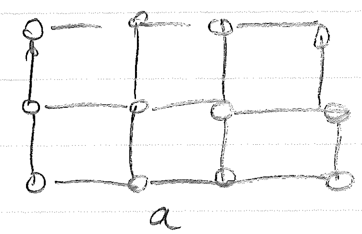
FIELD THEORIES AND CRITICAL POINTS

Typical problem is Stat mech



At a generic temperature the correlation length

$$\xi \sim a \quad (\text{upto } O(1))$$

In field theory - we typically have fields in CONTINUOUS space-time
More significantly
$\rightarrow$ Low energy laws do not depend sensitively on the physics at short distances
Usually do this by introducing a CUTOFF
However the PHYSICAL PARAMETERS in the theory are not of the cutoff scale

Whole idea is to be able to write down dynamics with a minimal reference to the high energy scale

The 'older' way of thinking about this $a \to 0$ or $\Lambda \to \infty$

What this really means is

$$ma \to 0 \qquad \xi/a \to \infty$$

In SM this happens at special points — parameters tuned so that

$$\xi = \infty$$

— FIELD THEORIES $\approx$ CRITICAL PHENOMENA

Q: How does one obtain a low energy theory from a theory with a cutoff

— RENORMALIZATION GROUP

RG TRANSFORMATIONS

At a critical point

$$\langle \phi(x)\,\phi(0)\rangle = \frac{c}{|x|^{d-2+\eta}}$$

Lattice spacing a $\qquad x = \frac{x_{phy}}{a}$

If we are interested in physics at scales much larger than lattice spacing

— Find a new theory with lattice spacing ba $\quad b > a$ which gives same answer

Block spins $\quad \tilde{\phi}(\tilde{x}) = \frac{1}{b^d} \sum_{x \in y} \phi(x)$

Then

$$\langle \tilde{\phi}(y)\,\tilde{\phi}(0)\rangle = \frac{c}{|\tilde{x}|^{d-2+\eta}}$$

Since new lattice spacing is ba

$$x_{phy} = \tilde{x}a = \left(\frac{\tilde{x}}{b}\right) ba.$$

$$\equiv (y)\, ba$$

$$\langle \tilde{\phi}(y)\,\tilde{\phi}(0)\rangle = b^{2-d-\eta}\,\frac{c}{|y|^{d-2+\eta}}$$

Now redefine

$$\phi(y) = b^{\frac{d-2+\eta}{2}}\,\tilde{\phi}(y)$$

$$\langle \phi(y)\,\phi(0)\rangle = \frac{c}{|y|^{d-2+\eta}}$$

Thus this quantity is invariant under a scale transformation

$$\left.\begin{array}{ll} (1) & x \to y/b \\ (2) & \tilde{\phi} \to b^{\frac{d-2+\eta}{2}}\,\phi \end{array}\right] \text{RG}$$

If $\xi \neq \infty$ these transformations will not lead to the same answer — but the answer of a different theory

TRANSFER MATRIX & Q MECHANICS

In classical statistical mechanics we compute the partition function.
Example: In LG theory we have

$$E = \frac{1}{2}\int d^dx \left[(\nabla\phi)^2 + r_0\phi^2 + u\phi^4 \right]$$

and we want

$$Z = \int \mathcal{D}\phi(x)\, e^{-\beta E}$$

Of the d dimensions — split

$$d = \underbrace{(d-1)}_{\uparrow\, \vec{x}} + \underbrace{1}_{\uparrow\, \tau}$$

$$E = \frac{1}{2}\int_0^T d\tau\, d^dx \left[(\partial_\tau\phi)^2 + (\nabla\phi)^2 + V(\phi) \right]$$

Now discretize τ variable.

$$i = 0 \quad 1 \quad 2 \quad 3 \quad \ldots\ldots \quad N$$

$$N\epsilon = T$$

$$\int d\tau (\partial_\tau \phi)^2 = \sum_{i=0}^{N-1} \epsilon \left(\frac{\phi(\vec{x}, i+1) - \phi(\vec{x}, i)}{\epsilon} \right)^2$$

$$\int d\tau (\nabla\phi)^2 = \sum_i \epsilon (\nabla\phi)^2$$

Now write

$$e^{-\frac{\beta}{2\epsilon}(\phi_{i+1}-\phi_i)^2}$$

$$= \int dp_i \exp\left[-\frac{\epsilon p_i^2}{2\beta} - i\, p_i(\phi_{i+1}-\phi_i)\right]$$

Thus write the partition function as

$$Z = \int \prod_{\substack{i=0 \\ x}}^{N} d\phi(\vec{x},i)\; e^{-\beta E}$$

$$= \prod_{\substack{i=0 \\ \vec{x}}}^{N} \int d\phi(\vec{x},i)\; e^{-\frac{\beta}{2\epsilon}(\phi_{i+1}-\phi_i)^2 - \epsilon F(\phi)}$$

$$= \prod_{\substack{i=0 \\ \vec{x}}}^{N} \int d\phi(\vec{x},i)\, dp(\vec{x},i)\; e^{-\frac{\epsilon p_i^2}{2\beta} - i\, p_i(\phi_{i+1}-\phi_i) - \epsilon F}$$

Now introduce a Hilbert space
Consider operators $\hat{\phi}(\vec{x})$, $\hat{P}(\vec{x})$.

$$\hat{\phi}(\vec{x})\,|\phi(x)\rangle = \phi(x)\,|\phi(x)\rangle$$

etc.

Use
$$e^{-iP(\vec{x})\phi(\vec{x})} = \langle \phi(\vec{x}) | P(\vec{x}) \rangle$$
$\Rightarrow$
$$e^{-\frac{\epsilon P_i^2}{2\beta}} e^{-iP_i(\phi_{i+1}-\phi_i)} e^{-\epsilon\beta V(\phi)}$$
$$= e^{-\frac{\epsilon P_i^2}{2\beta}} \langle \phi_{i+1} | P_i \rangle \langle P_i | \phi_i \rangle e^{-\epsilon\beta V}$$
$$= \langle \phi_{i+1} | e^{-\epsilon\left(\frac{\hat{P}^2}{2\beta}\right)} | P_i \rangle \langle P_i | e^{-\epsilon\beta V} | \phi_i \rangle$$

Hence
$$\int dP_i \, e^{-\frac{\epsilon P_i^2}{2\beta} - iP_i(\phi_{i+1}-\phi_i) - \epsilon V}$$
$$= \langle \phi_{i+1} | e^{-\epsilon\left[\frac{\hat{P}^2}{2\beta} + \beta V(\hat{\phi})\right]} | \phi_i \rangle.$$

Finally integrating over ϕ_i gives
$$Z = \mathrm{Tr}\left(\left(e^{-\epsilon \hat{H}}\right)^N\right)$$
$$Z = \mathrm{Tr}\, e^{-T\hat{H}}$$
where
$$H = \frac{\hat{P}^2}{2\beta} + \beta V(\hat{\phi})$$
Fields
$$H = \int d\vec{x} \left(\frac{\hat{P}(x)^2}{2\beta} + \beta V(\hat{\phi}(x)) \right)$$

Had a system
- Classical
- Temperature β^{-1}
- On a circle of size T

This is equivalent to
- Quantum mechanics in Euclidean time
- At finite temperature T^{-1}
- Coupling β

Extends to

(d+1) dim classical
$\equiv$ d dim quantum

Rules of Functional integration

$$\phi(x) = \frac{1}{V} \sum_n e^{-i\vec{k}\cdot\vec{x}} \phi(\vec{k})$$

$$K^\mu = \frac{2\pi n^\mu}{L} \qquad |k^\mu| < \pi/\epsilon$$

ϵ : lattice

$V = L^4$

$$\phi^*(\vec{k}) = \phi(-k)$$

Choose independent variables

$\mathrm{Re}\ \phi(\vec{k}) \quad k_0 > 0$

$\mathrm{Im}\ \phi(\vec{k}) \quad k_0 > 0$

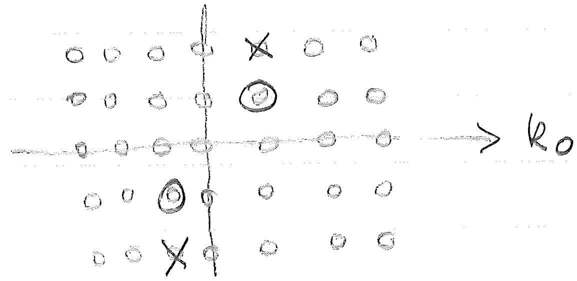
Change of variables has trivial jacobian

$$\mathcal{D}\phi(x) = \prod_{k_0 > 0} d(\mathrm{Re}\,\phi)\, d(\mathrm{Im}\,\phi)$$

Reason

$$\phi(k) = \phi_1(k) + i\phi_2(k)$$
$$\phi^*(+k) = \phi_1(k) - i\phi_2(k)$$
$$= \phi_1(-k) + i\phi_2(-k)$$

$$\Rightarrow \quad \phi_1(k) = \phi_1(-k)$$
$$\phi_2(k) = -\phi_2(-k)$$



$$\int dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 \right]$$

$$= \frac{1}{V} \sum_n \frac{1}{2}(m^2 + \vec{k}^2) \, |\phi(\vec{k})|^2$$

$$= \frac{1}{V} \sum_{\substack{n \\ k_0>0}} (m^2 + \vec{k}^2) \left[\phi_1^2(k) + \phi_2^2(k) \right]$$

Then the integral factors out and we can perform a Gaussian integral

$$\int \prod d\xi_k \, \exp\left[- \xi_i B_{ij} \xi_j \right]$$

$\xi_i = O_{ij} x_j$ $\qquad$ $O^T O = 1 \Leftarrow$ orthogonal matrix which diagonalizes

$$\int \prod dx_k \, e^{-\sum b_i x_i^2} \qquad b_i : \text{eigenvalues}$$

$$= \prod_i \left(\int dx \, e^{-b x_i^2} \right) = \prod_i \sqrt{\frac{\pi}{b_i}}$$

$$= (\text{const}) \, (\det B)^{-1/2}$$

Thus

$$Z = \prod_{k_0>0} \sqrt{\frac{\pi V}{m^2+k^2}} \sqrt{\frac{\pi V}{m^2+k^2}} = \prod_{\substack{\text{all} \\ k}} \sqrt{\frac{\pi V}{m^2+k^2}}$$

Generally write

$$Z = \left[\det(-\partial^2 + m^2)\right]^{-1/2}$$

Correlation functions:

$$Z[J] = \int \mathcal{D}\phi \, e^{-S_0(\phi) - \int J(x)\phi(x) d^dx}$$

This is like

$$\int \prod d\xi_k \, e^{-[\xi_i B_{ij} \xi_j + J_i \xi_i]}$$

$$\xi_i B_{ij} \xi_j = O_{ik} x_k B_{ij} O_{jl} x_l$$
$$= (O^T_{ki} B_{ij} O_{jl}) x_k x_l = \sum b_k x_k^2$$

Write

$$\xi_i B_{ij} \xi_j + J_i \xi_i$$
$$\sum b_k x_k^2 + J_i O_{ik} x_k$$

$$= \sum (b_k x_k^2 + n_k x_k) \qquad n_k = O^T_{ki} J_i$$
$$= \sum b_k (x_k^2 + \frac{n_k}{b_k} x_k)$$
$$= \sum b_k (x_k + \frac{n_k}{2b_k})^2 - \sum_k \frac{n_k^2}{2b_k}$$

Write $\sum_k \frac{n_k^2}{2b_k} = \sum n^T D^{-1} n$

where

$$D^{-1} = (O^T B O)^{-1} = O^{-1} B^{-1} (O^T)^{-1}$$
$$= O^T B^{-1} O$$

$$n^T D^{-1} n = J^T O O^T B^{-1} O O^T J = J^T B^{-1} J$$

Thus

$$Z[J] = \left[\det(-\partial^2+m^2)\right]^{-1/2} \exp\left[\frac{1}{2}\int dx\,dy\, J(x)\, G(x-y)\, J(y)\right]$$

where

$$(-\partial^2+m^2)\, G(x,y) = \delta(x-y)$$

RG: FREE THEORY

$$S = \int_0^1 d^dq \; f(q^2)\, \varphi(q)\varphi(-q) \quad \begin{pmatrix}\text{cutoff}\\ \text{units}\end{pmatrix}$$

$$f(q^2) = \tilde{c}_0 + \tilde{c}_1 q^2 + \tilde{c}_2 q^4$$
$$= c_0 + c_1 q^2 + \underline{c_2} q^4 + \cdots$$

Integrate out high momentum modes $b > 1$

$$S' = \int_0^{1/b} d^dq \left[c_0 + c_1 q^2 + \frac{c_2}{\Lambda^2} q^4 + \cdots \right] \varphi(q)\varphi(-q)$$

Now define new momenta

$$p = bq$$

$$S' = b^{-d} \int_0^1 d^dp \left[c_0 + c_1 \frac{p^2}{b^2} + c_2 \frac{p^4}{b^4} + \cdots \right] \varphi(p/b)\,\varphi(-p/b)$$

Define new field

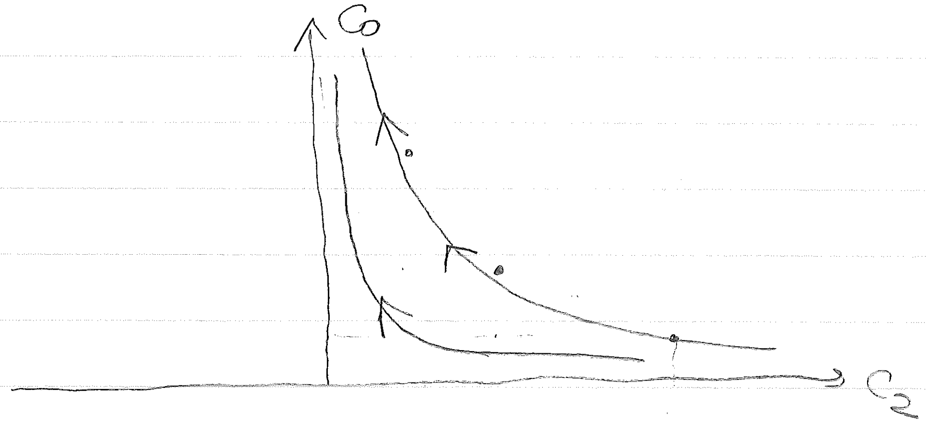
$$\varphi(p/b) = b^{(d+2)/2} \psi(p)$$

$$S' = b^2 \int_0^\Lambda d^dp \left[c_0 + c_1 \frac{p^2}{b^2} + \frac{c_2 p^4}{b^4} + \cdots \right] \psi(p)\psi(-p)$$

$$\Rightarrow \quad c_0' = b^2 c_0 \simeq (1 + 2\delta)\, c_0$$
$$c_1' = c_1 = c_1$$
$$c_2' = b^{-2} c_2 = (1 - 2\delta)\, c_2$$

and so on

Fixed point $c_0 = c_2 = c_4 = \cdots = 0$



c_0 : Relevant coupling
c_2, c_4 : Irrelevant coupling
c_1 : Marginal coupling

$$\langle \varphi(p)\varphi(-p)\rangle = \frac{1}{c_0\Lambda^2 + c_1 q^2 + \frac{c_2}{\Lambda^2} q^4}$$

