ Assume that $P_{x_0}(t+s,B)>0$ for some $x_0\in E$, and $B\in\mathcal{E}$. By Chapman-Kolmogorov equation,

$$P_{x_0}(t+s,B)=\int_E P_{x_0}(t,dy)P_y(s,B)$$

Therefore, there exists a y_0 such that $P_{y_0}(s,B)>0$. By the strong Feller property, $P_y(s,B)$ is a continuous function of y. Hence, there exists a neighborhood of y_0, say, $B_r(y_0)$ such that $P_y(s,B) > 0$ for all $y \in B_r(y_0)$. So, for any arbitrary $x \in E$,

$$\begin{aligned} P_x(t+s,B) &= \int_E P_x(t,dy)P_y(s,B) \\ &\geq \int_{B_r(y_0)} P_x(t,dy)P_y(s,B) > 0 \end{aligned}$$

since $P_x\{t, B_r(y_0)\} > 0$ by irreducibility. □

The main result that we need is the following theorem due to Doob.

Theorem 2.13. *Let μ be an invariant measure for the Markov family $\{X^x : x \in E\}$. If the corresponding semigroup S_t is irreducible and strongly Feller, then μ is the unique invariant measure, and hence ergodic.*

Proof. Let A be an invariant set so that

$$S_t 1_A = 1_A \quad \text{for all} \quad t > 0. \tag{2.11}$$

Suppose that $\mu(A) > 0$. We have to show that $\mu(A) = 1$. Since $\mu(A) > 0$, there exists an $x_0 \in A$ such that $P_{x_0}(t,A) = 1$. By the previous proposition the family of probability measures $P_x(t,\cdot)$ indexed by x are equivalent, so that $P_x(t,A) = 1$ for all $x \in E$. Using, in addition, invariance of μ, it follows that

$$\mu(A) = \int_E P_x(t,A)d\mu(x) = 1.$$

Thus μ is ergodic. If ν is another invariant measure, then repeating the above arguments, ν is also ergodic. Being invariant measures, both μ and ν are equivalent to $P_x(t,\cdot)$ for any x and t. Therefore, μ and ν are equivalent, and by Proposition 2.9, we can conclude that $\mu = \nu$. □

3. Linear Equations with Additive Noise

Let H be a real separable Hilbert space with norm denoted by $|\cdot|$, and inner product, by $(\cdot,\cdot)$. Consider the infinite-dimensional stochastic equation

$$du_t = Au_t dt + \sigma dW_t \tag{3.1}$$

with $u_0 = x \in H$. We will work under the following hypotheses:

Hypotheses A:

- A.1 The operator $A : D(A) \subset H \to H$, and A generates a strongly continuous semigroup e^{tA} on H.
- A.2 The noise coefficient $\sigma : H \to H$ be a continuous linear operator.

A.3 The linear operator $Q_t : H \to H$ defined by

$$Q_t h = \int_0^t e^{sA}\sigma\sigma^* e^{sA^*} h ds$$

is a nuclear operator for all $t \geq 0$.

A.4 The operator Q_t admits an inverse, and the operator L_t defined by

$$L_t = Q_t^{-1/2} e^{tA} \tag{3.2}$$

is a continuous linear operator on H.

The integrator W is taken to be a cylindrical Wiener process in H defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We can write, for any CONS $\{e_j\}$ in H,

$$W_t = \sum_{j=1}^{\infty} B_j(t) e_j$$

where B_j are independent standard one-dimensional Brownian motions. The mild solution of (3.1) is given by

$$u_t = e^{tA}x + \int_0^t e^{(t-s)A}\sigma dW_s. \tag{3.3}$$

We will use the notation

$$Z_t := \int_0^t e^{(t-s)A}\sigma dW_s$$

and call Z_t as the **stochastic convolution** integral. Note that Z_t is H-valued since

$$Z_t = \sum_{j=1}^{\infty} \int_0^t e^{(t-s)A}\sigma e_j dB_j(s)$$

and the series converges in $L^2(P)$ as H-valued random variables. Indeed, for any t,

$$\begin{aligned} E[|Z_t|^2] &= \sum_{j=1}^{\infty} \int_0^t \|e^{(t-s)A}\sigma e_j\|^2 \, ds \\ &= \int_0^t \|e^{(t-s)A}\sigma e_j\|_{HS}^2 \, ds \\ &= \int_0^t \mathrm{tr}(\sigma^* e^{(t-s)A^*} e^{(t-s)A}\sigma) ds \\ &= \int_0^t \mathrm{tr}(e^{(t-s)A}\sigma\sigma^* e^{(t-s)A^*}) ds \\ &= \mathrm{tr} \;\; Q_t < \infty \end{aligned}$$

by hypothesis (A.3). The H-valued random variable Z_t has a Gaussian distribution with mean 0 and variance given by tr Q_t.

Likewise, one can check that

$$E[(Z_{t_1}, Z_{t_2})] = \int_0^{t_1 \wedge t_2} \mathrm{tr}(e^{(t_1-s)A}\sigma\sigma^* e^{(t-2-s)A^*})ds.$$

and for any $h_1, h_2 \in H$,

$$E\big[(Z_t, h_1)(Z_t, h_2)\big] = (Q_t h_1, h_2).$$

So far, we have considered the solution Z for a fixed t. The next result views Z as a stochastic process.

Proposition 3.1. *Fix any $T > 0$. Let $L^2(\Omega; H)$ denote the space of H-valued random variables which are square integrable. Then, $Z : [0,T] \to L^2(\Omega; H)$ is a continuous map. That is, Z is in $C([0,T]; H)$ in the mean-square sense.*

Proof. Let $0 \le r < t \le T$. Then $Z_t - Z_r$

$$= \sum_{j=1}^{\infty} \int_0^r \big(e^{(t-s)A} - e^{(r-s)A}\big)\sigma e_j dB_j(s) + \sum_{j=1}^{\infty} \int_r^t \big(e^{(t-s)A}\big)\sigma e_j dB_j(s)$$
$$= I_1 + I_2 \text{ say.}$$

Since I_1 and I_2 are independent,

$$E|Z_t - Z_r|^2 = \int_0^r \mathrm{tr}\big\{(e^{(t-s)A} - e^{(r-s)A})\sigma\sigma^*(e^{(t-s)A^*} - e^{(r-s)A^*})\big\}ds + Q_{t-r}$$

which tends to zero as $r \to t$. □

The mild solution u_t given by (3.3) is known as the **Ornstein-Uhlenbeck** process. We will denote it by $u(t,x)$ as well. Define the semigroup S_t acting on $B_b(H)$ by

$$S_t\phi(x) = E[\phi(u(t,x))] = \int_H \phi(y)\mu_{x,t}(dy)$$

where $\mu_{x,t}$ is the $N(e^{tA}x, Q_t)$ distribution. When $x = 0$, we will write μ_t for $\mu_{0,t}$. By change of variables, we can also write

$$S_t\phi(x) = \int_H \phi(e^{tA}x + y)\mu_t(dy).$$

Theorem 3.2. *The semigroup S_t has the strong Feller property. In fact, S_t maps $B_b(H)$ to $C_b^1(H)$.*

Proof. STEP 1: The semigroup S_t is strong Feller.

Recall the operator $L_t = Q_t^{-1/2}e^{tA}$ defined in (3.2) which we have assumed to be continuous. It is clear that

$$\frac{d\mu_{x,t}}{d\mu_t}(y) = \exp\{-\frac{1}{2}(L_t x, L_t x) + (L_t x, Q_t^{-1/2}y)\}.$$

Hence,

$$(S_t\phi)(x) = \int_H \phi(y)\exp\{-\frac{1}{2}(L_t x, L_t x) + (L_t x, Q_t^{-1/2}y)\}d\mu_t(y). \tag{3.4}$$

Note that for any positive ϵ,

$$|(L_t x, Q_t^{-1/2}y)| \leq \frac{2}{\epsilon}|L_t x|^2 + \epsilon|Q_t^{-1/2}y|^2.$$

Take $\epsilon < 1/2$, and fix any $R > 0$. Consider any x in the open R-ball in H. There exists a corresponding constant C such that $|L_t x| \leq C$. Hence, the integrand in (3.4) in absolute value can be bounded by

$$\|\phi\|_\infty \exp\{\frac{2}{\epsilon}C^2\} \;\; e^{\epsilon|Q_t^{-1/2}y|^2}$$

which is integrable with respect to μ_t. This enables us to use LDCT in (3.4) to conclude that if $x_n \to x$ in H, then

$$\lim_{n\to\infty}(S_t\phi)(x_n) = (S_t\phi)(x).$$

STEP 2: The semigroup $S_t : B_b(H) \to C_b^1(H)$.