When $c_0 = 0$ pole at $q = 0$
This means position space
correlator is long range

$$\int d^dq \, \frac{e^{iq\cdot x}}{c_1 q^2 + \tilde{c}_2 q^4} \sim \frac{1}{x^{d-2}} \int \frac{d^dy \, e^{iy}}{c_1 y^2 + \tilde{c}_2 y^4/x^2}$$

Thus for $|x| > \tilde{c}_2$

Operator Dimensions

Under the scaling

$$\varphi(p) = b^{-\frac{d+2}{2}} \varphi(p/b)$$

$$\mathrm{Dim}\,[\varphi(p)] = -\frac{d+2}{2}$$

$$\phi(x) = \int d^d p \, e^{ipx} \phi(p)$$

$$\mathrm{Dim}\,[\phi(x)] = d - \frac{d+2}{2} = \frac{d-2}{2}$$

The operator $\quad O_n = q^{2n} \phi(q) \phi(-q)$

$$\sim \phi \, \partial^{2n} \phi$$

$$d_n = d - 2 + 2n = d + 2(n-1)$$

Def

$$\boxed{y_n = -2(n-1)} \quad d_n = d - y_n$$

Corresponding Couplings

$$c_n' = b^{y_n} c_n$$

RG transf :

$$x \to x/b \qquad O_n \to b^{-d_n} O_n$$

$$\langle O_n(x) \, O_n(0) \rangle \equiv G_n(x)$$

$$G_n(x, c_n) = b^{-2d_n} G\left(\frac{x}{b}, c_n b^{y_n}\right)$$

Choose $b = x$

$$G(x, c_n) = \frac{1}{x^{2d_n}} G(1, c_n x^{y_n})$$

Only y_n which is +ve is $n = 0, 1$
Thus for $x \to \infty$

$$G(x, c_n) = \frac{1}{x^{2d_n}} G(1, c_0 x)$$

In particular on critical surface $c_0 = 0$

$$G(x, c_n) = \frac{1}{x^{2d_n}} \quad \text{scale invariant}$$

Note: All the irrelevant operators go away

GENERAL RG-LOGY

$$S = \sum g_i O_i$$

$\{g_i\} \to$ Couplings (all chosen to be dimensionless)

Under RG transformation $a \to ba$

$$g_i \to g_i'(g_j)$$

Fixed point

$$g_i' = g_i = g_i^*$$

$$\delta g_i = g_i - g_i^*$$

$$g_i' = g_i^* + \delta g_i'$$
$$g_i = g_i^* + \delta g_i$$

$\Rightarrow$

$$\delta g_i' = L_i^j \, \delta g_j$$

$L_i^j(b, g_i^*)$

Eigenvalues of form $b^{y_i} = \Lambda_i$

Since $[\Lambda_i(b)]^n = \Lambda_i(b^n)$

Eigenvectors δh_i

$\Rightarrow$

$$\boxed{\begin{aligned} \delta h_i' &= b^{y_i} \delta h_i \\ &= (1+\delta)^{y_i} \delta h_i \end{aligned}}$$

Partition fn invariant

$\Rightarrow$ Free energy density becomes larger (

$$f(g_i) = b^{-d} f(g_i')$$

Near fixed point

$$f(\delta h_i) = b^{-d} f(b^{y_i} \delta h_i)$$

Suppose $i = 1 \ldots n$ are relevant couplings. Choose one of them δh_K

$$b^{y_K} \delta h_K = 1$$

$$f(\delta h_i) = (\delta h_K)^{\frac{d}{y_K}} f(\delta h_i (\delta h_K)^{-y_i/y_K})$$

As $\delta h_K \to 0$ only i with $y_i > 0$ contribute

Terms with $y_i < 0$ go away
— IRRELEVANT

The correlation length

$$b = (\delta h_k)^{-\frac{1}{y_k}}$$

$$\xi(g_i) = b\,\xi(g_i') \qquad b^{y_k}\delta h_k = 1$$

Thus

$$\xi(\delta h_i) = (\delta h_k)^{-\frac{1}{y_k}}\,\xi\left(\delta h_i (\delta h_k)^{-y_i/y_k}\right)$$

<u>Usual bi-critical systems</u>

1 relevant operator (e.g. T)

$$f(T) = (T - T_c)^{d/y_T}$$

EQUATIONS FOR GREEN'S FUNCTIONS

$$G_a(x, g_i) = \langle O_a(x)\, O_a(0) \rangle$$

$$G_a(x, g_i) = Z_a(g, b)\, G_a(x/b, g_i')$$

$b = 1 + \varepsilon$

$$G_a(x, g_i) = G_a(x, g_i) + \varepsilon\left[-x\frac{\partial}{\partial x}G_a + \beta_i \frac{\partial}{\partial g_i} G_a + \frac{\partial Z_a}{\partial \varepsilon} G_a\right]$$

$$\beta^i \equiv \frac{\partial g^i}{\partial \varepsilon}$$

Def $\quad \lambda_a \equiv -\frac{\partial Z_a}{\partial \varepsilon}$

$$\boxed{\left[x\frac{\partial}{\partial x} - \beta^i\frac{\partial}{\partial g_i} + \lambda_a\right]G_a = 0}$$

Clearly at fixed point $\beta^i = 0$

$$\left[x\frac{\partial}{\partial x} + \lambda_a(g^*)\right]G_a = 0$$

$$\boxed{G_a(x) = \frac{1}{|x|^{\lambda_a^*}}}$$

<u>Physical distances & Continuum Green's function</u>

Physical distance $z = xa$.

Continuum Green's fn

$$\tilde{G}_a(z,g) = a^{-2d_a}G_a(x,g)$$

Clearly

$$\left[z\partial_z - \beta^i\frac{\partial}{\partial g_i} + \lambda_a\right]\tilde{G}_a = 0$$

Since $G_a(x,g) = G_a\left(\frac{z}{a},g\right)$

$$z\partial_z G_a = -a\partial_a G_a$$

$$\Rightarrow \quad a^{2d_a} z\partial_z \tilde{G}_a = -a\partial_a\left(a^{+2d_a}\tilde{G}_a\right)$$

$$\Rightarrow \quad z\partial_z \tilde{G}_a = -\left[2d_a + a\partial_a \tilde{G}_a\right]$$

$$\Rightarrow$$

$$-a\partial_a \tilde{G}_a - 2d_a \tilde{G}_a - \beta^i \partial_i \tilde{G}_a + \lambda_a \tilde{G}_a = 0$$

$$\left[a\frac{\partial}{\partial a} + \beta^i \frac{\partial}{\partial g_i} - (\lambda_a - 2d_a)\right] \tilde{G}_a = 0$$

$$\gamma_a = \lambda_a - 2d_a \quad :: \ \text{Anomalous dimensions}$$

ε - EXPANSION

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2} r \phi^2 + \frac{1}{4} u \phi^4$$

$$\phi = \varphi_\ell + \varphi_s$$

$$\varphi_\ell = \begin{cases} \phi(k) & 0 < |k| < \Lambda/b \\ 0 & \Lambda/b < |k| < \Lambda \end{cases}$$

$$\varphi_s = \begin{cases} 0 & 0 < |k| < \Lambda/b \\ \phi(k) & \Lambda/b < |k| < \Lambda \end{cases}$$

$$S = S_0[\phi_\ell] + S_0[\varphi_s] + S_{int}[\varphi_s, \varphi_\ell]$$

$$S_0 = \frac{1}{2}\int \frac{d\vec{k}}{(2\pi)^d} \varphi(k)(r+k^2)\varphi(k)$$

$$S_{int} = \frac{u}{4}\int \Pi \frac{dk_i}{(2\pi)^d}(2\pi)^d \delta(\Sigma k_i) \prod_{i=1}^{4} \phi(k_i)$$

Rewrite

$$Z = \int \mathcal{D}\varphi_\ell \, e^{-S_{eff}(\varphi_\ell)}$$

where

$$e^{-S_{eff}(\varphi_\ell)} = \int \mathcal{D}\varphi_s \, e^{-(S_0[\varphi_s] + S_{int})}.$$

Thus

$$S_{eff}(\varphi_\ell) = S_0(\varphi_\ell) - \ln \langle \exp[-S_{int}] \rangle + \ln Z_s^0.$$

where

$$Z_s^0 = \int \mathcal{D}\varphi_s \, e^{-S_0(\varphi_s)}$$

Consider evaluating this in a perturbation theory

$$-\ln \langle \exp[-S_{int}] \rangle_{short}$$
$$\simeq \langle S_{int} \rangle - \frac{1}{2} \left\{ \langle S_{int}^2 [\varphi_\ell + \varphi_s] \rangle - \langle S_{int} \rangle^2 \right\}$$
$$+ \cdots$$

To linear order

$$\langle S_{int} \rangle = \ln \int \prod dk_i \left[\frac{1}{6} \varphi_s^3 \varphi_\ell + \frac{1}{6} \varphi_\ell^3 \varphi_s + \frac{1}{4} \varphi_s^2 \varphi_\ell^2 + \frac{1}{4!} \varphi \right]$$

$$\langle S_{int} \rangle = \frac{3}{2} u \int dk_1 dk_2 dk_3 dk_4 \, \delta(k_1+k_2+k_3+k_4) \, \varphi_s(k_1) \varphi_s(k_2) \varphi_\ell(k_3) \varphi_\ell(k_4)$$

$$= \frac{3}{2} u \int dk_1 dk_2 dk_3 dk_4 \, \delta(k_1+k_2+k_3+k_4) \, \varphi_\ell(k_3) \varphi_\ell(k_4) \langle \varphi_s(k_1) \varphi_s(k_2) \rangle$$

$$= \frac{3}{2} u \int \prod dk_i \, \delta(\Sigma k_i) \, \delta(k_1+k_2) \, \varphi_\ell(k_3) \varphi_\ell(k_4) \frac{1}{r+k_1^2}$$

$$= \frac{3}{2} u \int dk \, \varphi_\ell(k) \varphi_\ell(-k) \int d\vec{q} \, \frac{1}{r+q^2}$$

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{r+q^2} = \frac{\Omega_{d-1}}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} dq\, q^{d-1} \frac{1}{r+q^2}$$

When $b = 1 + \delta l$ we can approximate

$$\int_{\Lambda/b}^{\Lambda} dq\, q^{d-1} \frac{1}{r+q^2} = \frac{K_d \Lambda^d}{r+\Lambda^2} \delta l + O(\delta l^2).$$

Thus the coefficient of φ_l^2 is

$$\int_0^{\Lambda/b} \frac{d\vec{k}}{(2\pi)^2} \left[r + \frac{3u K_d \Lambda^d}{r^2+\Lambda^2} \delta l \right] \varphi_l(k) \varphi_l(-k).$$

~~Rescale with $b \to \Lambda$~~

Integration variable $k \to (q = b k)$

New field $\varphi_l(k) = b^{1+d/2} \varphi_l(q)$

Then

$$b^{-d}\, b^{2+d} \left(r + \frac{3u K_d \Lambda^d}{r+\Lambda^2} \delta l \right) \int_0^{\Lambda} dq\, \varphi_l(q) \varphi_l(-q)$$

Thus

$$r(l+\delta l) = (1+\delta l)^2 \left(r + \frac{3u K_d \Lambda^d}{r+\Lambda^2} \delta l \right)$$

$$r(l+\delta l) = r(l) + \left(2r + \frac{3u K_d \Lambda^d}{r+\Lambda^2} \right) \delta l$$

Define $r = r_0 \Lambda^2$

$$r_0(l+\delta l) = r_0(l) + 2r_0 + \frac{3u K_d \Lambda^{d-4}}{1 + r/\Lambda^2} \delta l$$

Now def $u = u_0 \Lambda^{d-4}$

$$\beta(r_0) = \frac{dr_0}{dl} = 2r_0 + \frac{3u_0 K_d}{1+r_0}$$

$$\boxed{\beta(r_0) = 2r_0 + \frac{3K_d u_0}{1+r_0}}$$

Now consider terms which are $O(u^2)$

$$\langle S_{int}^2 \rangle = 36u^2 \int \Pi dk_i \, \Pi dq_i \, \delta(\Sigma k_i)\delta(\Sigma q_i)$$

$$\Big[\tfrac{1}{6}\varphi_s(k_1)\varphi_s(k_2)\varphi_s(k_3)\varphi_\ell(k_4) \quad ①$$

$$+\tfrac{1}{6}\varphi_\ell(k_1)\varphi_\ell(k_2)\varphi_\ell(k_3)\varphi_s(k_4) \quad ②$$

$$+\tfrac{1}{4}\varphi_s(k_1)\varphi_s(k_2)\varphi_\ell(k_3)\varphi_\ell(k_4) \quad ③$$

$$+\tfrac{1}{4!}\varphi_s(k_1)\varphi_s(k_2)\varphi_s(k_3)\varphi_s(k_4)\Big] \quad ④$$

$$\Big[\tfrac{1}{6}\varphi_s(q_1)\varphi_s(q_2)\varphi_s(q_3)\varphi_\ell(q_4) \quad ①'$$

$$+\tfrac{1}{6}\varphi_\ell(q_1)\varphi_\ell(q_2)\varphi_\ell(q_3)\varphi_s(q_4) \quad ②'$$

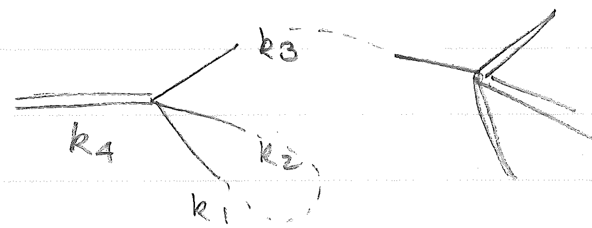
$$+\tfrac{1}{4}\varphi_s(q_1)\varphi_s(q_2)\varphi_\ell(q_3)\varphi_\ell(q_4) \quad ③'$$

$$+\tfrac{1}{4!}\varphi_s(q_1)\varphi_s(q_2)\varphi_s(q_3)\varphi_s(q_4)\Big] \quad ④'$$

Subtraction of $\langle S_{int} \rangle^2 \Rightarrow$ only contractions between k's and q's

① – ①′ → gives $O(u^2)$ correction to φ_ℓ^2

① – ②′



① – ③′ $= 0$ (odd # of φ_s)

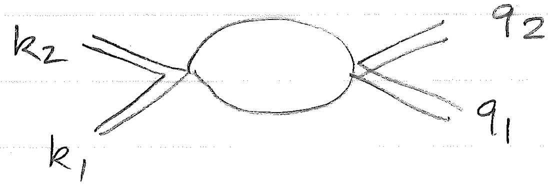
① – ④′ $= 0$ (")

② – ②′ ⟶ gives φ_ℓ^6

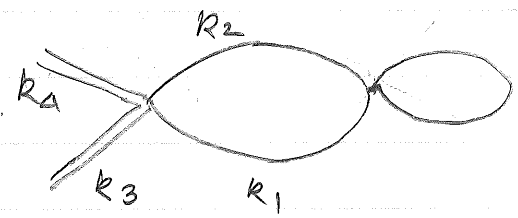
② – ③′ $= 0$ (odd # of φ_s)

② – ④′ $= 0$ (odd # of φ_s)

③ – ③′



③ – ④′



The main contribution comes as

$$\varphi_\ell(k_1)\,\varphi_\ell(k_2)\,\varphi_\ell(q_1)\,\varphi_\ell(q_2)\,\delta(k_1+k_2+k_3+k_4)$$
$$\delta(q_1+q_2+q_3+q_4)\,\delta(k_3+q_3)\,\delta(k_4+q_4)$$
$$\frac{1}{k_3^2\,k_4^2}$$

Keep integral over k_1, k_2, q_1, q_2 intact.
Remaining $k_3\ k_4\ q_3\ q_4$

2 δ fns $\Rightarrow$ $k_3\ k_4$

$\Rightarrow$

$$\delta(k_1+k_2+k_3+k_4)\,\delta(q_1+q_2-k_3-k_4)$$
$$=\delta(k_1+k_2+q_1+q_2)\,\delta(q_1+q_2-k_3-k_4)$$

Thus get

$$\int d\vec{k}_3\,\frac{1}{(k_3^2+r)\left((Q-k_3)^2+r\right)}$$

where $Q = q_1+q_2 = -(p_1+p_2)$

Since integral over $(\Lambda/b, \Lambda)$ and
$b = 1+\delta\ell$
Result for $Q \ll \Lambda$.

$$-\ \delta\ell\ \frac{9K_d\Lambda^d u^2}{(r+\Lambda^2)^2}$$

Now write $\varphi_\ell = b^{1+d/2} \varphi_\ell(q)$

$$k = q/b$$

$$u = u_0 \Lambda^{4-d}$$

Then the term in S_{eff} is

$$b^{-3d} \; b^{4+2d} \left(u - \frac{9u^2 K_d \Lambda^d}{(r+\Lambda^2)^2} \delta\ell \right) \int \varphi_\ell^4$$

3 remaining momentum integrals

fields.

$$b = 1 + \delta\ell$$

$$u' = (1+\delta\ell)^{4-d} \left(u - \frac{9u^2 K_d \Lambda^d}{(r+\Lambda^2)^2} \delta\ell \right)$$

$$= u + \delta\ell \left\{ (4-d)u - \frac{9u^2 K_d \Lambda^d}{(r+\Lambda^2)^2} \right\}$$

$$u_0' \Lambda^{4-d} = u_0 \Lambda^{4-d} + \delta\ell \left\{ (4-d) u_0 \Lambda^{4-d} - \frac{9 u_0^2 \Lambda^{8-2d} K_d \Lambda^d}{\Lambda^4 (r_0+1)^2} \right\}$$

$$\boxed{\frac{du_0}{d\ell} = (4-d)u_0 - \frac{9u_0^2 K_d}{(1+r_0)^2}}$$

Fixed points & exponents

Def $\quad \varepsilon = 4-d$

$$-\beta(r_0) = 2r_0 + \frac{3K_d u_0}{1+r_0}$$

$$-\beta(u_0) = \varepsilon u_0 - \frac{9u_0^2 K_d}{(1+r_0)^2}$$

Take both u_0, r_0 small & same order

$$\beta(r_0) = 2r_0 + 3K_d u_0$$

$$\beta(u_0) = \varepsilon u_0 - 9u_0^2 K_d$$

F.P $\quad$ Gaussian $\quad u_0 = r_0 = 0$

Wilson-Fisher

$$u_0^* = \frac{\varepsilon}{9K_d}$$

$$r_0^* = -\frac{3K_d\varepsilon}{2\cdot 9K_d} = -\frac{\varepsilon}{6}$$

· Clearly r_0, u_0 are both relevant at Gaussian.

$$\frac{dr_0}{dl} = +2r_0 + 3u_0 K_d$$

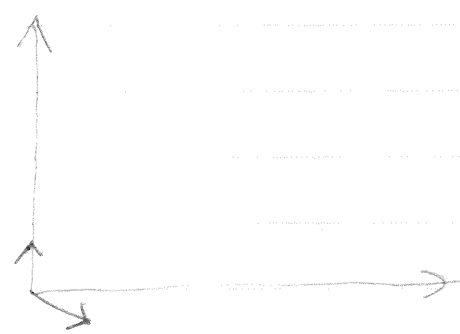
$$\frac{du_0}{dl} = \varepsilon u_0$$

Thus matrix

$$\begin{pmatrix} 2 & 3K_d \\ 0 & \varepsilon \end{pmatrix}$$

eigenvalues $(2, \varepsilon)$

Eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 3K_d \\ -\frac{2-\varepsilon}{3K_d} \end{pmatrix}$



· Around WF

$$r_0 = r_0^* + \delta r_0$$
$$u_0 = u_0^* + \delta u_0$$

$$-\beta(\delta r_0) = 2\,\delta r_0 + 3K_d\,\delta u_0$$

$$-\beta(\delta u_0) = \varepsilon(u_0^* + \delta u_0) - 9K_d(u_0^* + \delta u)^2$$

$$= (\varepsilon - 18K_d u_0^*)\,\delta u_0$$

$$= \left(\varepsilon - \frac{18K_d\,\varepsilon}{9K_d}\right)\delta u_0$$

$$= -\varepsilon\,\delta u_0$$

Thus

$$y_1 = 2$$

$$y_2 = -\varepsilon \quad \Leftarrow \text{Irrelevant}$$

O(N) NON-LINEAR SIGMA MODEL AT LARGE N

$$\mathcal{L} = \frac{1}{2g^2}\int d^dx\, (\partial\phi^i)^2$$

with condition

$$\phi^i\phi^i = 1$$

Rescale $\quad \phi^i \to \varphi^i = \frac{1}{g}\phi^i$

$$\mathcal{L} = \frac{1}{2}\int d^dx\, (\partial\phi^i)^2$$

$$\vec{\phi}^2 = \frac{1}{g^2}$$

$$Z = \int \mathcal{D}\phi^i \exp\left[-\frac{1}{2}\int d^dx\, (\partial\phi^i)^2\right] \delta\left[\vec{\phi}^2 - \frac{1}{g^2}\right]$$

- One way to deal with constraint is to solve it explicitly for say ϕ_1

$$\phi_1 = \sqrt{\frac{1}{g^2} - \phi_i\phi_i} \qquad (i = 2, \dots N-1)$$

$$\partial\phi_1 = -\frac{\phi_i\partial\phi_i}{\sqrt{\frac{1}{g^2} - \phi^2}}$$

$$\mathcal{L} = \frac{1}{2}\frac{(\phi_i\partial\phi_i)(\phi_j\partial\phi_j)}{\frac{1}{g^2} - \phi^2} + \frac{1}{2}(\partial\phi_i)^2$$

- Another way: Do an angular decomposition:

$$\begin{aligned}
\phi_1 &= \cos\theta_1 \\
\phi_2 &= \sin\theta_1 \cos\theta_2 \\
\phi_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\cdots\cdots \\
\phi_N &= \sin\theta_1 \ldots \sin\theta_{N-1}
\end{aligned}$$

Then

$$\mathcal{L} = g_{\mu\nu}(\theta_\mu)\, \partial\theta^\mu \partial\theta^\nu$$

In this form the model can be generalized to any manifold - particularly any group manifold.

- To find the phase structure it is better to keep the delta function, but write

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \sigma(x)\left(\phi^2 - \frac{1}{g^2}\right)$$

$$Z = \int \mathcal{D}\sigma(x)\, \mathcal{D}\phi(x)\, e^{-\int \mathcal{L}\, d^dx}$$

Integrate out ϕ

$$Z = \int \mathcal{D}\sigma(x) \exp\left[-\frac{N}{2} \mathrm{Tr} \log(-\partial^2 + \sigma(x)) + \int \frac{\sigma(x)}{g^2}\right]$$

Large-N limit

$$g \to 0 \quad N \to \infty \qquad g^2 N = \lambda = \text{fixed}$$

$$S[\sigma] = N\left\{ \frac{1}{2} \mathrm{Tr} \log(-\partial^2 + \sigma(x)) - \int \frac{\sigma(x)}{\lambda} \right\}$$

As $N \to \infty$ this can be evaluated by saddle point

$$\frac{1}{2} \mathrm{Tr} \frac{1}{-\partial^2 + \sigma(x)} = \frac{1}{\lambda}$$

- Since the underlying theory is translation invariant the saddle must be constant

$\Rightarrow$ In momentum space

$$\frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + \sigma_0} = \frac{1}{\lambda}$$

Integral $$\frac{1}{2} \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\Lambda \frac{dk\, k^{d-1}}{k^2 + \sigma_0} = \frac{1}{\lambda}$$

Define λ_c by

$$\frac{1}{2}\frac{\Omega_{d-1}}{(2\pi)^d}\int_0^\Lambda \frac{dk\, k^{d-1}}{k^2} = \frac{1}{\lambda_c},$$

Then

$$\frac{1}{\lambda_c} - \frac{1}{\lambda} = \frac{1}{2}\frac{\Omega_{d-1}}{(2\pi)^d}\int_0^\Lambda \frac{dk\, k^{d-1}}{k^2(k^2+\sigma_0)}$$

<u>$d=2$</u>:

$$\frac{1}{2}\frac{2\pi}{(2\pi)^2}\int_0^\Lambda \frac{dk\, k}{k^2+\sigma_0} = \frac{1}{\lambda}$$

$$\frac{1}{4\pi}\,\frac{1}{2}\left[\log(k^2+\sigma_0)\right]_0^\Lambda = \frac{1}{\lambda}$$

$$\frac{1}{8\pi}\log\left(\frac{\Lambda^2+\sigma_0}{\sigma}\right) = \frac{1}{\lambda}$$

Now Consider limit $\Lambda \gg \sqrt{\sigma}$

$$\frac{1}{4\pi}\log\left(\frac{\Lambda}{\sqrt{\sigma_0}}\right) = \frac{1}{\lambda}$$

Thus

$$\sqrt{\sigma_0} = \Lambda \exp\left[-\frac{4\pi}{\lambda}\right]$$

What is σ_0?

Suppose we want to calculate correlators of ϕ^i

$$Z[J_i] \equiv \int \mathcal{D}\phi^i \mathcal{D}\sigma \exp\left[-\frac{1}{2}\int(\partial\phi)^2 - \int \sigma\left(\phi^2 - \frac{1}{g^2}\right) + i\int J_i \phi_i\right]$$

Then integral over ϕ_i would give

$$\int \mathcal{D}\sigma \exp\left[-\frac{N}{2}\operatorname{Tr}\log(-\partial^2 + \sigma(x)) + \int \frac{\sigma(x)}{g^2}\right]$$
$$\exp\left[-\frac{1}{2}\int J_i(x)\, G(x-y)\, J_i(y)\right]$$

where

$$\left[-\partial_x^2 + \sigma(x)\right] G(x-y) = \delta(x-y)$$

At large N leading answer has $\sigma(x) = \sigma_0$

$\Rightarrow$ σ_0 is $(\text{MASS})^2$ of ϕ_i field.

Following our discussion of RG we want to find how $\lambda(\Lambda)$ keeping physical mass σ_0 fixed

$$\Lambda \frac{d}{d\Lambda}\left[\Lambda e^{-4\pi/\lambda}\right] = 0$$

$\Rightarrow$

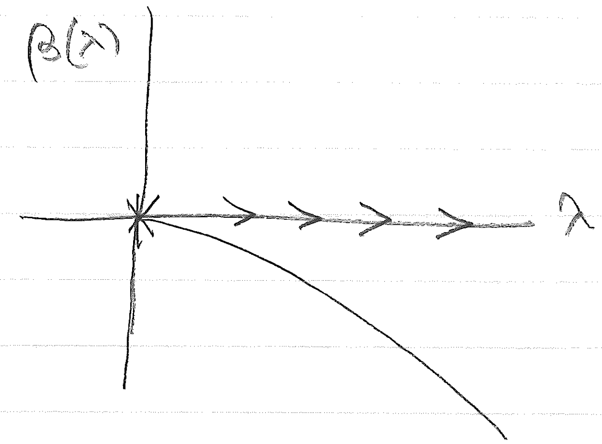
$$\Lambda e^{-4\pi/\lambda} + \Lambda^2 \frac{d\lambda}{d\Lambda}\frac{4\pi}{\lambda^2} e^{-4\pi/\lambda} = 0$$

$\Rightarrow$

$$\beta(\lambda) = \Lambda \frac{d\lambda}{d\Lambda} = -\frac{\lambda^2}{4\pi}$$

ASYMPTOTICALLY FREE. Massive for all values

Flow diagram.



<u>$d=3$</u>:

$$\frac{1}{2}\cdot\frac{4\pi}{8\pi^3}\int_0^\Lambda \frac{dk\,k^2}{k^2+\sigma_0} = \frac{1}{4\pi^2}\left[k - \sqrt{\sigma_0}\tan^{-1}\left(\frac{k}{\sqrt{\sigma_0}}\right)\right]_0^\Lambda$$

$$= \frac{1}{4\pi^2}\left[\Lambda - \frac{\pi}{2}\sqrt{\sigma_0}\right]$$

This gives

$$\frac{4\pi^2}{\lambda} = \Lambda - \frac{\pi}{2}\sqrt{\sigma_0}$$

$$\boxed{\sqrt{\sigma} = \frac{2}{\pi}\left(\Lambda - \frac{4\pi^2}{\lambda}\right)}$$

Clearly for this to happen

$$\lambda\Lambda > 4\pi^2.$$

Define dimensionless $\lambda_0 = \lambda\Lambda$.

Note: λ is a coupling of the ϕ^i fields, May be seen by the form of the lagrangian

$$\mathcal{L} = \frac{1}{2} \frac{(\phi_i \cdot \partial\phi_i)(\phi_j \partial\phi_j)}{\frac{1}{g^2} - \phi^2} + \frac{1}{2}(\partial\phi)^2$$

$$= \frac{g^2}{2}(\phi_i \cdot \partial\phi_i)(\phi_j \cdot \partial\phi_j)(1 - g^2\phi^2)^{-1} + \frac{1}{2}(\partial\phi)^2$$

In $d = 3$ $\quad [\phi] = 1/2$

$$[g^2] + 4 = 3 \Rightarrow [g^2] = -1$$

In common parlance this is an irrelevant operator

However For $\lambda_0 > 4\pi^2$

$$\frac{d}{d\Lambda}\left(\Lambda - \frac{4\pi^2}{\lambda_0}\Lambda\right) = \left(1 - \frac{4\pi^2}{\lambda_0}\right) + \frac{4\pi^2}{\lambda_0^2}\beta(\lambda_0)$$
$$= 0$$

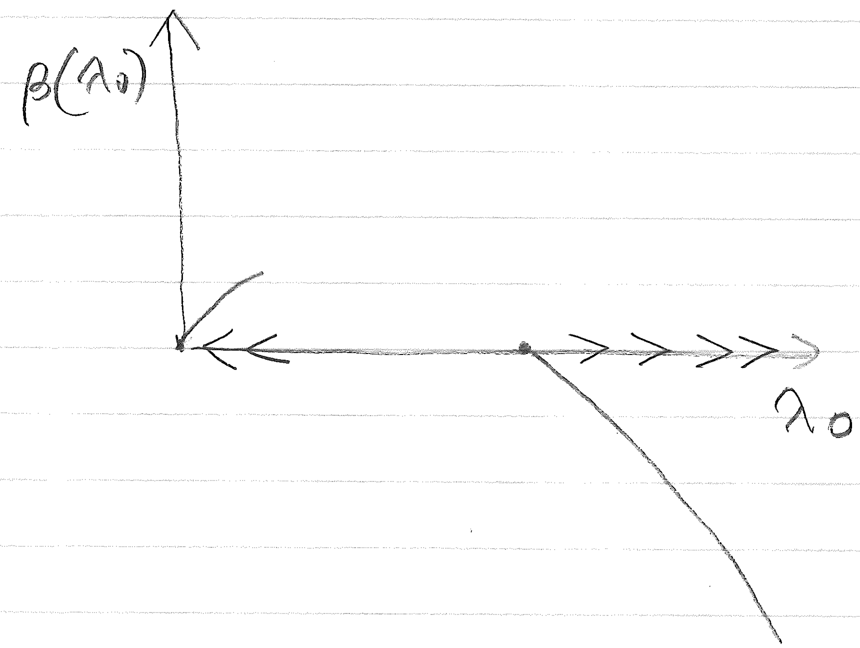
$\Rightarrow$

$$\beta(\lambda_0) = -\frac{\lambda_0^2}{4\pi^2}\left(1 - \frac{4\pi^2}{\lambda_0}\right)$$

$$\boxed{\beta(\lambda_0) = \lambda_0 - \frac{\lambda_0^2}{4\pi^2}}$$

By dimension counting, for small λ_0 must have

$$\beta(\lambda_0) \sim \lambda_0$$



To complete the calculation we need to compute for

$$\lambda_0 < 4\pi^2.$$

Physically in this regime we expect spotaneous symmetry breaking

Now write the field as.