Differentiating with respect to x in equation (3.4), the derivative can be taken inside the integral if the derivative of the integrand remains bounded uniformly in x in bounded balls. The derivative of the integrand on the right side of (3.4) with respect to x equals

$$\phi(y)\exp\{-\frac{1}{2}|L_t x|^2 + (L_t x, Q_t^{-1/2}y)\} \;\; (L_t, Q_t^{-1/2}y - L_t x). \tag{3.5}$$

Clearly, for any $x, y \in H$, we have $D_x|L_t x|^2 = 2(L_t x, L_t)$, and

$$D_x(L_t x, Q_t^{-1/2}y) = (L_t, Q_t^{-1/2}y).$$

Let $|x| < R$ so that $|L_t x| \leq C$ as in Step 1. Then

$$\begin{aligned}|(L_t, Q_t^{-1/2}y - L_t x)| &\leq \|L_t\|(|Q_t^{-1/2}y| + |L_t x|) \\ &\leq \|L_t\|(|Q_t^{-1/2}y| + C).\end{aligned} \tag{3.6}$$

Using (3.6), we can bound (3.5) by a function which is integrable with respect to μ_t. Such a bound holds for all x with $|x| < R$. We can therefore differentiate $S_t\phi(x)$ with respect to x and obtain

$$(D_x\phi(u(t,x)),z) = \int \phi(y)(L_t z, Q_t^{-1/2}y - L_t x)\exp\{-\frac{1}{2}|L_t x|^2 + (L_t x, Q_t^{-1/2}y)\}d\mu_t(y) \tag{3.7}$$

for all $z \in H$. Changing variables from y to $e^{tA}x + y$, we can write

$$(D_x\phi(u(t,x)),z) = \int \phi(e^{tA}x + y)(L_t z, Q_t^{-1/2}y)\mu_{x,t}(dy). \tag{3.8}$$

□

Next, we wil prove irreducibility of the Markov family $\{u(t,x)\}$.

Theorem 3.3. *Consider the mild solutions $u(t,x)$ given by equation* (3.3) *for all $x \in H$. Under the hypotheses A, the family is irreducible.*

Proof. Fix any $z \in H$ and $r > 0$. Let $B_r(z)$ be the r-ball with center at z. We need to show that $\mu_{x,t}(B_r(z)) > 0$ for all $x \in H$ and $t > 0$. Since $u(t,x)$ is a $N(e^{tA}x, Q_t)$ random variable, it follows that $u(t,x) - z$ has a $N(e^{tA}x - z, Q_t)$ distribution. We will call the latter distribution as ν, and $e^{tA}x - z$ as α for simplicity.

Since Q_t is a nuclear operator, and kernel of Q_t is $\{0\}$, there exists a CONS $\{e_j\}$ in H, and a sequence $\{\lambda_j\}$ of positive numbers such that

$$Q_t e_j = \lambda_j e_j.$$

We can expand any $y \in H$ in terms of e_j, and write $y = \sum_{j=1}^{\infty} y_j e_j$. Then,

$$\begin{aligned}\mu_{x,t}(B_r(z)) &= \nu(B_r(0)) \\ &\geq \nu(y : \sum_{j=1}^{n} y_j^2 < \frac{r^2}{2})\nu(y : \sum_{j=n+1}^{\infty} y_j^2 < \frac{r^2}{2}).\end{aligned} \tag{3.9}$$

Under ν, the random variables y_j are independent with $N(\alpha_j, \lambda_j)$ distribution. Therefore,

$$\nu(y : \sum_{j=1}^{n} y_j^2 < \frac{r^2}{2}) > 0.$$

Let A denote the set $\{y : \sum_{j=n+1}^{\infty} y_j^2 < \frac{r^2}{2}\}$. Then by the Chebyshev inequality,

$$\begin{aligned}\nu(A^c) &\leq \nu(\sum_{j=n+1}^{\infty}(y_j - \alpha_j)^2 \geq \frac{r^2}{4} - \sum_{j=n+1}^{\infty}\alpha_j^2) \\ &\leq \frac{1}{\beta}\sum_{j=n+1}^{\infty}\lambda_j\end{aligned}$$

where $\beta = \frac{r^2}{4} - \sum_{j=n+1}^{\infty} \alpha_j^2$ and n is large enough so that $\beta > 0$. Since

$$\sum_{j=n+1}^{\infty} \lambda_j \to 0 \tag{3.10}$$

as $n \to \infty$, we obtain that for n large enough,

$$\nu(y : \sum_{j=n+1}^{\infty} y_j^2 < \frac{r^2}{2}) > 0. \tag{3.11}$$

Thus from (3.9), it follows that $\mu_{x,t}(B_r(z)) > 0$. □

We turn our attention to explore the asymptotic behavior of the family of Markov processes, X^x. Under hypotheses A, it is clear that there exist constants $K > 0$ and $\beta \in \mathbb{R}$ such that

$$\|e^{tA}\| \leq K e^{\beta t}. \tag{3.12}$$

We will assume that $\beta < 0$. Define an operator Q_∞ on H by

$$Q_\infty(x) = \int_0^\infty e^{tA} \sigma\sigma^* e^{tA} x dt$$

Since $\beta < 0$, Q_∞ has finite trace. Define the process M_t by

$$M_t = \int_0^t e^{sA} dW_s. \tag{3.13}$$

It is clear that M_t and Z_t have the same distribution.

Lemma 3.4. *Under hypotheses A and the condition that $\beta < 0$, the following limit in $L^2(\Omega; H)$ holds:*

$$\lim_{t\to\infty} M_t = \int_0^\infty e^{sA} \sigma dW_s.$$

The limit is denoted by M_∞.

Proof. For any $s, t > 0$, consider

$$\begin{aligned}
\mathbb{E}\|M_{s+t} - M_s\|^2 &= \int_s^{s+t} \text{tr } (e^{rA} \sigma\sigma^* e^{rA^*}) dr \\
&= \int_0^t \text{tr } (e^{(s+r)A} \sigma\sigma^* e^{(s+r)A^*}) dr \\
&= \text{tr } \{e^{sA} Q_t e^{sA^*}\} \\
&\leq K e^{2\beta s} \text{tr } Q_\infty
\end{aligned}$$

Thus, M_t is Cauchy in $L^2(\Omega; H)$ so that $\lim_{t\to\infty} M_t$ exists. The proof is over by using the Itô isometry. □

In the next lemma, we will use the notation μ for the distribution of M_∞. Note that M_∞ is a $N(0, Q_\infty)$ random variable.

Lemma 3.5. *Under hypotheses A and the condition that $\beta < 0$, we have*

$$\lim_{t\to\infty} S_t\phi(x) = \int_H \phi(y)d\mu(y) \tag{3.14}$$

for all $\phi \in C_b(H)$ and $x \in H$.

Proof. It is clear that

$$S_t\phi(x) = \mathbb{E}[\phi(u(t,x)] = \mathbb{E}[\phi(e^{tA}x + Z_t)] = \mathbb{E}[\phi(e^{tA}x + M_t)]$$

since Z_t and M_t are identically distributed. Allowing $t \to \infty$, the result is obtained since $e^{tA}x \to 0$, and $M_t \to M_\infty$ in distribution. □

Theorem 3.6. *Under hypotheses A and the condition that $\beta < 0$, the measure μ is the unique invariant measure, and is ergodic.*

Proof. Let ξ be any initial random variable. Since the distribution of X_t^ξ converges weakly to μ, Proposition implies that μ is invariant. The uniqueness of invariant measures follows from Proposition.

Besides, μ is ergodic since the Markov family is irreducible and strong Feller. In fact, μ is the only ergodic measure for the family $\{X^x\}$ since the support of μ is full in H. □

4. Nonlinear Equations with Additive Noise

In this section, we will study the following stochastic differential equation:

$$du_t = \{Au_t + F(u_t)\}dt + \sigma dW_t \tag{4.1}$$

with $u_0 = x \in H$. We assume hypotheses A and a Lipschitz continuity condition on F:

A.5 There exists a constant $C > 0$ such that $|F(x) - F(y)| \leq C|x - y|$ for all $x, y \in H$.

With condition (A.5) appended to hypotheses A, we will call the full list of conditions as Hypotheses A'. The mild solution u of (4.1) on any given interval $[0, T]$ is defined to be a *continuous* map $u : [0, T] \to L^2(\Omega; H)$ such that

$$u_t = e^{tA}x + \int_0^t e^{(t-s)A}F(u_s)ds + Z_t \tag{4.2}$$

where, as before, Z_t is the stochastic convolution integral. The existence and uniqueness of a mild solution to equation (4.1) is well-known and is a simple consequence of the fixed point principle.

Consider the following two auxiliary problems on $[0,T]$. First, truncate the cylindrical Wiener process W and form the finite-dimensional process

$$W_n(t) = \sum_{j=1}^{n} B_j(t) e_j.$$

The first problem is to solve

$$du_n(t) = [Au_n(t) + F(u_n(t))]dt + \sigma dW_n(t) \tag{4.3}$$

with $u_n(0) = x$. We would expect u_n to converge to u in some sense.

The second problem consists in replacing A in (4.3) by the Yosida approximation A_m and solving the resulting equation

$$du_{n,m}(t) = [A_m u_{n,m}(t) + F(u_{n,m}(t)]dt + \sigma dW_n(t) \tag{4.4}$$

with $u_{n,m}(0) = x$.

By a fixed point argument, a unique mild solution exists for each of the above problems.

Theorem 4.1. *Let the hypotheses A' hold, and $x \in H$. Then, for any $n, m \in \mathbb{N}$, the following limits hold in $C([0,T]; L^2(\Omega; H))$:*

$$\lim_{n\to\infty} u_n(t) = u_t \quad \textit{and} \quad \lim_{m\to\infty} u_{n,m}(t) = u_n(t). \tag{4.5}$$

Proof. The first approximation in (4.5) is straightforward. For, consider

$$u_n(t) - u_t = \int_o^t e^{(t-s)A}[F(u_n(s)) - F(u_s)]ds + \sum_{j=n+1}^{\infty} \int_0^t e^{(t-s)A} \sigma e_j dB_j(s).$$

Then, by using the Lipschitz condition on F,

$$E|u_n(t) - u_t|^2 \leq k_T \int_0^t \mathbb{E}|u_n(s) - u_s|^2 ds + 2 \sum_{j=n+1}^{\infty} \int_0^t |e^{sA}\sigma e_j|^2 ds$$

for a suitable constant k_T that depends on T. By the Gronwall Lemma, it follows that

$$\lim_{n\to\infty} E|u_n(t) - u_t|^2 = 0.$$

We will now prove the second limit in (4.5). Consider

$$
\begin{aligned}
u_{n,m}(t) - u_n(t) &= [e^{tA_m} - e^{tA}]x + \int_0^t e^{(t-s)A_m}[F(u_{n,m}(s)) - F(u_n(s))]ds \\
&\quad + \int_0^t [e^{(t-s)A_m} - e^{(t-s)A}]F(u_n(s))ds \\
&\quad + \sum_{j=1}^n \int_0^t [e^{(t-s)A_m} - e^{(t-s)A}]\sigma e_j dB_j(s). \\
&= I_1 + I_2 + I_3 + I_4 \ \text{(say)}.
\end{aligned}
$$

The bound

$$
E|I_2|^2] \le C(T) \int_0^t E|u_{n,m}(s) - u_n(s)|^2\, ds \tag{4.6}
$$

is easy to obtain, where $C(T)$ is a constant that depends on T. Clearly, the term $I_1 \to 0$ uniformly in $t \in [0,T]$ as $m \to \infty$. Using a similar reasoning and LDCT, it follows that

$$
E(|I_3|^2) \le T \int_0^T E|(e^{(t-s)A_m} - e^{(t-s)A})F(u_n(s))|^2\, ds \tag{4.7}
$$

which tends to 0 as $m \to \infty$.

$$
E|I_4|^2] \le \sum_{j=1}^n \int_0^T |(e^{(t-s)A_m} - e^{(t-s)A})\sigma e_j|^2\, ds \tag{4.8}
$$

which tends to zero as $m \to \infty$. Using (4.6), (4.7), (4.8) in a Gronwall argument, the proof is completed. □

Differentiability of Solutions: It is a simple exercise to show that, for any $x, y \in H$, there exists a constant $C_T > 0$ such that

$$
|u(t,x) - u(t,y)| \le C_T|x-y|. \tag{4.9}
$$

We will prove differentiability of the mild solution with respect to initial data x. If the derivative $Du(t,x)$ exists, define $v_z(t,x) = Du(t,x)z$ for all $z \in H$. One would expect the process v_z to satisfy the equation:

$$
dv_z(t,x) = Av_z(t,x)dt + DF(u(t,x)) \cdot v_z(t,x) \tag{4.10}
$$

with $v_z(0,x) = z$. Hence, we need differentiability of F. For this purpose, we will *assume* the stronger condition that F is actually in $C_b^2(H;H)$ and denote the bounds on the derivatives of F by C.

As before, the mild solution of (4.10) can be shown to exist and is given by

$$
v_z(t,x) = e^{tA}z + \int_0^t e^{(t-s)A}DF(u(s,x)) \cdot v_z(s,x)ds.
$$

The solution is unique.

Theorem 4.2. *Let Hypotheses A hold, and F be in $C_b^2(H;H)$. Then $u(t,x)$ is differentiable with respect to x almost surely, and*

$$Du(t,x)z = v_z(t,x) \ \ a.s. \tag{4.11}$$

where $v_z(t,x)$ is the mild solution of (4.10)*. Moreover, there exists a constant C_t that depends on t such that*

$$|v_z(t,x)| \le C_T|z| \ \ for\ all \ \ z \in H. \tag{4.12}$$

Proof. First, using (3.12), it is easy to obtain

$$|v_z(t,x)| \le Ke^{\beta t}|z| + KC\int_0^t e^{\beta(t-s)}|v_z(s,x)|ds$$

where C is the bound for the derivative of F. By a Gronwall argument, the bound (4.12) follows.

Define $\Delta_z(t,x) := u(t,x+z) - u(t,x) - v_z(t,x)$. We will show that

$$|\Delta_z(t,x)| \le C_T|z|^2$$

for all $0 \le t \le T$. The process Δ_z satisfies

$$\begin{aligned}\Delta_z(t,x) = &\int_0^t e^{(t-s)A}[F(u(s,x+z)) - F(u(s,x))]ds \\ &- \int_0^t e^{(t-s)A}DF(u(s,x))\cdot v_z(s,x)ds.\end{aligned} \tag{4.13}$$

For any $0 \le a \le 1$ and any $t \in [0,T]$, define

$$\eta(a,t) = a\,u(t,x+z) + (1-a)\,u(t,x).$$

Then on the right side of (4.13) we can use chain rule to write

$$\begin{aligned}
\Delta_z(t,x) &= \int_0^t e^{(t-s)A} \int_0^1 DF(\eta(a,s))da[u(s,x+z)-u(s,x)]\,ds \\
&\quad - \int_0^t e^{(t-s)A} DF(u(s,x)) \cdot v_z(s,x)\,ds \\
&= \int_0^t e^{(t-s)A} \int_0^1 DF(\eta(a,s))da[\Delta_z(s,x)+v_z(t,x)]\,ds \\
&\quad - \int_0^t e^{(t-s)A} DF(u(s,x)) \cdot v_z(s,x) ds \\
&= \int_0^t e^{(t-s)A} \int_0^1 DF(\eta(a,s))da\Delta_z(s,x)\,ds \\
&\quad + \int_0^t e^{(t-s)A} \{\int_0^1 [DF(\eta(a,s)) - DF(u(s,x))]da\} \cdot v_z(s,x)\,ds.
\end{aligned}$$

Using the bound on the second derivative of F and (4.12), there exists a constant K such that

$$\begin{aligned}
\int_0^1 [DF(\eta(a,s)) - DF(u(s,x))]da &\leq K|u(s,x+z)-u(s,x)|\,|z| \\
&= K[|\Delta_z(s,x)| + |v_z(s,x)|]\,|z| \\
&\leq K[|\Delta_z(s,x)| + K|z|]\,|z|.
\end{aligned}$$

Therefore, for all $|z|<1$,

$$|\Delta_z(t,x)| \leq 2K \int_0^t |\Delta_z(s,x)|ds + K^2|z|^2.$$

A use of the Gronwall Lemma completes the proof. □

Along similar lines, the conclusions of the above theorem hold for the derivatives of the approximations $u_n(t,x)$ and $u_{n,m}(t,x)$ of $u(t,x)$.

Strong Feller Property: Consider the semigroup $S_t\phi(x) = \mathbb{E}[\phi(u_t(x)]$ for all $\phi \in B_b(H)$, $t>0$ and $x \in H$. Then S_t is a contraction semigroup on $B_b(H)$ equipped with the supremum norm. It has the Feller property. Indeed, if $\phi \in C_b^1(H)$, then by (4.9)

$$\begin{aligned}
|S_t\phi(x) - S_t\phi(y)| &\leq \|\phi\|_1 \mathbb{E}|u_t(x) - u_t(y)| \\
&\leq \|\phi\|_1 C_T |x-y|
\end{aligned}$$

for all $x,y \in H$. Thus $S_t\phi$ is in $C_b(H)$. Since $C_b^1(H)$ is dense in $C_b(H)$, S_t is Feller.

The above conclusions hold for the semigroups S_t^n and $S_t^{n.m}$ which correspond to the family of processes $u_n(t,x)$ and $u_{n,m}(t,x)$ respectively. By LDCT, we have, for all $\phi \in C_b(H)$, $t>0$ and $x \in H$,

$$\lim_{n\to\infty} S_t^n \phi(x) = S_t\phi(x)$$

and for each $n \in \mathbb{N}$,

$$\lim_{m\to\infty} S_t^{n,m}\phi(x) = S_t^n\phi(x).$$

By the above theorem, if we assume that $F \in C_b^2(H:H)$, then $S_t\phi$ is differentiable for any $\phi \in C_b(H)$, and

$$(DS_t\phi(x), z) = \mathbb{E}[(D\phi(u_t(x)), v_z(t,x))] \tag{4.14}$$

for all $x \in H$.

We will now prove a result known as the *Bismut-Elworthy* formula. Though it is a more general result, we present it below for the present context.

Proposition 4.3. *Let hypotheses A hold. Assume that F is a Lipschitz continuous $C^3(H;H)$ function with bounded derivatives. and that the inverse of σ exists and is a continuous operator. Then, for all $\phi \in C_b^2(H)$, we have*

$$(DS_t\phi(x), z) = \frac{1}{t}\mathbb{E}\big[\phi(u(t,x))\int_0^t \sigma^{-1}v_z(s,x)dW_s)\big] \tag{4.15}$$

for all $t>0$ and $z \in H$.

.

Proof. First, we will consider the approximation $u_{n,m}(t,x)$. Then $S_t^{n,m}\phi \in C_b^2(H)$ since $u(t,x)$ is twice differentiable with respect to x. when F has bounded derivatives of first, second and third order.