$$\vec{\phi} = (\sqrt{N}\, r,\ \pi_1, \pi_2 \cdots \pi_{N-1})$$

Then

$$\mathcal{L} = \frac{N}{2}(\partial r)^2 + \frac{1}{2}(\partial \vec{\pi})^2 + \sigma\left(\vec{\pi}^2 + N r^2 - \frac{1}{g^2}\right)$$

$$\mathcal{L} = \frac{N}{2}(\partial r)^2 + \frac{(N-1)}{2}\mathrm{Tr}\log(-\partial^2 + \sigma) + N\sigma\left(r^2 - \frac{1}{\lambda}\right)$$

Saddle point equations: $(N-1 \to N)$

r_0 = constant

$$\frac{\delta}{\delta\sigma_0} \qquad r_0^2 + \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+\sigma_0} = \frac{1}{\lambda}$$

$$\frac{\delta}{\delta r_0} \qquad \sigma_0 r_0 = 0$$

If $\sigma_0 \neq 0$ we have the same soln as above which fails for $\lambda_0 < \lambda_c$

$\Rightarrow$ Only solution must be $\sigma_0 = 0$

Then

$$r_0^2 = \frac{1}{\lambda} - \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2}$$

However

$$\frac{1}{2}\int_0^\Lambda \frac{d^3k}{(2\pi)^3}\frac{1}{k^2} = \frac{\Lambda}{4\pi^2}$$

Thus

$$\boxed{r_0^2 = \frac{\Lambda}{4\pi^2}\left(\frac{4\pi^2}{\lambda_0} - 1\right)}$$

Works when $\lambda_0 < 4\pi^2$.

r_0 is the expectation value of the order parameter

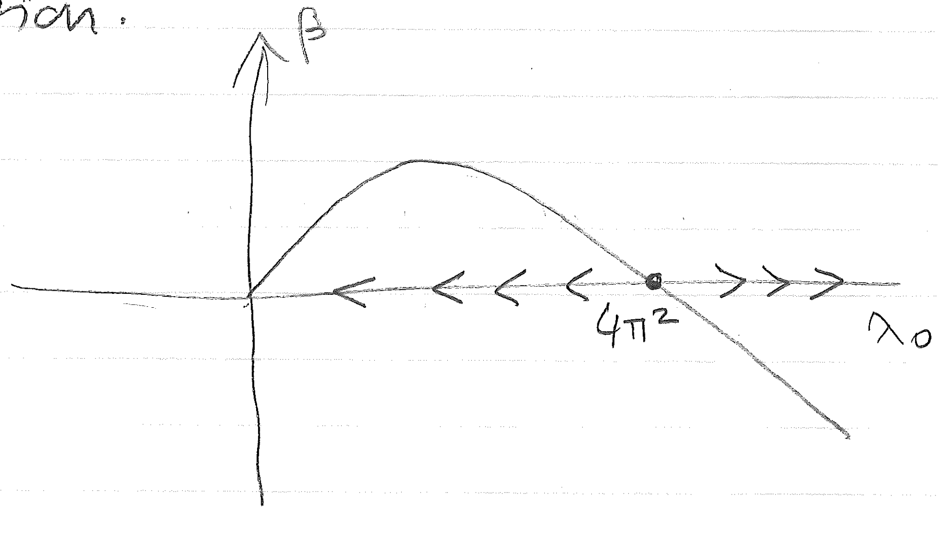
$\Rightarrow$ If r_0 independent of Λ

$$\frac{1}{4\pi^2}\left(\frac{4\pi^2}{\lambda_0} - 1\right) - \frac{\Lambda}{4\pi^2}\frac{4\pi^2}{\lambda_0^2}\frac{d\lambda_0}{d\Lambda} = 0$$

Thus $\beta(\lambda_0) = + \frac{\lambda_0^2}{4\pi^2}\left(\frac{4\pi^2}{\lambda_0} - 1\right)$

$$= \lambda_0 - \frac{\lambda_0^2}{4\pi^2}$$

Which is same as $\lambda_0 > 4\pi^2$ region.



- Compare with classical expectation
Classically
$$\vec{\phi}^2 = \frac{1}{g^2}$$
Means
$$N r_0^2 = \frac{1}{g^2} \qquad r_0^2 = \frac{1}{\lambda}$$
The other term is a quantum correction
The fields $\vec{\pi}$ are massless

Original symmetry $O(N)$ $\frac{N(N-1)}{2}$ generators

final symm $O(N-1)$ $\frac{(N-1)(N-2)}{2}$

# of Broken generators $= \frac{N-1}{2}[N-(N-2)]$
$= N-1$

These are the $\vec{\pi}$ fields

- At large N similar ($\pi\pi$) clearly massless.
$$r = r_0 + \delta r$$
$$\sigma = \sigma$$

$\underline{d=4}$:

$$\frac{1}{2}\frac{2\pi^2}{16\pi^4}\frac{1}{2}\left[k^2 - \sigma_0 \log\left(k^2+\sqrt{\sigma_0}\right)\right]_0^\Lambda = \frac{1}{\lambda}$$

$$\frac{1}{32\pi^2}\left[\Lambda^2 - \sigma_0 \log \Lambda^2 + \sigma_0 \log \sqrt{\sigma_0}\right] = \frac{1}{\lambda}$$

$$\Lambda^2 - \frac{32\pi^2}{\lambda} = \sigma_0 \log\left(\frac{\sqrt{\sigma_0}}{\Lambda}\right)$$

$$1 - \frac{32\pi^2}{\lambda_0} = \frac{1}{2}\, x \log x$$

Can we have σ_0 independent of the cutoff ?

$$\frac{32\pi^2}{\lambda_0^2}\frac{d\lambda_0}{d\Lambda} = -\frac{\sigma_0}{\Lambda}$$

$$\Rightarrow \qquad \beta(\lambda_0) = -\frac{\lambda_0^2}{32\pi^2}\sigma_0$$

But $\sigma_0 = \Lambda^2 f(\lambda_0)$.

So β function depends on Λ

– Does not make sense.

CP^{N-1} MODELS

Consider a $O(3)$ vector

$$\vec{\phi} = z^{+} \vec{\sigma} z$$

Where z: 2 component complex spinor

Note if we ~~perform~~

$$z \to e^{i\theta(x)} z \qquad z^{+} \to z^{+} e^{-i\theta(x)}$$

$\vec{\phi}$ remains the same

Furthermore $\vec{\phi}^2 = 1$ means

$$\vec{\phi}^2 = z^{+}_i \sigma^a_{ij} z_j \, z^{+}_k \sigma^a_{kl} z_l$$

Use

$$\sum_a \sigma^a_{ij} \sigma^a_{kl} = \delta_{il} \delta_{jk}$$

$$\vec{\phi}^2 = z^{+}_i z_j z^{+}_k z_l \delta_{il} \delta_{jk} = (z^{+} z)^2$$

Thus

$$\vec{\phi}^2 = 1 \text{ requires } z^{+} z = 1$$

With this parametrization

$$(\partial \vec{\phi})^2 = \left(\partial_\mu z^{+} \vec{\sigma} z + z^{+} \vec{\sigma} \partial_\mu z \right)^2$$

$$= (\partial_\mu z^{+} \vec{\sigma} z) \cdot (\partial_\mu z^{+} \vec{\sigma} \cdot z)$$
$$2 (\partial_\mu z^{+} \vec{\sigma} z) \cdot (z^{+} \vec{\sigma} \partial_\mu z)$$
$$+ (z^{+} \vec{\sigma} \cdot \partial_\mu z)(z^{+} \sigma \partial_\mu z)$$

$$= [(\partial_\mu z^{+}) z]^2 + 2 (\partial_\mu z^{+})(\partial_\mu z)$$
$$+ [z^{+} \partial_\mu z]^2$$

where we used $z^{+} z = 1$

Compare with

$$D_\mu z^\dagger D_\mu z \qquad D_\mu = \partial_\mu - iA_\mu$$

$$= (\partial_\mu z^\dagger + iA_\mu z^\dagger)(\partial_\mu z - iA_\mu z)$$

$$= (\partial_\mu z^\dagger)(\partial_\mu z) + iA_\mu(z^\dagger \partial_\mu z - \partial_\mu z^\dagger z) + A_\mu^2 z^\dagger z$$

Integrate out A_μ:

$$J_\mu = z^\dagger \partial_\mu z - \partial_\mu z^\dagger z$$

$$iJ_\mu + 2A_\mu = 0 \qquad A_\mu = -\frac{i}{2} J_\mu$$

Then

$$iA_\mu J_\mu + A_\mu^2$$

$$= \frac{1}{2} J_\mu^2 - \frac{1}{4} J_\mu^2 = \frac{1}{4} J_\mu^2$$

$$\frac{1}{4} J_\mu^2 = \frac{1}{4}\Big((z^\dagger \partial_\mu z)(z^\dagger \partial_\mu z) + [(\partial_\mu z^\dagger) z]^2 - 2(\partial_\mu z^\dagger) z\, z^\dagger \partial_\mu z$$

However $z^\dagger \partial_\mu z = -(\partial_\mu z^\dagger) z$

$$\Rightarrow 2(\partial_\mu z^\dagger) z \cdot z^\dagger \partial_\mu z = (\partial_\mu z^\dagger z)^2 + (z^\dagger \partial_\mu z)^2$$

Thus entire action becomes

$$\partial_\mu z^\dagger \partial_\mu z + \frac{1}{2}\left[(z^\dagger \partial_\mu z)^2 + ((\partial_\mu z^\dagger) z)^2 \right]$$
$$= \frac{1}{2}(\partial \vec{\phi})^2.$$

$\Rightarrow$ O(3) sigma model $\equiv$ CP^1 model
($z^\dagger z = 1$
z : 4 real numbers
2 equations)

Generalize to N.

$$\boxed{\mathcal{L} = D_\mu z^\dagger D_\mu z + \sigma\left(z^\dagger z - \frac{1}{g^2}\right)}$$

Consider this model in $d=3$
Integrate out z

$$\boxed{S_{eff} = N \operatorname{tr} \log\left(-D_\mu D^\mu + \sigma\right) - \frac{N\sigma}{g^2}}$$

The saddle point for $\sigma = \sigma_0$ is the same as before
$\rightarrow$ gives a mass to z-particles

However now there appears to be photon field.

Effective action Calculation

$$\mathrm{Tr}\log(\mathcal{O}_x + m^2) = \int \frac{d\tau}{\tau} e^{-m^2\tau}\, \mathrm{Tr}\, e^{-\tau\mathcal{O}_x}$$

$$\mathcal{O}_x = -D_\mu D^\mu$$

Now consider

$$h(\tau, x, x') = \langle x | e^{-\tau \mathcal{O}_x} | x' \rangle$$
$$= e^{-\tau\mathcal{O}_x} \langle x | x' \rangle$$
$$= \int dk\, e^{-\tau\mathcal{O}_x} \langle x|k\rangle\langle k|x'\rangle$$
$$= \int dk\, e^{-ikx'} e^{-\tau\mathcal{O}_x} e^{ikx}$$

Now use

$$e^{-ikx} D_\mu e^{ikx} = ik_\mu + D_\mu$$

where check

$$D_\mu e^{ikx} f(x) = \partial_\mu (e^{ikx} f) + iA_\mu e^{ikx} f$$
$$= ik_\mu e^{ikx} f + e^{ikx}\partial_\mu f + iA_\mu e^{ikx} f$$
$$= (ik_\mu + D_\mu) f$$

Then

$$e^{-ikx}\mathcal{O}_x e^{ikx} f = \left(m^2 - (e^{-ikx} D_\mu e^{ikx})(e^{-ikx} D_\mu e^{ikx})\right) f$$

$$= \quad -(ik_\mu + D_\mu)(ik_\mu f + D_\mu f)$$

$$= \quad +k^2 f - ik_\mu D_\mu f - ik_\mu D_\mu f - D_\mu D_\mu f$$

$\Rightarrow$

$$e^{-ikx} \mathcal{O}_x e^{ikx} = k^2 - 2ik_\mu D_\mu + \mathcal{O}_x$$

Similarly

$$e^{-ikx} \mathcal{O}_x^2 e^{ikx} = (k^2 - 2ik_\mu D_\mu + \mathcal{O}_x)^2$$

Thus

$$h(\epsilon, x, x) = \int \frac{d^d k}{(2\pi)^d} \exp\left[-\varepsilon(k^2 - 2ik_\mu D_\mu + \mathcal{O}_x)\right]$$

Now rescale $k = q/\sqrt{\varepsilon}$

$$h = \frac{1}{\varepsilon^{d/2}} \int \frac{d^d q}{(2\pi)^d} e^{-q^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sqrt{\varepsilon}\, 2iq \cdot D_x - \varepsilon \mathcal{O}_x\right)^n$$

Thus we get expansion in ε

Suppose

$$e^{-\varepsilon m^2} h(\varepsilon x, x) = \frac{e^{-\varepsilon m^2}}{\varepsilon^{d/2}} \sum a_n(x)\, \varepsilon^n.$$

Then action is

$$\int dx \int_0^\infty \frac{d\varepsilon}{\varepsilon^{1+d/2}} e^{-m^2 \varepsilon} \sum_{n=1}^{\infty} a_n(x)\, \varepsilon^n$$

Perform ε integral as

$$\int_0^\infty d\varepsilon\ \varepsilon^{\alpha-1} e^{-a\varepsilon} = a^{-\alpha}\,\Gamma(\alpha)$$

Then Lagrangian density is

$$\sum_{n=1}^{\infty} a_n(x)\ \Gamma(n-\tfrac{d}{2})\ \frac{1}{m^{2(n-d/2)}}$$

To evaluate a_n:

$n=1$ term $\qquad (\mathcal{O}_x = -D^2)$

$$a_1(x) = \int \frac{d^dk}{(2\pi)^d}\, e^{-k^2}\left(\cancel{\sqrt{\varepsilon}\,2ik\cdot D} + \varepsilon D^2\right)$$

$$+\int \frac{d^dk}{(2\pi)^d}\, \frac{e^{-k^2}}{2!}\left(\varepsilon\,(2i)^2 q_\mu q_\nu D_\mu D_\nu\right)$$

$$\int \frac{d^dk}{(2\pi)^d}\, e^{-k^2} = 1$$

$$\int \frac{d^dk}{(2\pi)^d}\, e^{-k^2} k_\mu k_\nu = \frac{1}{2}\,\delta_{\mu\nu}$$

$$a_1(x) = D^2 - \frac{4}{2}\,\frac{1}{2}\,D^2 = 0 \qquad \checkmark$$

The coefficient $a_2(x)$
(Note ½ integer powers vanish by k integration)