If we denote $S_t^{n,m}\phi(x)$ by $a_{n,m}(t,x)$, then $a_{n,m}(t,x)$ satisfies the Kolmogorov equation:

$$\begin{aligned}\frac{\partial a_{n,m}(t,x)}{\partial t} &= (A_m x + F(x), Da_{n,m}(t,x)) + \frac{1}{2}\text{tr }\,[\sigma_n^*\sigma_n D^2 a_{n,m}(t,x)]\\ a_{n,m}(0,x) &= \phi(x).\end{aligned}$$

Fixing t, consider $a_{n,m}(t-s, u_{n,m}(s,x))$ as a function of s and $u_{n,m}(s,x)$. Then by the Itô formula, one obtains

$$a_{n,m}(0, u_{n,m}(t,x)) = a_{n,m}(t,x) + \int_0^t (Da_{n,m}(t-s, u_{n,m}(s,x)), \sigma_n dW_s).$$

Since $\phi \in C_b^2(H)$, we can take, by LDCT, the limit $n, m \to \infty$ inside the integral and obtain

$$\phi(u(t,x)) = S_t\phi(x) + \int_0^t (DS_{t-s}\phi(u(s,x)), \sigma dW_s). \tag{4.16}$$

Multiply (4.16) by $\int_0^t \sigma^{-1} v_z(s,x)dW_s$ and take expectation to obtain

$$\begin{aligned}
\mathbb{E}[\phi(u(t,x)) \int_0^t \sigma^{-1} v_z(s,x)dW_s] &= \mathbb{E}[\int_0^t (\sigma^* DS_{t-s}\phi(u(s,x)), \sigma^{-1} v_z(s,x))ds] \\
&= \mathbb{E}[\int_0^t (DS_{t-s}\phi(u(s,x)), v_z(s,x))ds] \\
&= \mathbb{E}[\int_0^t D_x S_{t-s}\phi(u(s,x)) \cdot z ds] \\
&= \int_0^t D_x S_s S_{t-s}\phi(x) \cdot z ds \\
&= t D_x S_t \phi(x) \cdot z.
\end{aligned}$$

which finishes the proof. □

Any Lipschitz continuous $F : H \to H$ can be approximated pointwise by a Lipschitz continuous $C^3(H; H)$ function with bounded derivatives by convolution.

Indeed, consider a self-adjoint, negative definite operator $B : D(B) \subset H \to H$ such that B^{-1} is nuclear. The unique mild solution of

$$dX(t,x) = BX(t,x)dt + IdW(t)$$

with $X(0,x) = x \in H$ exists.

The distribution of $X(t,x)$ is $N(e^{tB}x, \frac{1}{2}B^{-1}(e^{2tB} - I))$. The corresponding semigroup is strong Feller. Define for any $\delta > 0$,

$$F_\delta(x) = E[e^{\delta B} F(X(\delta, x))].$$

Then, F_δ is Lipschitz continuous and is C^∞ with bounded derivatives. Since $X(t,x)$ is continuous in time,

$$\lim_{\delta \to 0} F_\delta(x) = F(x)$$

for all $x \in H$. We thus have a smooth approximation of F.

Let $u^\delta(t,x)$ be the unique mild solution of

$$du^\delta(t,x) = [Au^\delta(t,x) + F_\delta(u^\delta(t,x))]dt + \sigma dW_t \tag{4.17}$$

with $u^\delta(0,x) = x$. Then, for all $t > 0$ and $x \in H$, we have

$$\lim_{\delta\to 0} u^\delta(t,x) = u(t,x). \tag{4.18}$$

Let S_t^δ be the semigroup corresponding to the family $u^\delta(t,x)$. Equation (4.18) implies that

$$\lim_{\delta\to 0} S_t^\delta\phi(x) = S_t\phi(x) \tag{4.19}$$

for all $\phi \in B_b(H)$, $x \in H$, and $t > 0$.

Theorem 4.4. *Assume hypotheses A'. Let σ admit a continuous inverse. Then the Markov family $\{u(t,x)\}$ indexed by $x \in H$ (or equivalently S_t) is strong Feller.*

Proof. STEP 1: We will assume that $\phi \in C_b^2(H)$. Fix any $\delta > 0$. Using (4.15) first, and then (4.12), we obtain, for any $z \in H$,

$$\begin{aligned} |(DS_t^\delta\phi(x), z)|^2 &\le \frac{1}{t^2}\||\|\phi|^2\ \|\sigma^{-1}\|^2\mathbb{E}[\int_0^t |v_z^\delta(s,x)|^2 ds] \\ &\le \frac{1}{t^2}|\phi|^2\ \|\sigma^{-1}\|^2 C_t^2|z|^2 \end{aligned}$$

where $v_z^\delta(s,x) = Du^\delta(t,x)\cdot z$. Thus, we can conclude that

$$|DS_t^\delta\phi(x)| \le \frac{1}{t}|\phi|\ \|\sigma^{-1}\|C_t.$$

Hence, for any $x, y \in H$,

$$|S_t^\delta\phi(x) - S_t^\delta\phi(y)| \le \frac{1}{t}|\phi|\ \|\sigma^{-1}\|C_t|x-y|$$

Allow $\delta \to 0$ so that

$$|S_t\phi(x) - S_t\phi(y)| \le \frac{1}{t}|\phi|^2\ \|\sigma^{-1}\|C_t|x-y|. \tag{4.20}$$

STEP 2: Suppose that $\phi \in B_b(H)$. For any $t > 0$, and $x, y \in H$, let μ_x^t and μ_y^t denote the distributions of $u(t,x)$ and $u(t,y)$ respectively. Let $\eta_{x,y}$ denote the signed measure $\mu_x^t - \mu_y^t$. Choose a sequence ϕ_n in $C_b^2(H)$ such that

$$|\phi_n| \le |\phi| \ \ \forall n$$

and

$$\phi_n(h) \to \phi(h)\ \ \eta_{x,y} \quad \text{a.a. } \ h \in H.$$

By LBCT,

$$S_t\phi(x) - S_t\phi(y) = \int \phi(h)\ d\eta_{x,y}(h) = \lim_{n\to 0}\int \phi_n(h)\ d\eta_{x,y}(h).$$

The proof is completed by a use of the bound (4.20). □

Irreducibility: We will prove irreducibility of the Markov family $u(t,x)$ via the notion of *approximate controllability* of differential systems. Consider the equation

$$\frac{dx(t)}{dt} = Ax(t) + F(x(t)) + \sigma\theta(t) \tag{4.21}$$

with $x(0) = \xi \in H$ and $\theta \in L^2(0,T;H)$. The mild solution $x(t)$ exists and is unique. The system (4.21) is approximately controllable in time T if for any $\epsilon > 0$, and any $\xi_1, \xi_2 \in H$, there exists a parameter or control $\theta \in L^2(0,T;H)$ such that

$$|x(T,\xi_1;\theta) - \xi_2| < \epsilon. \tag{4.22}$$

A well-known result on approximate controllability is given below, and its proof is omitted.

Proposition 4.5. *Under hypotheses A', the system* (4.21) *is approximately controllable in any time $T > 0$ provided that the range of σ is dense in H.*

Theorem 4.6. *Assume the hypotheses of the previous proposition. Then, the family $u(t,x)$ is irreducible.*

Proof. Let $T > 0$ and $\epsilon > 0$. Take any ξ_1, ξ_2 in H. We need to show that

$$P\{|u(T,\xi_1) - \xi_2| < \epsilon\} > 0$$

which is equivalent to showing that $P\{|u(T,\xi_1) - \xi_2| \geq \epsilon\} < 1$. We know that there exists a $\theta \in L^2(0,T;H)$ such that

$$|x(T,\xi_1;\theta) - \xi_2| < \frac{\epsilon}{2}.$$

Since

$$|u(T,\xi_1) - \xi_2| \leq |u(T,\xi_1) - x(T,\xi_1;\theta)| + |x(T,\xi_1;\theta) - \xi_2|$$

it suffices to show that

$$P\{|u(T,\xi_1) - x(T,\xi_1;\theta)| \geq \frac{\epsilon}{2}\} < 1. \tag{4.23}$$

It is easy to obtain, for all $t \in [0,T]$, the bound

$$|u(t,\xi_1) - x(t,\xi_1;\theta)| \leq C_T \int_0^t |u(s,\xi_1) - x(s,\xi_1;\theta)| ds + |Z_t - \int_0^t \sigma\theta(s) ds|$$

where Z_t is the stochastic convolution integral that appears in the mild solution $u(t,\xi_1)$. Let us call $\int_0^t \sigma\theta(s)ds$ as H_t. Hence, Gronwall Lemma yields

$$|u(T,\xi_1) - x(T,\xi_1;\theta)| \leq e^{C_T} \int_0^T |Z_s - H_s| ds.$$

Using Jensen's inequality, it follows that

$$u(T,\xi_1) - x(T,\xi_1;\theta)|^2 \le Te^{C_T}\|Z-H\|^2_{L^2(0,T;H)}.$$

By (4.23), it suffices to show that

$$P\{\|Z-H\|_{L^2(0,T;H)} \ge \frac{\epsilon}{2\sqrt{Te^{C_T}}}\} < 1.$$

This is obvious since Z is a normal random variable in $L^2(0,T;H)$. □

Ergodicity: With strong Feller property and irreducibility established, one would expect ergodicity once we show the existence of an invariant measure. For this we need additional hypotheses. We will pool together all the assumptions and call them as Hypotheses B.

Hypotheses B:

B.1 Hypotheses A' holds.
B.2 For all positive t, $\|e^{tA}\| \le e^{\beta t}$.
B.3 There exists a constant κ such that

$$(F(x)-F(y), x-y) \le \kappa |x-y|^2 \tag{4.24}$$

for all $x, y \in H$.
B.4 $\beta + \kappa < 0$.

The usefulness of condition [B.2] lies in obtaining the bound

$$(Ah,h) \le \beta |h|^2 \ \ \forall \ \ h \in D(A).$$

We introduce the notation $\rho = |\beta+\kappa|$. To prove the existence of an invariant measure, consider the following problem with initial time being negative:

$$\begin{aligned} dy(t) &= [Ay(t) + F(y(t))]dt + \sigma dN_t \ \ \forall \ \ t > -s \qquad (4.25)\\ y(-s) &= x \in H \end{aligned}$$

where $s > 0$ and N_t is defined for all t as follows. Take a cylindrical Wiener process $\hat{W}$ independent of W and define

$$N_t = \begin{cases} W_t & \text{if } t \ge 0 \\ \hat{W} & \text{if } t \le 0 \end{cases}$$

For all t, let $\mathcal{F}_t$ denote the σ-field generated by $\{N_r : r \le t\}$ augmented by P-null sets. Equation (4.25) has a unique mild solution. The distribution of y_0 coincides with that of $u(s,x)$. Let us call the random variable y_0 started at time $-s$ and initial state x as $y(0;-s,x)$.

Then, one can show that under Hypotheses B, $\lim_{s\to\infty} y(0;-s,x)$ exists in $L^2(\Omega;H)$. Let us call the limit as Y. The law of Y can easily be shown to be an invariant measure for S_t. In fact, it is a unique invariant measure. We can gather all of the observations that have been made and state the following theorem:

Theorem 4.7. *Assume Hypotheses B. Additionally, let σ admit a continuous inverse, and let the range of σ be dense in H. Then the law of Y is an ergodic measure for the Markov family $\{u(t,x) : t \in \mathbb{R}^+ \text{ and } x \in H\}$.*

5. **Strong Solutions and Invariant Measures**

Let H be a real separable Hilbert space which contains a reflexive Banach space V. Let V^* denote the dual space of V. With the identification of H^* with H, we have the following inclusions:

$$V \subset_{\rightarrow} H = H^* \subset_{\rightarrow} V^*.$$

We will assume that the inclusions are dense and compact. The norms on H, V and V^*, the norms will be denoted by $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_*$ respectively. The inner product on H and the dual pairing between V and V^* will be denoted by $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ respectively.

Consider the stochastic the infinite-dimensional stochastic differential equation on the time interval $[0,T]$:

$$du_t = Au_t dt + F(u_t)dt + G(u_t)dW_t \tag{5.1}$$

with $u_0 = \xi$.

In the above equation, let A be is a closed linear operator with domain $D(A)$ dense in H, and $A : V \to V^*$. Let $F : V \to H$. The process W is a H-valued Wiener process with a trace class operator Q as the covariance form. The noise coefficient $G : V \to L^2_Q$ which means that

$$\|G\|_Q := \{T\mathrm{tr}(G(v)QG(v)^*)\}^{1/2} < \infty \ \ \forall \ v \in V.$$

The initial random variable ξ is in $L^2(\Omega; H)$ and is $\mathcal{F}_0$-measurable

Definition 5.1. An adapted V-valued process u is a **strong solution** of the equation (5.1) if $u \in L^2([0,T]\times\Omega; V)$, and for any $f \in V$, the following equality holds a.s. for each $t \in [0,T]$:

$$(u_t, f) = (h,f) + \int_0^t \langle Au_s, f\rangle ds + \int_0^t (F(u_s), f)ds + \int_0^t (f, G(u_s)dW_s). \tag{5.2}$$

A strong solution u is called unique if $u(t,\omega) = \tilde{u}(t,\omega)$ for a.e. $(t,\omega) \in \Omega_T$ (here Ω_T denotes $[0,T]\times\Omega$) where $\tilde{u}$ is any other strong solution of (5.1).

The meaning of Equation (5.1) is made clear by Equation (5.2). The differential equation is actually an integral equation, and the equality sign in (5.1) is meant as an equality in V^*. To obtain the existence of a unique strong solution u, we work with the following hypotheses:

Hypotheses H:

H.1 (linear growth) For all $v \in V$, there exists a constant $C > 0$ such that

$$|F(v)|^2 + \|G(v)\|_Q^2 \leq C(1 + \|v\|^2).$$

H.2 (coercivity) There exists constants $a > 0$, b and c such that for any $v \in V$,

$$\langle Av, v\rangle + (F(v), v) + \|G(v)\|_Q^2 \leq -a\|v\|^2 + b|v|^2 + c$$

H.3 (monotonicity) For any $u, v \in V$, there exists a constant $K > 0$ such that

$$2\langle [Au + F(u)] - [Av + F(v)], u - v\rangle + \|G(u) - G(v)\|_Q^2 \leq K|u - v|^2$$

for all $u, v \in V$ and $0 \leq t \leq T$.

Under Hypotheses H, one can show that there exists a unique strong solution of Equation (5.1) such that $u \in L^2(\Omega : C([0,T]; H) \cap L^2(\Omega_T; V)$. The solution u can be shown to be a time-homogeneous strong Markov process so that the transition probability function, for all $0 \leq s \leq t$, Borel sets B in H and $\xi \in H$, satisfies

$$P_{s,t}(\xi, B) = P(u(t) \in B \,|\, u(s) = \xi) = P(u(t-s) \in B \,|\, u_0 = \xi) = P_{t-s}(\xi, B) \tag{5.3}$$

Theorem 5.2. *Let the coefficients of Equation* (5.1) *satisfy the hypotheses H.1, H.3 and the following coercivity condition which is stronger than H.2: There exist positive constants a and c such that for all $v \in V$,*

$$2\langle Av, v\rangle + 2(F(v), v) + \|G(v)\|_Q \leq c - a\|v\|^2. \tag{5.4}$$

Then there exists a unique invariant measure for the solution u of equation (5.1).

Proof. Step 1: We will first establish the Feller property. Suppose that u_t^h and u_t^g are the solutions started at h and g respectively where h and g are in H. By the energy equality, we obtain

$$\begin{aligned}\mathbb{E}|u_t^h - u_t^g|^2 &= |h-g|^2 + \mathbb{E}\int_0^t [2\langle A(u_s^h - u_s^g), u_s^h - u_s^g\rangle \\ &\quad + 2(F(u_s^h) - F(u_s^g), u_s^h - u_s^g) + \|G(u_s^h) - G(u_s^g)\|_Q^2] ds \\ &\leq |h-g|^2 + K\int_0^t E\{|u_s^h - u_s^h|^2\} ds\end{aligned}$$

where we have used the monotonicity condition H.3. By the Gronwall inequality,

$$\mathbb{E}|u_t^h - u_t^g|^2 \leq e^{Kt}|h-g|^2. \tag{5.5}$$

It suffices to consider bounded, Lipschitz continuous functions ϕ in verifying the Feller property. Let the Lipschitz constant be L. Consider

$$\begin{aligned}|P_t\phi(h) - P_t\phi(g)|^2 &\leq \mathbb{E}|\phi(u_t^h) - \phi(U_t^g)|^2 \\ &\leq L\mathbb{E}\|u_t^h - u_t^g\|^2 \\ &\leq Le^{Kt}|h-g|^2.\end{aligned}$$

STEP 2: By the energy equality and (5.4), one obtains

$$\begin{aligned}\mathbb{E}|u_t^h|^2 &= |h|^2 + \mathbb{E}\int_0^t \{2\langle Au_s^h, u_s^h\rangle + 2(F(u_s^h), u_s^h) + \|u_s^h\|_Q^2\} ds \\ &\leq |h|^2 + ct - a\int_0^t \mathbb{E}\|u_s^h\|^2 ds.\end{aligned}$$

Hence,

$$\mathbb{E}\{\int_0^t \|u_s^h\|^2 ds\} \leq \frac{1}{a}(ct + |h|^2). \tag{5.6}$$

Since u_t^h is a V-valued process, $P(h,t,C_R) = P\{\|u_t^h\| > R\}$. Hence, we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T P(h,t,C_R)dt &= \frac{1}{T}\int_0^T P\{\|u_t^h\| > R\} dt \\ &\leq \frac{1}{N^2 T}\int_0^T E\|u_t^h\|^2 dt \\ &\leq \frac{1}{aN^2T}(ct + |h|^2).\end{aligned}$$

Therefore, $\lim_{R\to\infty} \frac{1}{T}\int_0^T P(h,t,C_R)dt = 0$ uniformly in $T \geq 1$. We can conclude that there exists a measure μ such that $\mu_t^h \to \mu$ as $t \to \infty$ for any $h \in H$. Hence, μ is the unique invariant measure. □

6. Two-Dimensional Stochastic Navier-Stokes Equation

Let G be a bounded open domain in $\mathbf{R}^2$ with a smooth boundary ∂G. For $t \in [0,T]$, consider the stochastic Navier-Stokes equation for a viscous incompressible flow with no-slip condition at the boundary. That is,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} + \sigma(t)\,\frac{d\mathbf{W}(t)}{dt} \tag{6.1}$$

and

$$\nabla\cdot\mathbf{u} = 0 \tag{6.2}$$

with

$$\mathbf{u}(t,x) = 0 \ \forall\ x \in \partial G,$$
$$\mathbf{u}(0,x) = \mathbf{u}_0(x) \ \forall\ x \in G.$$