From $n=2$

$$\frac{1}{2!}\,\varepsilon^2\,\mathcal{O}_x^2$$

From $n=3$ get $\varepsilon^{3/2}$, ε^2, $\varepsilon^{5/2}$, ε^3

The ε^2 term is

$$\frac{1}{3!}\,\varepsilon^2(2i)^2\Big[q_\mu D_\mu q_\nu D_\nu D^2 + q_\mu D_\mu D^2 q_\nu D_\nu + D^2 q_\mu D_\mu q_\nu D_\nu\Big]$$

COMPACT QED IN 3 DIMENSIONS

$$L = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \quad \text{(euclidean)}$$

Equations of motion are

$$\partial_\mu F_{\mu\nu} = 0$$

while Bianchi identities are

Bianchi identity

$$\varepsilon^{\mu\nu\lambda} \partial_\mu F_{\nu\lambda} = 0$$

In 3 dimensions $F_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} C_\gamma$

Then $\partial_\gamma C_\gamma = 0$

Eqn of motion becomes

$$\partial_\alpha F_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \partial_\alpha C_\gamma = 0$$

Solved by $C_\gamma = \partial_\gamma \phi$

Bianchi + eqn of motion $\Rightarrow$ $\partial^2 \phi = 0$

Instantons: $\partial^2 \phi = \rho$

Solve by $\phi(k) = -\frac{\rho(k)}{k^2}$

Action is now

$$S = \int \frac{1}{4} \varepsilon_{\mu\nu\alpha} \varepsilon_{\mu\nu\beta} \partial_\alpha \phi \partial_\beta \phi$$

$$S = \frac{1}{2e^2} \int (\partial\phi)^2 = \int d^3k \; k^2 \phi(k) \phi(-k)$$

Substitute

$$S = \frac{1}{2e^2}\int d^3k \; \frac{\rho(k)\,\rho(-k)}{k^2}$$

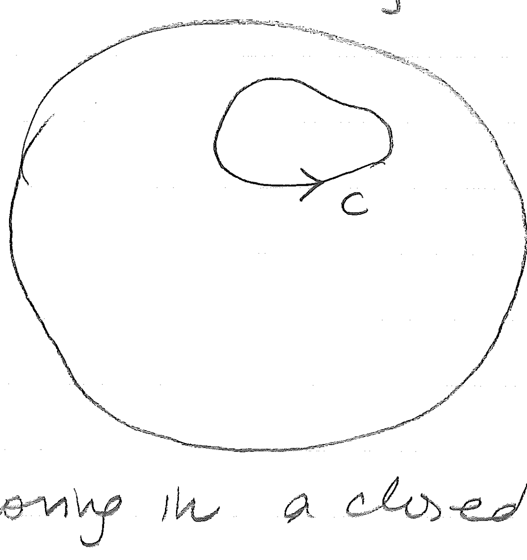
Monopoles:

When the gauge group is compact there are configurations which have a monopole charge.
This has

Check normalizations

$$C_\mu = \frac{g\, x_\mu}{|x|^2} = \partial_\mu\left(\frac{g}{|x|}\right)$$

Consider a particle moving on a sphere enclosing this monopole



and moving in a closed path

Then the action is

$$\int dt \left(\tfrac{1}{2} \dot{x}^2 + eA_i \dot{x}^i \right)$$

Then the wavefn aquires a phase

$$\exp\left[ie \oint_C d\vec{x} \cdot \vec{A} \right]$$

Stokes theorem says this is

$$\exp\left[ie \int \vec{B} \cdot d\vec{s} \right]$$

However which surface ?
Ambiguity removed if

$$e \left(\int_1 \vec{B}\, d\vec{s} + \int_2 \vec{B}\, d\vec{s} \right) = 2\pi n$$

$$4\pi eg = 2\pi n \qquad eg = \tfrac{1}{2}$$

$\Rightarrow$ Dirac quantization rule.

<u>Euclidean configurations & Monopoles</u>

A slightly different argument also shows that there are euclidean configurations

This means we can have

$$\rho(x) = \sum_i q_i \, \delta^{(3)}(x - x_i)$$

So that

$$S = \frac{1}{2e^2} \sum_{ij} q_i q_j \, G(x_i - x_j)$$

and

$$\partial^2 G(x-y) = -\delta^{(3)}(x-y)$$

Monopole gas:

The partition function needs a sum over all possible monopoles and all possible charges

$$Z = \sum_{\substack{N \\ \{q_i\}}} \frac{J^N}{N!} \int \prod d\vec{x}_i \, \exp\left[-\frac{1}{2e^2} \sum_{i \neq j} q_i q_j \, G(x_i - x_j) \right]$$

J : fugacity comes from the self energy

Now use

$$\exp\left[-\frac{1}{e^2} \sum q_i q_j \, G(x_i - x_j) \right]$$

$$= \int \mathcal{D}\chi \, \exp\left[-\frac{e^2}{2} \int (\partial\chi)^2 + i \sum q_i \chi(x_i) \right]$$

Contribution from $q_i = 0, \pm 1$ gives

$$Z = \int \mathcal{D}\chi \exp\left[-\frac{e^2}{2}\int (\partial\chi)^2\right] \sum_N \frac{\zeta^N}{N!} \int \prod dx_i \sum_{q_i = 0, \pm 1} e^{i \sum q_i \chi(x_i)}$$

$$= \int \mathcal{D}\chi \exp\left[-\frac{e^2}{2}\int (\partial\chi)^2\right] \sum_N \frac{\zeta^N}{N!} \left(\int d^3x \, (1 + 2\cos\chi(x))\right)^N$$

$$= \int \mathcal{D}\chi \exp\left[-\frac{e^2}{2}\int d^3x \left[(\partial\chi)^2 + 2M^2 \cos\chi\right]\right]$$

(ignored const)

$$M^2 = \frac{\zeta}{e^2}$$

$$\zeta = \frac{e}{a^{7/2}} e^{-\frac{1}{e^2 a}}$$

<u>Field Strength propagator</u>

We want to compute

$$\langle C_\mu(k) C_\nu(-k) \rangle$$

$$* \quad \langle C_\mu(k) C_\nu(-k) \rangle^{(0)} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$ zero monopole sector

From the monopole gas use

$$C_\mu(x) = \partial_\mu \phi(x)$$

Momentum space

$$C_\mu(k) = i k_\mu \phi(k) = -i k_\mu \frac{\rho(k)}{k^2}$$

$$\langle C_\mu(k)\, C_\nu(-k) \rangle = (-i k_\mu)(+i k_\nu) \frac{\langle \rho(k)\, \rho(-k) \rangle}{k^4}$$

$$= \frac{k_\mu k_\nu}{k^4} \langle \rho(k) \rho(-k) \rangle$$

To compute $\langle \rho(k) \rho(-k) \rangle$ consider

$$\langle e^{i \int \rho(x) \eta(x)\, dx} \rangle$$

$$= \sum_{\substack{N \\ \{q_i\}}} \frac{\zeta^N}{N!} \int \prod dx_i\, e^{-\frac{1}{2e^2} \sum q_i q_j G(x_i - x_j)} e^{i \int \rho \eta}$$

$$= \sum \frac{\zeta^N}{N!} \int \prod dx_i \int \mathcal{D}\chi\, e^{-\frac{1}{2} e^2 \int (\partial \chi)^2} e^{i \int \rho \chi + i \int \rho \eta}$$

Shift $\chi \to (\chi - \eta)$

$$= \sum_{\substack{N \\ \{q_i\}}} \frac{1}{N!}^N \int \prod dx_i \int \mathcal{D}\chi \, e^{-\frac{e^2}{2}\int(\partial(\chi-\eta))^2} e^{i\sum q_i \chi(x_i)}$$

Sum over i, sum over N

$$= \int \mathcal{D}\chi \, e^{-\frac{1}{2}e^2 S'}$$

$$S' = \frac{1}{2}e^2\int dk \Big[(\chi(k)-\eta(k))(\chi(-k)-\eta(-k))\, k^2 + 2M^2 \cos\chi \Big]$$

$$\frac{\delta Z}{\delta\eta(k)} = \int \mathcal{D}\chi e^{-S}\left[-\chi(-k)k^2 + \eta(-k)\, k^2\right]$$

$$\frac{\delta^2 Z}{\delta\eta(k)\,\delta\eta(-k)} = -\int \mathcal{D}\chi e^{-S}[k^2] + \int \mathcal{D}\chi e^{-S}\left[-\chi(+k)k^2 + \eta(k)k^2\right]\left[-\chi(-k)k^2 + \eta(-k)k^2\right]$$

$$\frac{1}{Z}\frac{\delta^2 Z}{\delta\eta\,\delta\eta}\bigg|_{\eta=0} = -k^2 + k^4\langle\chi(k)\chi(-k)\rangle_{\eta=0}$$

To lowest order

$$\langle\chi(k)\,\chi(-k)\rangle_{\eta=0} = \frac{1}{k^2+M^2}$$

Thus

$$\langle C_\mu(k) C_\nu(-k)\rangle^{(1)} = \frac{k_\mu k_\nu}{k^4} \langle \rho(k)\rho(-k)\rangle$$

$$= - \frac{k_\mu k_\nu}{k^4}\left(\frac{1}{Z}\frac{\delta^2 Z}{\delta\eta(k)\delta\eta(-k)}\right)$$

$$= + \frac{k_\mu k_\nu}{k^4}\left(k^2 - \frac{k^4}{k^2+M^2}\right)$$

Thus

$$\langle C_\mu(k) C_\nu(-k)\rangle^{(0)} + \langle C_\mu(k) C_\nu(-k)\rangle^{(1)}$$

$$= \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \frac{k_\mu k_\nu}{k^4}\left(k^2 - \frac{k^4}{k^2+M^2}\right)$$

$$= \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+M^2}$$

$\Rightarrow$ There is no pole at $k^2=0$

Wilson loop and Confinement

Need to calculate

$$\tilde{e}\oint \vec{A}\cdot d\vec{x} = \tilde{e}\int_S \vec{C}\cdot d\vec{s}$$

For a given source $\rho(x)$ the potential is

$$\phi = \int d^3y\, \rho(y)\, G(x-y)$$

$$\nabla^2 G(x-y) = -\delta(x-y)$$

Thus $$C_\mu(x) = \int d^3y\, \rho(y)\, \frac{\partial}{\partial x^\mu} G(x-y)$$

$$\int ds_x^\mu C_\mu = \int d^3y\, \rho(y) \int ds_x^\mu \frac{\partial}{\partial x^\mu} G(x-y)$$

Same as calculation of generating function for correlators, but with

$$\eta(y) = \int ds_x^\mu \frac{\partial}{\partial x^\mu} G(x-y)$$

We found

$$\tilde{e}\eta = \Upsilon$$

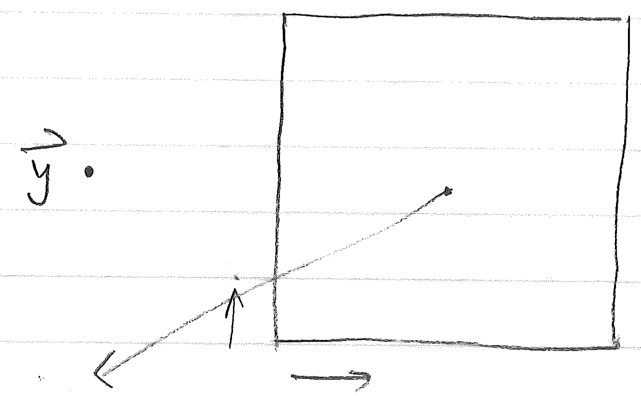
$$\left\langle e^{i\tilde{e}\int d^3y\, \rho(y)\eta(y)} \right\rangle$$

$$= \int \mathcal{D}\chi \exp\left[-\tfrac{1}{2}e^2 \int d^3x \left[(\partial(\chi-\eta))^2 + 2M^2 \cos\chi \right]\right]$$

Evaluate by saddle

$$\nabla^2(\chi_0 - \eta) = M^2 \sin\chi_0$$

Consider a rectangular loop in the x-y plane



Then

$$\eta(\vec{y}) = \frac{\partial}{\partial y_3} \int_S dx_1 dx_2 \; G(\vec{x}-\vec{y})$$

Write the integral as a 3d integral

$$\int d^3x \; \delta(x_3) \, \Theta_S(x_1, x_2) \, G(\vec{x}-\vec{y})$$

where

$$\Theta_S(x_1, x_2) = [\theta(x_1) - \theta(x_1+L)][\theta(x_2) - \theta(x_2+T)]$$

$\Theta_S \neq 0$ only inside the surface whose boundary is the loop.

Thus

$$\nabla^2 \eta(\vec{y}) = \frac{\partial}{\partial y_3} \int d^3x \; \delta(x_3) \, \Theta_S(x_1, x_2) \, \delta(\vec{x}-\vec{y})$$

$$= \delta'(y_3) \, \Theta_S(y_1, y_2)$$

Thus the saddle point equation is

$$\nabla^2 \chi_0 = \nabla^2 \eta + M^2 \sin \chi_0$$

~~If~~ If the loop is big χ_0 depends only on $x_3 \equiv z$

$$\Rightarrow \quad \frac{d^2 \chi_0}{d z^2} = \delta'(z)\, \Theta_S(xy) + M^2 \sin \chi_0$$

LINEAR O(N) MODEL

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{2N}(\phi^2)^2$$

To solve model - Hubbard-Stratonavich:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \sigma\phi - \frac{N}{2g}\sigma^2$$

Equ. of motion for σ

$$\phi = \frac{N}{g}\sigma_0$$

$$\sigma_0\phi - \frac{N}{2g}\sigma_0^2 = \frac{g}{N}\phi^2 - \frac{g}{2N}(\phi^2)^2 = \frac{g}{2N}(\phi^2)^2$$

Integrate out ϕ

$$S_{eff} = \frac{N}{2}\mathrm{Tr}\log\left(-\partial^2 + m^2 + \sigma\right) - \frac{N}{2g}\sigma^2$$

Saddle point equ. for σ

$$\boxed{\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2+\sigma} = \frac{1}{g}\sigma}$$

Clearly $m^2+\sigma = m_{eff}^2$ is the mass of the scalar field fluctuations

Critical surface has

$$\sigma = \frac{g}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}$$

Factorization:

The fact that a saddle of σ controls the physics shows that correlators factorize.

Saddle point shows that

$$W = -\log Z \sim O(N)$$

Now consider

$$Z[J] = \int \mathcal{D}\phi \, e^{-S_0 - N\int J(x) B(x)}$$

where $B(x) = \frac{1}{N} \vec{\phi}^{\,2}(x)$

Same analysis holds here.
Thus

$$\langle B_1 \dots B_n \rangle_c = \frac{1}{N^n} \left[\frac{\delta^n W}{\delta J(x_1) \dots \delta J(x_n)} \right]$$

$$= O(N^{1-n}).$$

e.g.

$$\langle B_1 B_2 \rangle_c = \langle B_1 B_2 \rangle - \langle B_1 \rangle \langle B_2 \rangle$$

$$= O\left(\frac{1}{N}\right)$$

This means
$$\langle B_1 B_2 \rangle_c = \langle B_1 \rangle \langle B_2 \rangle + O\left(\frac{1}{N}\right)$$

→ Pretty much like Classical limit

Consider now

$$\langle B(x_1)\ B(x_2)\ B(x_3) \rangle_c$$
$$= \langle B(x_1) B(x_2) B(x_3) \rangle$$
$$- \langle B(x_1) \rangle \langle B(x_2) \rangle \langle B(x_3) \rangle$$
$$- \left[\langle B(x_1) B(x_2) \rangle \langle B(x_3) \rangle + \text{perm} \right]$$

⇒
$$\langle B(x_1) B(x_2) B(x_3) \rangle = \langle B(x_1) \rangle \langle B(x_2) \rangle \langle B(x_3) \rangle$$
$$+ \left[\langle B(x_1) B(x_2) \rangle \langle B(x_3) \rangle + \text{perm} \right]$$
$$+ \langle B(x_1) B(x_2) B(x_3) \rangle_c$$
$$= O(1) + \frac{1}{N} + \frac{1}{N^2}.$$

If we consider normal ordered operators: $- \langle B \rangle = 0$

$$\langle B(x_1)\ B(x_2)\ B(x_3) \ldots\ B(x_{2n}) \rangle$$
$$= \langle B(x_1) B(x_2) \rangle \langle B(x_3) B(x_4) \rangle \cdots$$
$$+ \text{higher orders in } 1/N$$

General Green's function

$$S = S_0 + \frac{1}{2} g \int B^2(x)$$

$$S_0 = \int dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$$

Then

$$\langle B(k)\, B(-k) \rangle = \left\langle B(k)\, B(-k)\, e^{-\frac{1}{2} g \int B(q) B(-q)} \right\rangle_0$$

$$= G_0(k) - \frac{1}{2} g \int dq \, \langle B(k) B(-k) B(q) B(-q) \rangle_0$$

$$+ \frac{1}{8} g^2 \int dq_1 dq_2 \, \langle B(k) B(-k) B(q_1) B(-q_1) B(q_2) B(-q_2) \rangle_0$$

$$+ \ldots$$

Combinatoric factors

2 for 1st

4.2 = 8 for 2nd

. . . .

$$= G_0(k) - g\,(G_0(k))^2 + g^2 (G_0(k))^3$$

$$\boxed{G_g(k) = \frac{G_0(k)}{1 + g\, G_0(k)}}$$

This leads to the definition of a RUNNING COUPLING

$$\boxed{g(k) = \frac{g_0}{1 + g_0 G_0(k)}}$$

Since

$$G_g(k) \equiv G_0(k) - g(k)\, G_0^2(k)$$

<u>Flow along the critical surface</u> $m=0$

$$G_0(k) = \int_{dp}^{dq} \langle (\phi^i(q)\, \phi^i(k-q)) \\ (\phi^j(p)\, \phi^j(-k-p)) \rangle$$

$$= 2\int d\vec{q}\, d\vec{p}\; \delta(p+q)\, \frac{1}{q^2} \frac{1}{(k-q)^2}$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 (k-q)^2}$$

Feynman formula

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[xa + (1-x) b]^2}$$

$$G_0(k) = \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{[x(k-q)^2 + (1-x) q^2]^2}$$

Denominator

$$x\left(k^2+q^2-2\vec{q}\cdot\vec{k}\right)+(1-x)q^2$$
$$= q^2-2xq\cdot k+xk^2$$
$$= (q-kx)^2-k^2x^2+xk^2$$
$$= q'^2+x(1-x)k^2$$

Now do integral

$$\int\frac{d^dq'}{(2\pi)^d}\frac{1}{(q'^2+m^2)^2}$$

$$= \frac{\Omega_{d-1}}{(2\pi)^d}\int\frac{dq\, q^{d-1}}{(q^2+m^2)^2}$$

$$z=q^2 \qquad = \frac{1}{2}\frac{\Omega_{d-1}}{(2\pi)^d}\int_0^\infty\frac{dz\, z^{\frac{d}{2}-1}}{(z+m^2)^2}$$

$$w=\frac{m^2}{z+m^2} \qquad 1-w=\frac{z}{(z+m^2)}$$

$$\frac{1}{2}\frac{\Omega_{d-1}}{(2\pi)^{d-1}}\int_0^\infty dw\, w^{1-\frac{d}{2}}(1-w)^{\frac{d}{2}-1}$$

$$\int dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Finally

$$\frac{1}{(2\pi)^d}\int \frac{d^d q}{(q^2+m^2)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \left(\frac{1}{m^2}\right)^{2-\frac{d}{2}}$$

Special case $d=3$:

$$\int_0^1 dx \left(\frac{1}{(x(1-x))k^2}\right)^{1/2} \frac{1}{(4\pi)^{3/2}} \frac{\Gamma(1/2)}{\Gamma(2)}$$

$$= \frac{\pi}{16\pi^2 k} \qquad \left[\int_0^1 dx \frac{1}{\sqrt{x(1-x)}} = \pi\right]$$

This gives

$$\boxed{g(k) = \frac{g}{1+\frac{g}{16\pi k}}}$$

Note that $[g] = 1$ in $d=3$

Thus when

$$k/g \ll 1 \quad \text{we get}$$

$$g(k) = 16\pi k$$

regardless of the initial value of the bare coupling g.

Define renormalized coupling at $k = \mu$

$\Rightarrow$

$$g_R(\mu) = g(k=\mu)/\mu$$

This is dimensionless

$$g_R(\mu) = \frac{g}{\mu + \frac{g}{16\pi}}$$

This means

$$g = \frac{\mu g_R(\mu)}{1 - b g_R(\mu)}$$

Since g is independent of μ

$$\frac{g_R}{1-bg_R} + \frac{\mu \dot{g}_R(\mu)}{1-bg_R} + \frac{\mu g_R}{(1-bg_R)^2} b \dot{g}_R = 0$$

$$\beta(g_R) = \mu \partial_\mu g_R$$

$$\frac{\beta(g_R)}{(1-bg_R)^2}\left[1 - bg_R + bg_R\right] + \frac{g_R}{1-bg_R} = 0$$

$$\boxed{\beta(g_R) = - g_R(1-bg_R)}$$

ISOTROPIC OSCILLATOR

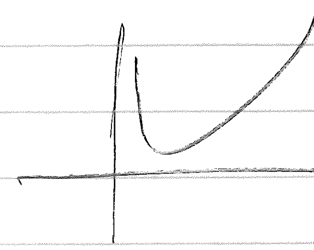
$$\sum_{i=1}^{N} \frac{1}{2}\left(-\frac{d^2}{dx_i^2} + k^2 x_i^2\right) = E\psi$$

$$r = \frac{1}{N}\sum x_i^2 \qquad \Psi = r^{-\frac{N-1}{2}}\chi$$

$$\left[-\frac{1}{2N^2}\frac{d^2}{dr^2} + \frac{(N-1)(N-3)}{8N^2r^2} + \frac{1}{2}k^2r^2\right] = \frac{E}{N}\chi$$

$\Rightarrow$ $\frac{1}{N}$ is $\hbar$

$$V_{eff} = \frac{1}{8r^2} + \frac{1}{2}k^2r^2$$



Solve classical problem

$c=1$ MATRIX MODEL

$$H = -\frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial M_{ij}\partial M_{ij}} + V(M_{ij})$$

Diagonalize:

$$\int dM_{ij} = \int d\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^2 d\Omega$$

$\Rightarrow$ Fermions:

$$H = -\frac{1}{2}\frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i)$$

$$\rho(x) = \frac{1}{N}\sum_i \delta(x - \lambda_i) = \partial_x \varphi$$

Then we get a

$$H =$$

2D CFT's

Simplest CFT in 1+1 dimension is

$$S = \frac{1}{8\pi}\int dx\,dt\,\left[(\partial_t\phi)^2 + (\partial_x\phi)^2\right]$$

Define

$$z = t + ix \qquad \bar{z} = t - ix$$

Metric $\quad ds^2 = d\bar{z}\,dz$

Eqn of motion

$$\partial_z \partial_{\bar{z}} \phi = 0$$

Any holomorphic or anti-holomorphic fn solves eqn. of motion.

$$\langle \phi(z,\bar{z})\, \phi(w,\bar{w}) \rangle = -\log\left(m^2|z-w|^2\right)$$

$m \to$ IR cutoff.

Thus

$$\langle \partial_z \phi(z)\, \phi(w,\bar{w}) \rangle = -\frac{1}{z-w}$$

$$\langle \partial_z \phi(z)\, \partial_w \phi(w) \rangle = -\frac{1}{(z-w)^2}$$

Action $\quad S = \frac{1}{2\pi}\int dz\,d\bar{z}\; \partial_z\phi\, \partial_{\bar{z}}\phi$

invariant under

$$z \to z' = z'(z)$$
$$\bar{z} \to \bar{z}' = \bar{z}'(z)$$

Under these coordinate transformations

$$ds^2 = dz\,d\bar{z} = \left(\frac{\partial z}{\partial z'}\,\frac{\partial z}{\partial \bar{z}'}\right) dz'\,d\bar{z}'$$

OPE's Wick's Theorem

$$T\{:\phi(x_1)\phi(x_2)\phi(x_3):\ :\phi(y_1)\phi(y_2)\phi(y_3):\}$$
$$= :\phi(x_1)\ldots\phi(x_3)\phi(y_1)\ldots\phi(y_3):$$
$$+ \langle\phi(x_1)\phi(y_1)\rangle :\phi(x_2)\phi(x_3)\phi(y_1)\phi(y_3):$$
+ all other ~~one~~ pair contractions
$$+ \langle\phi(x_1)\phi(y_1)\rangle\langle\phi(x_2)\phi(y_2)\rangle :\phi(x_3)\phi(y_3):$$
+ all others

Consider e.g.

$$T\left(\partial_z\phi(z) :e^{ik\phi(w)}:\right)$$

$$= \sum_n \frac{(ik)^n}{n!} T\left(\partial_z\phi(z) :\phi(w)\phi(w)\ldots:\right)$$

$$= :\partial_z\phi(z) e^{ik\phi(w)}:$$
$$+ \sum_n \frac{(ik)^n}{n!} n \langle\partial_z\phi(z)\phi(w)\rangle \underbrace{:\phi(w)\ldots\phi(w):}_{(n-1)}$$

$$= :\partial_z\phi\, e^{ik\phi(w)}: - \frac{ik}{z-w} :e^{ik\phi(w)}:$$

Want to express everything in terms of operators at $(w, \bar{w})$

$$\partial_z \phi(z) = \partial_w \phi(w) + O(z-w)$$

Thus

$$\lim_{z \to w} T\left(\partial_z \phi \, e^{ik\phi} \right)$$

$$= :\partial_w \phi(w) e^{ik\phi(w)}: - \frac{ik}{z-w} :e^{ik\phi(w)}$$

Example of OPE

$$T[O_i(x) \, O_j(y)] = \sum_k C_{ij}^{k}(x-y) \, O_k(y)$$

<u>Radial Quantization</u>

Usually in Minkowski space

$$\phi(t,x) = \int_{-\infty}^{\infty} dk \left[\frac{a(k)}{\sqrt{2|k|}} e^{-i(|k|t - kx)} + c.c. \right]$$

$$[a(k), a^\dagger(k')] = \delta(k-k')$$

Only 2 possibilites

$|k| = k \rightarrow$ Right movers

$|k| = -k \rightarrow$ Left movers

Define
$$\alpha(k) = \sqrt{|k|} \, a(k) \qquad k > 0$$
$$\tilde{\alpha}(-k) = \sqrt{|k|} \, a(k) \qquad k < 0$$

Also define $\quad t + x = x_+$
$\qquad\qquad\quad t - x = x_-$

$$\partial_+\phi = \int_0^\infty dk\,\left[\alpha(k)\,e^{-ik(t+x)} + h.c\right]$$

$$\partial_-\phi = \int_0^\infty dk\,\left[\tilde{\alpha}(k)\,e^{-ik(t-x)} + h.c\right]$$

$$[\alpha(k), \alpha^+(k')] = k\,\delta(k-k')$$
$$[\alpha(k), \tilde{\alpha}^+(k')] = 0$$
$$[\tilde{\alpha}(k), \tilde{\alpha}^+(k')] = k\,\delta(k-k')$$

× Compactify x on a circle $0 \leq x \leq 2\pi$

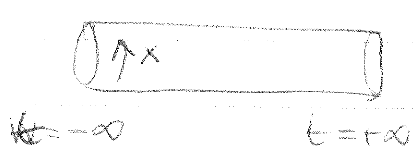
$$k = \frac{2\pi n}{2\pi} = n \qquad \int dk \to \sum_n$$

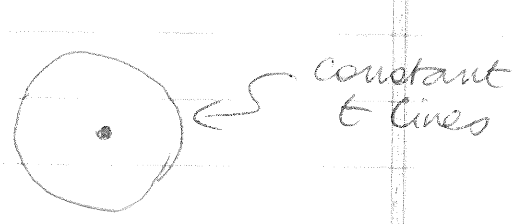
× Euclidean

$$\begin{aligned} in(t+x) &= in(-it+x) \\ &= nt + inx \\ &= n(t+ix) \equiv nw \end{aligned}$$

· Now make a conformal transformation

$$z = e^{w} \qquad \frac{\partial z}{\partial w} = z$$





$$\partial_z \phi = \frac{\partial w}{\partial z} \partial_w \phi = \frac{1}{z} \partial_w \phi$$

$\Rightarrow$

$$\partial_z \phi = \sum \alpha_n z^{-(n+1)}$$

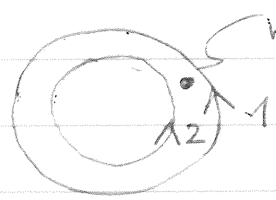
Thus time t becomes radius

Time ordering $\Rightarrow$ Radial ordering

$$R\left(A(z) B(w)\right) = \begin{cases} A(z) B(w) & |z| > |w| \\ B(w) A(z) & |z| < |w| \end{cases}$$

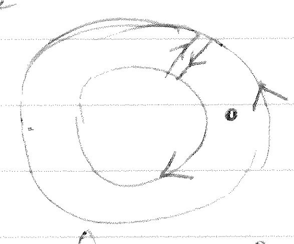
Consider now an operator

$$A_f = \oint dz\, f(z) A(z)$$

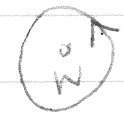


$$\oint_1 dz\, f(z)\, R\left[A(z) B(w)\right] = A_f B(w)$$

$$\oint_2 dz\, f(z)\, R\left[A(z) B(w)\right] = B(w) A_f$$



$$\oint_1 - \oint_2 = \oint_C$$



$$\oint dz\, f(z)\, R(A(z) B(w)) = [A_f, B(w)]$$

Since the contour integral only picks up singular terms
Singular terms of OPE $\Rightarrow$ Commutators

eg $\qquad \partial_z \phi = \sum \alpha_n z^{-(n+1)}$

$$\oint \frac{dz}{2\pi i} z^n \partial_z \phi = \alpha_n$$

Thus consider

$$\oint \frac{dz}{2\pi i} z^n R\left(\partial_z \phi \; B(w) \right) = [\alpha_n, B(w)]$$