To study the stochastic Navier-Stokes equations, we first write the stochastic partial differential equation in the abstract (variational) form on suitable function spaces. Let $\mathcal{V}$ be the space of 2-dimensional vector functions $\mathbf{u}$ on G which are infinitely differentiable with compact support strictly contained in G, satisfying $\nabla\cdot\mathbf{u} = 0$. Let V_α denote the closure of $\mathcal{V}$ in $W^{\alpha,2}$ for all real α. In particular, let $H = V_0$ and $V = V_1$. The notation $L^2(G)$, $W_0^{1,2}(G)$ etc. would mean vector functions with each coordinate in $L^2(G)$, $W_0^{1,2}(G)$ etc. The following characterizations of the spaces H and V are well-known:

$$\begin{aligned} H &= \{\mathbf{u}\in L^2(G); \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n}|_{\partial G} = 0\}, \\ V &= \{\mathbf{u}\in W_0^{1,2}(G) : \nabla\cdot\mathbf{u} = 0\}. \end{aligned}$$

where $W_0^{1,2}(G) = \{\mathbf{u}\in L^2(G) : \nabla\mathbf{u}\in L^2(G),\ \mathbf{u}|_{\partial G} = 0\}$, and $\mathbf{n}$ is the outward normal. Let V' be the dual of V. We will denote the norm in H by $|\cdot|$, and the inner product in H by $(\cdot,\cdot)$. We have the dense, continuous embedding

$$V \subset_{\rightarrow} H = H^{'} \subset_{\rightarrow} V^{'}.$$

Let $\mathcal{D}(\mathbf{A}) = W^{2,2}(G)\cap V$. Define the linear operator $\mathbf{A} : \mathcal{D}(\mathbf{A}) \to H$ by $\mathbf{A}\mathbf{u} = -\Delta\mathbf{u}$. Since $V = \mathcal{D}(\mathbf{A}^{1/2})$, we can endow V with the norm $||\mathbf{u}|| = |\mathbf{A}^{1/2}\mathbf{u}|$. The V-norm is equivalent to the $W^{1,2}$-norm by the Poincare inequality. The operator $\mathbf{A}$ is known as the Stokes operator and is positive, self-adjoint with compact resolvent. The eigenvalues of $\mathbf{A}$ will be denoted by $0 < \lambda_1 < \lambda_2 \le \cdots$, and the corresponding eigenfunctions by $e_1, e_2, \cdots$. The eigenfunctions form a complete orthonormal system for H. From now on, $||.||$ will denote the V-norm. Note that $||\mathbf{u}||^2 \ge \lambda_1\,|u|^2$.

Define $b(\cdot,\cdot,\cdot)\colon V\times V\times V\to\mathbf{R}$ by

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=\sum_{i,j=1}^{d}\int_G u_i\partial v_j\omega\partial x_i w_j\,dx$$

using which we can define $\mathbf{B}\colon V\times V\ \to\ V'$ as the continuous bilinear operator such that

$<\mathbf{B}(\mathbf{u},\mathbf{v}),\ \mathbf{w}>=b(\mathbf{u},\mathbf{v},\mathbf{w})$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$. Note that $b(\mathbf{u},\mathbf{v},\mathbf{w})=-b(\mathbf{u},\mathbf{w},\mathbf{v})$.

$\mathbf{B}(\mathbf{u})$ will be used to denote $\mathbf{B}(\mathbf{u},\mathbf{u})$. $\mathbf{B}(\mathbf{u})$ satisfies the following estimate:

$$||\mathbf{B}(\mathbf{u})||_{V'}\le 2\,|\mathbf{u}|\,||\mathbf{u}|| \tag{6.3}$$

Let $\sigma(t):=\{\sigma_1(t),\sigma_2(t),\cdots\}$ be $l_2(H)$-valued for all t, and $\mathbf{W}=\{W_1,W_2,\cdots\}$ where W_j be independent copies of a standard one-dimensional Wiener process. Thus $\sigma(t)d\mathbf{W}(t)=\sum_j\sigma_j(t)\,dW_j(t)$ is an H-valued noise term.

We assume that $\mathbf{f}(t)$ and $\mathbf{u}_0$ to be H-valued. Let Π denote the Leray projection of $L^2(G)$ into H. By applying this projection to each term of the Navier-Stokes system, and invoking the Helmholtz decomposition of $L^2(G)$ into divergence free and irrotational components, we can write the system (6.1) and (6.2) as

$$d\mathbf{u}+[\nu\mathbf{A}\,\mathbf{u}+\mathbf{B}(\mathbf{u})]\,dt=\mathbf{f}\,dt+\sigma(t)\,d\mathbf{W}(t). \tag{6.4}$$

The following is a statement on the monotonicity of the coefficients on bounded L^4-balls in V and is easy to prove.

Proposition 6.1. *For a given $r>0$, let $B_r=\{\mathbf{v}\in V:||\mathbf{v}||_{L^4(G,R^2)}\le r\}$. The nonlinear operator $\mathbf{u}\to\mathbf{A}\mathbf{u}+\mathbf{B}(\mathbf{u})$ is monotone in B_r; that is, for any $\mathbf{u}\in V$, $\mathbf{v}\in B_r$ and with $\mathbf{w}=\mathbf{u}-\mathbf{v}$, we have*

$$<\mathbf{A}\mathbf{w},\mathbf{w}>+<\mathbf{B}(\mathbf{u})-\mathbf{B}(\mathbf{v}),\mathbf{w}>+\frac{27}{2\nu^3}|\mathbf{w}|^2||\mathbf{v}||^4_{L^4(G,R^2)}\ge\frac{\nu}{2}||\mathbf{w}||^2 \tag{6.5}$$

Using the Galerkin approximations and the above monotonicity of coefficients, one can invoke the Minty-Browder argument to establish the following result on the existence and uniqueness of strong solutions to (6.4).

Theorem 6.2. *Let* $f \in L^p(0,T;H)$, $\sigma \in L^p(0,T;l_2(H))$ *and* $\mathbf{u}_0 \in L^p(\Omega, H)$ *for some* $p \geq 6$. *Then there exists a process* $\mathbf{u}(t,x,\omega)$ *adapted to the filtration* $\mathcal{F}_t$ *with* $\mathbf{u} \in L^p(\Omega; C(0,T;H)) \cap L^2(\Omega; L^2(0,T;V) \cap L^4(G \times (0,T) \times \Omega)$ *such that* $\mathbf{u}$ *solves (6.4) in the following sense:*

$$\begin{aligned} & d(\mathbf{u}(t), \mathbf{v}) + < \mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t)), \mathbf{v} > dt \\ & = (\mathbf{f}(t), \mathbf{v}) dt\, dt + \sum_j (\sigma_j(t), \mathbf{v})\, dW_j(t) \end{aligned} \tag{6.6}$$

in $(0,T)$, *with the initial condition*

$$(\mathbf{u}(0), \mathbf{v}) = (\mathbf{u}_0, \mathbf{v}) \tag{6.7}$$

for any $\mathbf{v}$ *in the space* V. *The solution is pathwise unique.*

7. Exponential Stability

In what follows, the conditions of the previous theorem are assumed so that a pathwise unique strong (in the sense of both stochastic analysis and partial differential equations) solution is guaranteed. By an application of the Itô Lemma, we get

$$\begin{aligned} |\mathbf{u}(t)|^2 &= |\mathbf{u}(0)|^2 - 2\nu \int_0^t ||\mathbf{u}(s)||^2\, ds + 2 \int_0^t (f(s), \mathbf{u}(s))\, ds \\ &\quad + \sum_j \int_0^t |\sigma_j(s)|^2\, ds + 2 \sum_j \int_0^t (\sigma_j(s), \mathbf{u}(s))\, dW_j(s) \end{aligned} \tag{7.1}$$

by using $b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$. Using the Poincaré inequality in (7.1), we get upon taking expectation,

$$E|\mathbf{u}(t)|^2 \leq E|\mathbf{u}(0)|^2 - \nu\lambda_1 \int_0^t E|\mathbf{u}(s)|^2\, ds + \int_0^t \sum_j |\sigma_j(s)|^2\, ds$$

where λ_1 is the first eigenvalue of the Stokes operator $\mathbf{A}$. By the Gronwall inequality, it follows that

$$E|\mathbf{u}(t)|^2 \leq E|\mathbf{u}(0)|^2\, e^{-2\nu\lambda_1 t} + \int_0^t e^{-\nu\lambda_1(t-s)} \sum_j |\sigma_j(s)|^2\, ds \leq C \tag{7.2}$$

where C is a constant independent of t. A few more estimates are needed and are stated below:

Proposition 7.1. *Under the conditions of Theorem 6.2 the following estimates hold:*

1. For any $0 \le t \le T$,

$$E|\mathbf{u}(t)|^p \le E|\mathbf{u}(0)|^p + C_p \int_0^T \{|\mathbf{f}(s)|^p + (\sum_j |\sigma_j(s)|^2)^{p/2}\}ds \tag{7.3}$$

2.

$$E \sup_{0\le t\le T} |\mathbf{u}(t)|^{2p} \le E|\mathbf{u}(0)|^p + C_{p,T} \int_0^T \{|\mathbf{f}(s)|^p + (\sum_j |\sigma_j(s)|^2)^{p/2}\}\, ds \tag{7.4}$$

Proof. Using the Ito Lemma in (7.1) for the function $f(x) = x^{p/2}$, one gets that

$$\begin{aligned} |\mathbf{u}(t)|^p &= |\mathbf{u}(0)|^p - \nu p \int_0^t ||\mathbf{u}(s)||^2\, |\mathbf{u}|^{p-2}\, ds \\ &+ p/2 \int_0^t \{2(\mathbf{f}(s), \mathbf{u}(s)) + \sum_j |\sigma_j(s)|^2\}\, |\mathbf{u}(s)|^{p-2}\, ds \\ &+ p/2 \sum_j \int_0^t (\sigma_j(s), \mathbf{u}(s)) |\mathbf{u}(s)|^{p-2}\, dW_k(s) \\ &+ \frac{p(p-2)}{2} \sum_j \int_0^t |\mathbf{u}(s)|^{p-4}\, (\sigma_j(s), \mathbf{u}(s))^2\, ds \end{aligned} \tag{7.5}$$

Therefore (7.3) follows by using (7.5).

Let $J = E\{\sup_{0\le t\le T} | \sum_j \int_0^t (\sigma_j(s), \mathbf{u}(s))\, |\mathbf{u}(s)|^{p-2} dW_j(s)|\}$.

By the the Burkholder-Davis-Gundy inequality, one obtains

$$\begin{aligned} J &\le \sqrt{2} E\{\int_0^T \sum_j (\sigma_j(s), \mathbf{u}(s))^2\, |\mathbf{u}(s)|^{2p-4}\, ds\}^{1/2} \\ &\le \sqrt{2} E\{\int_0^T \sum_j |\sigma_j(s)|^2\, |\mathbf{u}(s)|^{2p-2}\, ds\}^{1/2} \\ &\le \sqrt{2} E\{\sup_{0\le s\le T} |\mathbf{u}(s)|^{p-1}\}\, (\int_0^T \sum_j |\sigma_j(s)|^2\, ds)^{1/2} \\ &\le \epsilon E\{\sup_{0\le s\le T} |\mathbf{u}(s)|^p\} + C_{\epsilon,p,T} \int_0^T (\sum_j |\sigma_j(s)|^2)^{p/2}\, ds \end{aligned} \tag{7.6}$$

Using (7.5) and (7.6),it is easy to show (7.4). □

It should be noted that $\sum_j \int_0^t (\sigma_j(s), \mathbf{u}(s))\, dW_j(s)$ is indeed a martingale for $t \in [0, T]$ as can be seen by a stopping time argument. The next Lemma is stated without a proof. One can see a proof in [9].

Lemma 7.2. *Let $B(t)$ be an increasing, progressively measurable process with $B(0) > 0$ a.s. Let $M(t)$ be a continuous martingale with $M(0) = 0$. If $Z(t) := \displaystyle\int_0^t \frac{1}{B(s)}\, dM(s)$ converges a.s. to a finite limit as $t \to \infty$, then $\displaystyle\lim_{t\to\infty} \frac{M^*(t)}{B(t)} = 0$ a.s. on the set $\{B(\infty) = \infty\}$ where $M^*(t) = \sup_{0\le s\le t} |M(s)|$.*

Lemma 7.3. *Let $\lim_{t\to\infty} \sum_j |\sigma_j(t)|^2 = L_1 < \infty$. Then $\lim_{t\to\infty} \frac{M^*(t)}{t} = 0$ a.s. where $M(t) = \sum_j \int_0^t (\sigma_j(s), \mathbf{u}(s))\, dW_j(s)$.*

Proof. For any $\epsilon > 0$, set $B(t) = \epsilon + t$ in Lemma 7.2 so that in the notation of that Lemma $Z(t) = \displaystyle\int_0^t \frac{1}{(\epsilon + s)}\, dM(s)$. Therefore,

$$\langle Z\rangle_t = \sum_j \int_0^t \frac{(\sigma_j(s), \mathbf{u}(s))^2}{(\epsilon+s)^2}\, ds \le \sum_j \int_0^t \frac{|\sigma_j(s)|^2\, |\mathbf{u}(s)|^2}{(\epsilon+s)^2}\, ds.$$

so that upon letting t to ∞ and then taking expectation,

$$\begin{aligned} E\,\langle Z\rangle_\infty &\le \sum_j \int_0^\infty \frac{|\sigma_j(s)|^2\, E|\mathbf{u}(s)|^2}{(\epsilon+s)^2}\, ds \\ &\le C \int_0^\infty \frac{1}{(\epsilon+s)^2}\, ds \\ &< \infty \end{aligned}$$

by using the estimate (7.2). Hence $\langle Z\rangle_\infty < \infty$ a.s. which allows us to conclude that $\lim_{t\to\infty} Z(t)$ exists and is finite almost surely. Invoking Lemma (7.2), the proof is completed. □

Theorem 7.4. *Let $\mathbf{u}$ and $\mathbf{v}$ be the solutions of (6.4) started at $\mathbf{u}_0$ and $\mathbf{v}_0$ respectively. Assume that $\lim_{t\to\infty} \sum_j |\sigma_j(t)|^2 = L_1 < \infty$ and $\lim_{t\to\infty} |\mathbf{f}(t)|^2 = L_2$. Let L denote $L_1 + \dfrac{L_2}{4\nu}$. If $\nu^3 > \dfrac{4L}{\lambda_1}$, then $\lim_{t\to\infty} |\mathbf{u}(t) - \mathbf{v}(t)| = 0$ a.s.*

Proof. From (6.4),

$$\mathbf{u}(t)-\mathbf{v}(t)+\nu\int_0^t \mathbf{A}(\mathbf{u}(s)-\mathbf{v}(s))\,ds+\int_0^t (\mathbf{B}(\mathbf{u}(s))-\mathbf{B}(\mathbf{v}(s)))\,ds=\mathbf{u}_0-\mathbf{v}_0.$$

By the Itô formula,

$$|\mathbf{u}(t)-\mathbf{v}(t)|^2+\int_0^t 2(\nu\,||(\mathbf{u}(s)-\mathbf{v}(s))||^2+\langle \mathbf{B}(\mathbf{u}(s))-\mathbf{B}(\mathbf{v}(s)),\mathbf{u}(s)-\mathbf{v}(s)\rangle)\,ds$$
$$=|\mathbf{u}_0-\mathbf{v}_0|^2 \tag{7.7}$$

Let $\mathbf{w}=\mathbf{u}-\mathbf{v}$. By the bilinearity of the operator $\mathbf{B}$,

$$\langle \mathbf{B}(\mathbf{u}),\mathbf{w}\rangle = b(\mathbf{u},\mathbf{u},\mathbf{w}) = b(\mathbf{u},\mathbf{w},\mathbf{w})+b(\mathbf{u},\mathbf{v},\mathbf{w}).$$

Therefore by using the fact that $b(\mathbf{u},\mathbf{w},\mathbf{w})=0$, we get

$$\langle \mathbf{B}(\mathbf{u})-\mathbf{B}(\mathbf{v}),\mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w})-b(\mathbf{v},\mathbf{v},\mathbf{w})=b(\mathbf{w},\mathbf{v},\mathbf{w}). \tag{7.8}$$

Using (7.8) and a basic estimate of the nonlinear term in the Navier-Stokes equation yields

$$|\langle \mathbf{B}(\mathbf{u})-\mathbf{B}(\mathbf{v}),\mathbf{w}\rangle| \le 2||\mathbf{w}||\,|\mathbf{w}|\,||\mathbf{v}||. \tag{7.9}$$

The bound (7.9) when used in (7.7) gives the following estimate:

$$|\mathbf{w}(t)|^2+2\nu\int_0^t ||\mathbf{w}(s)||^2\,ds$$
$$\le \quad |\mathbf{w}(0)|^2+4\int_0^t ||\mathbf{w}(s)||\,|\mathbf{w}(s)|\,||\mathbf{v}(s)||\,ds \tag{7.10}$$

Using Young's inequality on the right side of (7.10), and continuing:

$$\le \quad |\mathbf{w}(0)|^2+\nu\int_0^t ||\mathbf{w}(s)||^2\,ds+\frac{4}{\nu}\int_0^t |\mathbf{w}(s)|^2\,||\mathbf{v}(s)||^2\,ds \tag{7.11}$$
$$\tag{7.12}$$

Using the Poincaré inequality $\lambda_1|\mathbf{w}|^2\le||\mathbf{w}||^2$ in (7.11) and rearranging the terms ,we arrive at

$$|\mathbf{w}(t)|^2+\nu\,\lambda_1\int_0^t |\mathbf{w}(s)|^2\,ds \le |\mathbf{w}(0)|^2+\frac{4}{\nu}\int_0^t |\mathbf{w}(s)|^2\,||\mathbf{v}(s)||^2\,ds \tag{7.13}$$

By the Gronwall Lemma, we thus get

$$|\mathbf{w}(t)|^2 \le |\mathbf{w}(0)|^2\,\exp\{\frac{4}{\nu}\int_0^t ||\mathbf{v}(s)||^2\,ds-\nu\lambda_1 t\} \tag{7.14}$$

The equation (7.1) can be rewritten as

$$
\begin{aligned}
&|\mathbf{v}(t)|^2 + 2\nu \textstyle\int_0^t ||\mathbf{v}(s)||^2\,ds \\
&\quad = |\mathbf{v}(0)|^2 + 2\int_0^t (f(s), \mathbf{v}(s))\,ds + \sum_j \int_0^t |\sigma_j(s)|^2\,ds \\
&\qquad +2\sum_j \int_0^t (\sigma_j(s), \mathbf{v}(s))\,dW_j(s) \qquad (7.15)
\end{aligned}
$$

Upon noting that

$$
|\int_0^t (\mathbf{f}(s), \mathbf{v}(s))\,ds| \leq \int_0^t \frac{|\mathbf{f}(s)|^2}{4\nu}\,ds + \nu \int_0^t |\mathbf{v}(s)|^2\,ds,
$$

one obtains from (7.15) the following:

$$
\begin{aligned}
&\nu \textstyle\int_0^t ||\mathbf{v}(s)||^2\,ds \\
&\quad \leq \frac{1}{t}(|\mathbf{v}(0)|^2 - |\mathbf{v}(t)|^2) + \frac{1}{4\nu}\int_0^t |\mathbf{f}(s)|^2\,ds + \sum_j \int_0^t |\sigma_j(s)|^2\,ds \\
&\qquad +2\sum_j \int_0^t (\sigma_j(s), \mathbf{v}(s))\,dW_j(s) \qquad (7.16)
\end{aligned}
$$