If $\quad B(w) = \partial_w \phi$

$$\oint \frac{dz}{2\pi i} z^n R\left(\partial_z \phi \; \partial_w \phi \right)$$

$$= - \oint \frac{dz}{2\pi i} z^n \frac{1}{(z-w)^2}$$

$$= - \frac{d}{dw} \oint \frac{dz}{2\pi i} \frac{z^n}{(z-w)} = -\frac{d}{dw} w^n$$

$$= - n w^{n-1}$$

Thus

$$[\alpha_n, \partial_w \phi] = - n w^{n-1}$$

$$[\alpha_n, \alpha_m] = \oint \frac{dw}{2\pi i} w^m [\alpha_n, \partial_w \phi]$$

$$= -n \oint \frac{dw}{2\pi i} w^{m+n-1} = m \, \delta_{m+n,0}$$

Energy Momentum Tensor

$$S = S_g[g] + S_M[g, \phi]$$

$$\delta S = \delta S_g - \frac{1}{4\pi}\int T_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{g}$$

defines $T_{\mu\nu}$

If $\delta g^{\mu\nu}$ is a diffeomorphism

$$\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu$$

S_g, S_M are invariant $\Rightarrow$ $\delta S = 0$

$\delta S_g = 0$ separately

$\Rightarrow$

$$\int d^2\xi\,\sqrt{g}\;T_{\mu\nu}(\nabla^\mu \xi^\nu) = 0$$

$$\Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0 \qquad \nabla_\mu T^{\mu\nu} = 0$$

$T_{\mu\nu}$ clearly generates coordinate transformations

Consider now a conformal coordinate transformation in flat space

$$\delta S = -\frac{1}{2\pi}\int \Big\{ T_{zz}\,\partial^z \xi^z + T_{\bar z\bar z}\,\partial^{\bar z}\xi^{\bar z} + T_{z\bar z}\left(\partial^z \xi^{\bar z} + \partial^{\bar z}\xi^z\right)\Big\}$$

Note $g^{z\bar z} = g^{\bar z z} = \frac{1}{2}$

Thus

$$\partial^z \xi^z = g^{z\bar{z}} \partial_{\bar{z}} \xi^z = \frac{1}{2} \partial_{\bar{z}} \xi^z = 0$$

since $\xi^z(z)$ only

Similarly $\partial^{\bar{z}} \xi^{\bar{z}} = 0$

$\Rightarrow$

$$\delta S = -\frac{1}{2\pi} \int T_{z\bar{z}} \left(\partial_z \xi^z + \partial_{\bar{z}} \xi^{\bar{z}} \right)$$

If $\delta S = 0$ ie action is invariant under Conformal transformations

$$\theta = \cancel{2} T_{z\bar{z}} = 0 \qquad \text{VANISHING TRACE}$$

Scale invariance. These are specific conformal transformations

$$\xi^z = \alpha z \qquad \xi^{\bar{z}} = \bar{\alpha} \bar{z}$$

$$\int T_{z\bar{z}} \, \partial_z \xi^z = \alpha \int T_{z\bar{z}}$$

Thus

Scale invariance $\Rightarrow \int \theta = 0$

- Holomorphicity

If $\theta = 0$ $\quad T_{zz}(z) \quad T_{\bar{z}\bar{z}}(\bar{z})$

$$\partial^\mu T_{\mu\nu} = \frac{1}{2} \partial_{\bar{z}} T_{zz} + \frac{1}{2} \partial_z T_{\bar{z}z} = 0$$

$$\Rightarrow \partial_{\bar{z}} T_{zz} = 0$$

EM Tensor for free scalar field

$$S_M = \frac{1}{4\pi}\int d^2x \sqrt{g}\, g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$$

$$g = (\det g) = \exp[\mathrm{Tr}\log g]$$

Thus

$$\delta\sqrt{g} = \delta\left(\exp\left[\tfrac{1}{2}\mathrm{Tr}\log g\right]\right)$$

$$\delta(\log g) = \frac{1}{2} g^{-1}\delta g \quad \text{as matrices}$$

But g^{-1} is what we call $g^{\mu\nu}$

$\Rightarrow$

$$\delta\sqrt{g} = \frac{1}{2} g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2} g_{\mu\nu}\delta g^{\mu\nu}$$

$$\delta S_M = \frac{1}{4\pi}\int d^2x\sqrt{g}\left\{\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right\}\delta g^{\mu\nu}$$

Then by definition

$$T_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right)$$

In $z\,\bar{z}$ coordinates

$$(\partial\phi)^2 = 2g^{z\bar{z}}(\partial_z\phi)(\partial_{\bar{z}}\phi)$$

$$T_{zz} = -\frac{1}{2}(\partial_z\phi)^2$$

$$T_{\bar{z}\bar{z}} = -\frac{1}{2}(\partial_{\bar{z}}\phi)^2$$

$$T_{z\bar{z}} = 0$$

OPE's:

Normal order T_{zz}

$$:T_{zz}: = -\frac{1}{2} \lim_{z \to w} \left[\partial_z \phi \partial_w \phi - \langle \partial_z \phi \partial_w \phi \rangle \right]$$

$$= -\frac{1}{2} \lim_{z \to w} \left[\partial_z \phi \partial_w \phi + \frac{1}{(z-w)^2} \right]$$

Consider now

$$T\left[:T_{zz}(z): :T_{ww}(w): \right]$$

By Wick's Theorem this is

$\cancel{:T_{zz}(z) T_{ww}(w): + \ldots}$

$$T\left[\frac{1}{4} :\partial_z \phi \partial_z \phi: \; :\partial_w \phi \partial_w \phi: \right]$$

$$= \frac{1}{4} : :\partial_z \phi \partial_z \phi: :\partial_w \phi \partial_w \phi: :$$

$$+ \frac{1}{4} \cdot 4 \langle \partial_z \phi \partial_w \phi \rangle :\partial_z \phi \partial_w \phi:$$

$$+ \frac{1}{4} \cdot 2 \langle \partial_z \phi \partial_w \phi \rangle^2$$

Since $\langle \partial_z \phi \partial_w \phi \rangle = -\frac{1}{(z-w)^2}$

and

$$\partial_z \phi \partial_w \phi = \left(\partial_w \phi + (z-w) \partial_w^2 \phi + \cdots \right) \partial_w \phi$$

get

$$\frac{1}{2}\frac{1}{(z-w)^4} + \frac{1}{(z-w)^2}\left\{:\partial_w\phi\,\partial_w\phi: + \frac{(z-w)}{2}\partial_w(\partial_w\phi\partial_w\phi)\right\} + \text{finite}$$

$$= \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{finite}$$

Consequences:

- $T(z)$ is not quite holomorphic

$$\frac{1}{(z-w)^4} = -\frac{1}{3!}\partial_z^3\left(\frac{1}{z-w}\right)$$

However

$$\partial_{\bar w}\left(\frac{1}{z-w}\right) = -\pi\,\delta^{(2)}(\sigma_z-\sigma_w)$$

Since

$$\partial_w\partial_{\bar w}\log|z-w| = -\pi\,\delta^{(2)}(\sigma_z-\sigma_w)$$

Thus

$$\partial_{\bar w}\langle T(z)T(w)\rangle = -\frac{\partial_{\bar w}}{2\cdot 3!}\partial_z^3\left(\frac{1}{z-w}\right)$$

$$= \frac{\pi}{12}\partial_z^3\,\delta^{(2)}(\sigma_z-\sigma_w)$$

Other OPE

$$T(z)\partial_w\phi \sim \frac{\partial_w\phi}{(z-w)^2} + \frac{\partial_w^2\phi}{z-w}$$

$$T(z)\,e^{ik\phi} \sim \frac{\frac{k^2}{2}e^{ik\phi}}{(z-w)^2} + \frac{\partial_w(e^{ik\phi})}{(z-w)}$$

Generalizations

$$T(z)\,X(w) = \frac{h\,X(w)}{(z-w)^2} + \frac{\partial_w X}{z-w}$$

$X(w)$: Primary field

EM tensor generates conformal transf.

Consider $z \to z + \varepsilon(z) = z'$

Claim: Generator is

$$\oint \varepsilon(z)\,T(z)$$

For free scalar fields, under this transformation

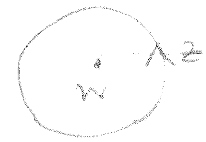
$$\phi(z) \to \phi(z) + \epsilon(z)\,\partial_z\phi$$

$$\partial_z\phi \to \partial_z\phi + (\partial_z\varepsilon)(\partial_z\phi) + \varepsilon\,\partial_z^2\phi$$

Now consider the OPE

$$T(z)\,\partial_w\phi \sim \frac{\partial_w\phi}{(z-w)^2} + \frac{\partial_w^2\phi}{(z-w)}$$

Then this means



$$\oint dz\, \varepsilon(z)\, R\left[T(z)\, \partial\phi(w) \right] = \left[T_\varepsilon ,\ \partial\phi(w) \right]$$

LHS

$$\oint dz\, \varepsilon(z) \left[\frac{\partial_w \phi}{(z-w)^2} + \frac{\partial_w^2 \phi}{(z-w)} + \cdots \right]$$

$$= (\partial_w \varepsilon)(\partial_w \phi) + \varepsilon\, \partial_w^2 \phi$$

which is exactly what we want

For a general $X(w)$ this gives

$$\left[T_\varepsilon ,\ X(w) \right] = h(\partial_w \varepsilon)\, X + \varepsilon\, \partial X$$

In other words

$$X'(z') = \left(\frac{\partial z}{\partial z'} \right)^h X(z).$$

Check

$$z' = z + \varepsilon(z)$$

$$\frac{\partial z}{\partial z'} = 1 - \partial_z \varepsilon$$

$$\left(\frac{\partial z}{\partial z'} \right)^h = 1 - h(\partial_z \varepsilon)$$

$\Rightarrow$

$$X'(z) + \varepsilon\, \partial X = (1 - h\, \partial_z \varepsilon)\, X$$

$$X'(z) - X(z) = - h\, \partial_z \varepsilon - \varepsilon\, \partial_z X$$

Virasoro Algebra

$$T(z) = \sum L_n z^{-(n+2)}$$

Then

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$$

To compute $[L_n, X(w)]$

$$= \oint \frac{dz}{2\pi i} z^{n+1} [T(z), X(w)]$$

$$= \oint_w \frac{dz}{2\pi i} z^{n+1} R[T(z) X(w)]$$

$$= \oint_w \frac{dz}{2\pi i} z^{n+1} \left\{ \frac{hX}{(z-w)^2} + \frac{\partial X}{z-w} \right\}$$

$$= \cancel{\partial_w [w^{n+1} h X] + w^{n+1} \partial X}$$

$$= h(\partial_w w^{n+1}) X + w^{n+1} \partial X$$

$$[L_n, X(w)] = h(n+1) w^n X + w^{n+1} \partial X$$

Same technology

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}$$

Meaning of L_n

Recall
$$\delta X(z) = + h\, \partial_z \varepsilon + \varepsilon\, \partial_z X$$
Compare with action of L_n
$$\varepsilon(z) \sim z^{n+1}$$
Thus L_n generates these conformal transformations

$$\boxed{L_n = z^{n+1} \frac{\partial}{\partial z}}$$

$n = \cancel{0} -1$ $\quad L_{-1} = \partial_z$: translations

$n = 0$ $\quad L_0 = z\partial_z$: Scale transf

$n = 1$ $\quad L_1 = z^2 \partial_z$ special conf.

It is useful to rewrite these as transformations in x, y
$$z = x + iy$$

$$\left.\begin{aligned} L_{-1} + \bar{L}_{-1} &= \partial_x \\ i(L_{-1} - \bar{L}_{-1}) &= \partial_y \end{aligned}\right\} \text{translations.}$$

$$L_0 + \bar{L}_0 = x\partial_x + y\partial_y \quad \text{scaling}$$

$$L_0 - \bar{L}_0 = x\partial_y - y\partial_x \quad \text{rotations.}$$

The last ones are special

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$
$$\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$$

$$z^2\partial_z = \frac{1}{2}(x^2 - y^2 + 2ixy)(\partial_x - i\partial_y)$$
$$\bar z^2 \partial_{\bar z} = \frac{1}{2}(x^2 - y^2 - 2ixy)(\partial_x + i\partial_y)$$

Add

$$z^2\partial_z + \bar z^2\partial_{\bar z} = (x^2-y^2)\partial_x + 2xy\,\partial_y$$
$$z^2\partial_z - \bar z^2\partial_{\bar z} = 2ixy\,\partial_x - i(x^2-y^2)\partial_y$$

Compare with the transformations

$$\delta x = x^2 - y^2 \qquad \delta x = 2xy$$
$$\delta y = 2xy \qquad \delta y = y^2 - x^2$$

Combine to get

$$\delta x^i = 2(\vec\varepsilon \cdot \vec x)x^i - \vec x^2 \varepsilon^i$$

For $\varepsilon^x = \varepsilon \quad \varepsilon^y = 0$ we get one
$\varepsilon^y = \varepsilon, \; \varepsilon^x = 0$ we get the other

Meaning of central charge c

$c \neq 0$ means that $T(z)$ does not transform as a tensor

OPE $\Rightarrow$

$$[T_\varepsilon, T(z)] = 2(\partial\varepsilon)T + \varepsilon(\partial T) + \frac{c}{12}\partial^3\varepsilon$$

This is infinitesimal version of

$$T(z) \to \left(\frac{\partial f}{\partial z}\right)^2 T[f(z)] + \frac{c}{12} S(f,z)$$

where $z \to f(z)$ arbitrary

$$S(f,z) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

This relation shows that c is related to free energy.

Consider some CFT on an infinite line at temp T.