By Lemma 7.3,

$$
\lim_{t\to\infty} \frac{1}{t}\sum_j \int_0^t (\sigma_j(s), \mathbf{v}(s))\,dW_j(s) = 0. \qquad (7.17)
$$

By the hypotheses of the theorem,

$$
\lim_{t\to\infty} \frac{1}{t}\sum_j \int_0^t |\sigma_j(s)|^2\,ds = L_1. \qquad (7.18)
$$

and

$$
\lim_{t\to\infty} \frac{1}{t}\int_0^t |\mathbf{f}(s)|^2\,ds = L_2. \qquad (7.19)
$$

Let $L := L_1 + \frac{L_2}{4\nu}$. Using (7.17), (7.18) and (7.19) in (7.16), one obtains

$$
\lim_{t\to\infty} \frac{\nu}{t}\int_0^t ||\mathbf{v}(s)||^2\,ds \leq L \qquad (7.20)
$$

Thus the inequality (7.14) in conjunction with the bound (7.20) completes the proof. □

Remark: Exponential stability doesn't imply that the process admits stationary measures. In fact, the above theorem pertains to time-inhomogeneous processes since $\mathbf{f}$ and σ are time dependent.

8. **Uniqueness of Invariant Measures**

Throughout this section, assume that $\mathbf{f}(t,x) = \mathbf{f}(x)$ and $\sigma(t,x) = \sigma(x)$ with $\mathbf{f} \in L^6(H)$ and $\sigma \in L^6(l_2(H))$. A pathwise unique strong solution $\mathbf{u}$ is guaranteed by theorem 2.2 with $\mathbf{u}(0) = \mathbf{u}_0 \in H$. The existence of a unique attractor follows from the previous section. The solution is a time-homogeneous strong Markov process so that the transition probability function, for all $0 \le s \le t$, Borel sets B in H and $\xi \in H$, satisfies

$$P_\xi(s;tB) = P(\mathbf{u}(t) \in B \,|\, \mathbf{u}(s) = \xi) = P(\mathbf{u}(t-s) \in B \,|\, \mathbf{u}_0 = \xi) = P_\xi\,(t-s, B) \tag{8.1}$$

The Navier-Stokes transition probability function has the Feller property. For, if ξ_n converges to ξ in H as $n \to \infty$, and if $\mathbf{u}_n$ and $\mathbf{u}$ refer to the solutions of (6.4) started at ξ_n and ξ respectively, then by theorem 7.4, $\mathbf{u}_n(t) \to \mathbf{u}(t)$ a.s. for any t. Therefore, $\phi(\mathbf{u}_n(t)) \to \phi(\mathbf{u}(t))$ a.s. by the continuity of ϕ. Thus, by the dominated convergence theorem, $E\phi(\mathbf{u}_n(t)) \to E\phi(\mathbf{u}(t))$.

Theorem 8.1. *Let $\sum_j |\sigma_j|^2 = L_1 < \infty$ and $|\mathbf{f}|^2 = L_2$. Let $L = L_1 + \dfrac{L_2}{4\nu}$. Under the hypotheses of the previous theorem, there exists a unique stationary measure, with support in V, for the solution $\mathbf{u}$ of the equation (6.4).*

Proof. The method for showing the existence of a stationary measure is by averaging:

It follows from (7.1) that

$$\frac{\nu}{t}\int_0^t E||\mathbf{u}(s)||^2\,ds \le \frac{E|\mathbf{u}_0|^2}{t} + \frac{|\mathbf{f}|^2}{\nu} + \sum_j |\sigma_j|^2 \le C \tag{8.2}$$

where C is an appropriate constant. Therefore

$$\lim_{N\to\infty} \sup_t \frac{1}{t}\int_0^t P\{||\mathbf{u}(s)|| > N\}\,ds = 0 \tag{8.3}$$

Let $\{t_n\}$ be any increasing sequence of positive numbers with $\lim_{n\to\infty} t_n = \infty$. Define probability measures μ_n as follows:

$$\mu_n(\Gamma) := \frac{1}{t_n}\int_0^{t_n} P_s(\xi,\Gamma)\,ds$$

for all $\Gamma \in \mathcal{B}(H)$. By (8.3), the sequence $\{\mu_n\}$ is tight in the space of probability measures on $(H, \mathcal{B}(H))$ equipped with the topology of weak convergence. By Prokhorov's theorem there exists a subsequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} \to \mu$.

Using this and the Feller property, if $\phi \in C_b(H)$,

$$\begin{aligned}\int_H S_t\phi(\eta)d\mu(\eta) &= \lim_{k\to\infty}\int_H S_t\phi(\eta)\,d\mu_{n_k}(\eta)\\ &= \lim_{k\to\infty}\frac{1}{t_{n_k}}\int_0^{t_{n_k}}\int_H S_t\phi(\eta)S_s(\xi,d\eta)\,ds\\ &= \lim_{k\to\infty}\int_H \phi(\eta)d\mu_{n_k}(\eta)\\ &= \int_H \phi(\eta)d\mu(\eta) \qquad (8.4)\end{aligned}$$

Thus μ has been shown to be a stationary measure.

2. By lower semi-continuity of the H-norm, or simply using the Skorohod representation theorem

$$\int_H |\xi|^2 d\mu(\xi) \le \lim\inf_{k\to\infty}\int_H |\xi|^2 d\mu_{n_k}(\xi)$$

so that the second moment for the measure μ exists.

Let $\mathbf{u}$ denote the solution of (6.4) started at $\mathbf{u}_0$ where the distribution of $\mathbf{u}_0$ is given by μ. By the estimate (8.2), it follows that $\mathbf{u}(s)$ is almost surely V-valued for almost all s. In particular, μ has support in V.

3. uniqueness: Suppose μ_1 and μ_2 are two probability measures on H that are stationary for the equation (6.4). To show that $\mu_1 = \mu_2$, it suffices to establish that

$$\int_H \phi(\xi)\,d\mu_1(\xi) = \int_H \phi(\xi)\,d\mu_2(\xi) \qquad (8.5)$$

for all $\phi \in C_b(H)$. Let $\mathbf{u}^\xi$ denote the solution of (6.4) started at $\xi \in H$ at time 0. Define

$$\mu_T^\xi(B) = \frac{1}{T}\int_0^T P_\xi(t,B)\,dt$$

for all $B \in \mathcal{B}(H)$. By stationarity,

$$\begin{aligned}&|\int_H \phi(\xi)\,d\mu_1(\xi) - \int_H \phi(\xi)\,d\mu_2(\xi)|\\ =\ & |\int_H\int_H \phi(\xi)d\mu_T^\lambda(\xi)\,d\mu_1(\lambda) - \int_H\int_H \phi(\eta)\,d\mu_T^\zeta(\eta)\,d\mu_2(\zeta)]|\\ =\ & |\frac{1}{T}\int_H\int_0^T E(\phi(\mathbf{u}^\lambda(t))\,dt\,d\mu_1(\lambda)) - \frac{1}{T}\int_H\int_0^T E(\phi(\mathbf{u}^\zeta(t))\,dt\,d\mu_2(\zeta))|\\ \le\ & \int_H\int_H\frac{1}{T}\int_0^T E|\phi(\mathbf{u}^\lambda(t)) - \phi(\mathbf{u}^\zeta(t))|\,dt\,d\mu_1(\lambda)d\mu_2(\zeta) \qquad (8.6)\end{aligned}$$

By Theorem 7.4 and continuity of ϕ, $|\phi(\mathbf{u}^\lambda(t)) - \phi(\mathbf{u}^\zeta(t))| \to 0$ a.s. as $t \to \infty$. Therefore,

$$\frac{1}{T}\int_0^T |\phi(\mathbf{u}^\lambda(t)) - \phi(\mathbf{u}^\zeta(t))|\, dt \to 0$$

as T tends to ∞. By the dominated convergence theorem, the bound (8.6) tends to 0 as $T \to \infty$. The proof is thus complete. □

Corollary 8.2. *Let μ denote the unique stationary measure for (6.4). If $\mathbf{u}^\eta$ denotes the solution of (6.4) started at η, then $E\frac{1}{t}\int_0^t \phi(\mathbf{u}^\eta(s))\, ds \to \int_H \phi(\xi)\, d\mu(\xi)$ for all $\phi \in C_b(H)$.*

Proof: For any fixed $\eta \in H$, let $\mu_t^\eta(B) = \frac{1}{t}\int_0^t P_\eta t(s, B) ds$ for all $B \in \mathcal{B}(H)$. From step 1 of the proof of theorem 8.1, the weak limit as of any weakly convergent subsequence of the measures $\{\mu_t^\eta\}$ indexed by t is a stationary measure for the stochastic differential equation (6.4). By the uniqueness of stationary measures, it follows that all weakly convergent subsequences of $\{\mu_t^\eta\}$ converge to the same limit μ. The proof is thus completed. □

Remark: Recall that the existence of a unique stationary measure guarantees that the measure is ergodic. We thus have a unique ergodic measure for the solution of the two dimensional Navier Stokes equation under the stated hypotheses.

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Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

E-mail address: sundar@math.lsu.edu