Then $\Rightarrow$ path integral over a cylinder with ~~to~~ circumference $L = 1/T$

u : Complex coordinates on plane

v : Complex coordinates on cylinder

$$u = e^{+\frac{2\pi}{L} v}$$

$$u = x + iy$$
$$v = \tau + i\sigma$$
$$\Rightarrow \quad u = e^{+\frac{2\pi\tau}{L}} e^{+\frac{2\pi i\sigma}{L}}$$

Then

$$\tau = -\infty \quad \text{is} \quad u = 0$$
$$\tau = +\infty \quad \text{is} \quad u = \infty$$

$$T(v) = \left(\frac{\partial u}{\partial v}\right)^2 T(u) + \frac{c}{12}\left[\frac{u'''}{u'} - \frac{3}{2}\left(\frac{u''}{u'}\right)^2\right]$$

$$\frac{\partial u}{\partial v} = \frac{2\pi}{L} u \qquad u'' = \left(\frac{2\pi}{L}\right)^2 u$$
$$u''' = \left(\frac{2\pi}{L}\right)^3 u$$

Thus

$$T(v) = \frac{4\pi^2}{L^2} u^2 T(u) + \frac{c}{12}\left[\left(\frac{2\pi}{L}\right)^2 - \frac{3}{2}\left(\frac{2\pi}{L}\right)^2\right]$$

$$= \frac{4\pi^2}{L^2} u^2 T(u) - \frac{c}{24}\left(\frac{2\pi}{L}\right)^2$$

Thus in the vacuum $\langle T(u)\rangle = 0$

$$\langle T(v)\rangle = -\frac{c}{24}\left(\frac{2\pi}{L}\right)^2$$

From definition of e.m. tensor

Consider cylinder metric

$$ds^2 = d\tau^2 + d\sigma^2$$

$$-\Delta(\log Z) = \frac{1}{2\pi}\int d^2\xi \sqrt{g}\,\langle T_{\mu\nu}\rangle \delta g^{\mu\nu}$$

Consider scaling of $\sigma \to \varepsilon\sigma + \sigma$

Then $\Delta L = \varepsilon L$

$\delta g^{\sigma\sigma} = \varepsilon$ others vanish

$\Rightarrow$

$$-\Delta(\log Z) = -\frac{\varepsilon}{2\pi} V L \frac{c}{12}\left(\frac{2\pi}{L}\right)^2$$

V volume of τ

since $T_{11} = T(v) + \bar{T}(\bar{v}) = \frac{c}{12}\left(\frac{2\pi}{L}\right)^2$

But $\varepsilon = \Delta L / L$

$\Rightarrow$

$$-L\frac{\partial \log Z}{\partial L} = -\frac{V L \pi c}{6 L^2}$$

$\Rightarrow$

$$-\log Z = \frac{c V \pi}{6 L} + \log Z(\infty)$$

If σ direction $\to$ temperature

$LF = -\log Z$

$\Rightarrow$ Central charge gives coefficient in free energy

Zamoldchikov c Theorem

Define $\quad x^2 = \bar{z} z$

Away from conformal point $T, \bar{T}, \Theta$
all nonzero – depend on $z, \bar{z}$
However by rotation + translation sym
Correlators depend only on x
Define

$$C(x, g) = 2 z^4 \langle T_{zz}(z, \bar{z}) \, T(0) \rangle$$
$$E(x, g) = x^2 z^2 \langle T_{zz}(z, \bar{z}) \, \Theta(0) \rangle$$
$$F(x, g) = x^4 \langle \Theta(z, \bar{z}) \, \Theta(0) \rangle$$

Conservation of stress tensor $\Rightarrow$

$$\partial_z \Theta = - \partial_{\bar{z}} T \qquad \partial_{\bar{z}} \Theta = - \partial_z \bar{T}$$

$$\cancel{\partial_{\bar{z}} C = 2 z^4 \langle \partial_{\bar{z}} T \, T(0) \rangle}$$
$$\cancel{= - 2 z^4 \langle \partial_z \Theta(z, \bar{z}) \, T(0) \rangle}$$
$$\cancel{\partial_z C = - 2 z^4 \langle \partial_{\bar{z}} \Theta(z, \bar{z}) \, T(0) \rangle}$$

~~Thus~~ $\cancel{x \partial_x C = z \partial_z C + \bar{z} \partial_{\bar{z}} C}$

$= -$

C, E, H are real

$$C = 2\bar{z}^4 \langle \bar{T}(z,\bar{z})\, \bar{T}(0)\rangle \quad \text{etc.}$$

$$\begin{aligned}
\partial_{\bar z} C &= \partial_{\bar z}\left(2z^4 \langle T(z,\bar z)\, T(0)\rangle\right) \\
&= 2z^4 \langle \partial_{\bar z} T(z,\bar z)\ T(0)\rangle \\
&= -2z^4 \langle \partial_z \Theta\ T\rangle
\end{aligned}$$

$$\bar z \partial_{\bar z} C = -2z^4 \bar z \langle \partial_z \Theta\ T\rangle = -2x^2 z^3 \langle \partial_z \Theta\ T\rangle$$

Similarly

$$z\partial_z C = -2x^2 \bar z^3 \langle \partial_{\bar z}\Theta\ \bar T\rangle$$

$\Rightarrow$

$$x\partial_x C = (z\partial_z + \bar z \partial_{\bar z})\, C$$

$$\boxed{x\partial_x C = -2x^2 \left\{ z^2 \langle z\partial_z \Theta\ T\rangle + \bar z^2 \langle \bar z \partial_{\bar z}\Theta\ \bar T\rangle \right\}}$$

$$E = z^3 \bar z \langle T(z,\bar z)\, \Theta(0)\rangle$$

$$\begin{aligned}
\bar z \partial_{\bar z} E &= \bar z \left[z^3 \langle T(z,\bar z)\, \Theta(0)\rangle + z^3 \bar z \langle \partial_{\bar z} T\ \Theta(0)\rangle \right] \\
&= x^2 z^2 \langle T(z,\bar z)\, \Theta(0)\rangle - z^3 \bar z \langle \partial_z \Theta . \Theta\rangle
\end{aligned}$$

$$\begin{aligned}
z\partial_z E &= x^2 \bar z^2 \langle \bar T(z,\bar z)\, \Theta(0)\rangle \\
&\quad - \bar z^3 z \langle \partial_{\bar z}\Theta \cdot \Theta\rangle
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
x\partial_x E &= 2E + x^2 z^2 \langle \partial_{\bar z} T\ \Theta\rangle \\
&\quad + x^2 \bar z^2 \langle \partial_z \bar T\ \Theta\rangle
\end{aligned}$$

Now start with

$$H = z^2 \bar{z}^2 \langle \theta(z,\bar{z})\, \theta \rangle$$

$$z \partial_z H = 2 z^2 \bar{z}^2 \langle \theta(z,\bar{z})\, \theta(0) \rangle + z^3 \bar{z}^2 \langle \partial_z \theta \;\, \theta(0) \rangle$$

$$= 2H - z^4 \bar{z}^2 \langle \partial_{\bar{z}} T \;\, \theta \rangle$$

$$x \partial_x H = 4H - x^2 \bar{z}^2 \langle \partial_{\bar{z}} T \;\, \theta \rangle - x^2 z^2 \langle \partial_z T \;\, \theta \rangle$$

Add $x \partial_x E$

$$\boxed{x \partial_x E + x \partial_x H = 4H + 2E}$$

On the other hand write

$$E = z^3 \bar{z} \langle \theta(z,\bar{z})\; T(0) \rangle = \bar{z}^3 z \langle \theta(z,\bar{z})\; \bar{T}(0) \rangle$$

$$z \partial_z E = 3E + z^4 \bar{z} \langle \partial_z \theta \;\, T \rangle$$

$\Rightarrow$

$$x \partial_x E = 6E + x^2 z^2 \langle \partial_z \theta \;\, T \rangle + x^2 \bar{z}^2 \langle \partial_{\bar{z}} \theta \;\, \bar{T} \rangle$$

$\Rightarrow$

$$\boxed{x \partial_x E = 6E - \tfrac{1}{2} x \partial_x C}$$

This gives

$$\tfrac{1}{2} x \partial_x (C - 6H - 4E) = -12H$$

Since C, H, E are dimensionless, dependence on x through μx

$$x \partial_x \begin{pmatrix} C \\ H \\ E \end{pmatrix} = +\mu \partial_\mu \begin{pmatrix} C \\ H \\ E \end{pmatrix}$$

$\Rightarrow$

Def $\tilde{C} = C \big|_{x=\mu}$ etc.

$$\boxed{\mu \partial_\mu (\tilde{C} - 6\tilde{H} - 4\tilde{E}) = -24\tilde{H}}$$

Thus the function

$$c(g) \equiv \tilde{C} - 6\tilde{H} - 4\tilde{E}$$

$$\mu \partial_\mu c(g) = -24\tilde{H}$$

Callan Symanzik eqn. for $c(g)$.

General CS equation for Green's function

$$\left(\mu \partial_\mu + \beta_i \cdot \frac{\partial}{\partial g_i} + n\gamma \right) G^{(n)} = 0$$

However for a conserved quantity $\gamma = 0$

$$\Rightarrow \quad \mu \partial_\mu C = -\beta_i \cdot \frac{\partial}{\partial g_i} = 0$$

Thus

$$\boxed{\beta_i(g)\frac{\partial}{\partial g_i} c = +24\tilde{H}}$$

However $\tilde{H} = \frac{1}{\mu^4}\langle \theta(z,\bar{z})\,\theta(0)\rangle_\mu$

For a unitary theory this is positive

$$\langle A(x)A(0)\rangle = \langle 0|e^{ipx}A(0)e^{-ipx}A(0)|0\rangle$$

$$= \langle 0|A(0)|p\rangle e^{-ipx}\langle p|A(0)|0\rangle$$

Thus

$$\beta_i \frac{\partial}{\partial g_i} c > 0$$

$\Rightarrow$ $\boxed{\text{RG flow to IR decreases } c}$

At fixed point $\beta = 0 \Rightarrow \tilde{H} = 0$

$\Rightarrow$ $\theta = 0$

$\Rightarrow$ Conformal invariance

Conformal Transformations in higher dimensions

Translations $P_\mu = -i\partial_\mu$

Dilatations $D = -ix^\mu\partial_\mu$

Rotations $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$

Special $K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$

Now combine them to define

$$J_{ab} \quad a = \underbrace{0, -1, 1 \ldots d}_{d+2}$$

$$J_{\mu\nu} = L_{\mu\nu}$$
$$J_{-10} = D$$
$$J_{0\mu} = \frac{1}{2}(P_\mu + K_\mu)$$
$$J_{-1\mu} = \frac{1}{2}(P_\mu - K_\mu)$$

Then

$$[J_{ab}, J_{cd}] = i(g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} - g_{bd}J_{ac})$$

where

$$g_{ab} = \mathrm{diag}(1, 1, 1, \ldots, -1)$$
$$g_{-1,-1} = -1 \text{ here}$$

This is the group $SO(d+1, 1)$

CONFORMAL TRANSF & AdS

Euclidean conformal group is
$O(d+1,1)$ in d space-time dimensions

This is the symmetry group of a hyperboloid

$$\underbrace{X_1^2 + X_2^2 + X_3^2 + \cdots}_{d+1} - U^2 = -1$$

To write generators:

$$X_1 = iY_1 \quad X_2 = iY_2 \quad . \text{ etc}$$

$$Y_1^2 + Y_2^2 + \cdots Y_{d+2}^2 = 1 \qquad Y_{d+2} = U.$$

Generators are angular momenta

$$L_{ij} = Y^i \frac{\partial}{\partial Y^j} - Y^j \frac{\partial}{\partial Y^i}$$

Count number

$(d+2) \times (d+2)$ antisymmetric matrix

$$N \times N \qquad \frac{N(N-1)}{2} = \frac{(d+2)(d+1)}{2}$$

Conformal group in d dimensions

Rotation	$\frac{d(d-1)}{2}$
Dilatation	1
Translations	d
Special conf	d

$$2d + 1 + \frac{d^2 - d}{2} = \frac{d^2 + 3d + 1}{2} = \frac{(d+1)(d+2)}{2}$$

This hyperboloid $\Rightarrow$ $EAdS_{d+1}$

Eliminate by Choosing (AdS_3)

$$U = \sec\nu \cosh\tau$$
$$X_1 = \sec\nu \sinh\tau$$
$$X_2 = \tan\nu \cos\phi$$
$$X_3 = \tan\nu \sin\phi$$

$$U = \frac{1}{2z}\left[1 + (y^2 + z^2 + t^2)\right]$$
$$X_1 = \frac{t}{z}$$
$$X_2 = \frac{1}{2z}\left[1 - (y^2 + z^2 + t^2)\right]$$
$$X_3 = \frac{x}{z}$$

Check $U^2 - X_1^2 = \sec^2\nu$
$X_2^2 + X_3^2 = \tan^2\nu$ ✓

$$ds^2 = -dU^2 + dX_1^2 + dX_2^2 + dX_3^2$$
$$= \sec^2\nu\left[+d\tau^2 + d\nu^2 + \sin^2\nu\, d\phi^2\right]$$
$$= \frac{1}{z^2}\left[dt^2 + dy^2 + dz^2\right]$$

Symmetry Generators (Lorentzian)

$$w = t + x \qquad \bar{w} = t - x$$

$$L_0 = -\frac{z}{2}\partial_z - w\partial_w$$
$$L_- = i\partial_w$$
$$L_+ = -i\left[wz\partial_z + w^2\partial_w + z^2\partial_{\bar{w}}\right]$$

Analytic continuation $t = i\tau$

$$w = i\tau + x = i(\tau - ix) = iu$$
$$\bar{w} = i\tau - x = i(\tau + ix) = i\bar{u}$$

$$\boxed{\begin{aligned} L_0 &= -\tfrac{1}{2}z\partial_z - u\partial_u \\ L_- &= \partial_u \\ L_+ &= uz\partial_z + u^2\partial_u - z^2\partial_{\bar{u}} \end{aligned}}$$

Suppose this acts on a function

$$f(z, w, \bar{w}) \sim z^{\Delta_-} f_0(u, \bar{u})$$

$$\boxed{\begin{aligned} L_0 &= -\frac{\Delta_-}{2} - u\partial_u \\ L_- &= \partial_u \\ L_+ &= u\Delta_- + u^2\partial_u \end{aligned}}$$

These act as L_0, L_{-1}, L_{+1} on $f(u, \bar{u})$ as if it was a field of $\dim = \Delta_-$

AdS/CFT: SCALARS

$$ds^2 = \left(\frac{L}{z}\right)^2 \left[d\vec{x}^2 + dz^2 \right] \qquad AdS_{d+1}$$

$$S = \frac{1}{2G_N} \int_\epsilon^\infty dz \int d^d x \, \sqrt{g} \left[g^{ab} \partial_a \phi \partial_b \phi + m^2 \phi^2 \right]$$

$$= \frac{1}{2G_N} \int_\epsilon^\infty dz \int d^d x \left(\frac{L}{z}\right)^{d+1} \left[\left(\frac{z}{L}\right)^2 \left[(\partial_z \phi)^2 + (\nabla\phi)^2 \right] + m^2 \phi^2 \right]$$

Evaluate on shell:

Eqns of motion

$$(\nabla^2 - m^2)\phi = 0$$

$$\frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g}\, g^{ab} \partial_b \phi \right) - m^2 \phi = 0$$

$$\frac{1}{z^{d+1}} \partial_a \left(z^{d-1} \partial_a \phi \right) - L^2 m^2 \phi = 0$$

Then

$$S = \frac{1}{2} \left(\frac{L^{d-1}}{G_N} \right) \int_\epsilon d^d x \; z^{1-d} \phi \, \partial_z \phi$$

Solution of wave equation in momentum space

$$\frac{1}{z^{d+1}} \partial_z \left(z^{d-1} \partial_z \phi \right) - \left(\frac{k^2}{z^2} + m^2 L^2 \right) \phi = 0$$

$$\phi(z, k) = \left(\frac{z}{\epsilon} \right)^{d/2} \frac{K_\nu(kz)}{K_\nu(k\epsilon)} \, \phi_0(k)$$

$$\nu = \sqrt{\frac{d^2}{4} + m^2 L^2}$$

For $kz \gg 1$ the solution is regular at $z = \infty$

$$K_\nu(x) \sim e^{-x} \sqrt{\frac{\pi}{2x}}$$

For $kz \ll 1$ $\quad x = kz$

$$K_\nu(x) = \frac{1}{2} \left\{ \Gamma(\nu) \left(\frac{x}{2}\right)^{-\nu} [1 + O(x^2)] \right.$$
$$\left. + \Gamma(-\nu) \left(\frac{x}{2}\right)^{\nu} [1 + O(x^2)] \right\}$$

Write the solution as

$$\phi(z,k) = \frac{A z^{\Delta_-}(1 + O(z^2)) + B z^{\Delta_+}(1 + O(z^2))}{A \epsilon^{\Delta_-}(1 + O(\epsilon^2)) + B \epsilon^{\Delta_+}(1 + O(\epsilon^2))} \phi_0(k)$$

where we took

$$A(k) = \frac{k^{-\nu}}{2^{1-\nu}} \Gamma(\nu) \quad \Big| \quad B(k) = \frac{k^{\nu}}{2^{1+\nu}} \Gamma(-\nu)$$

$$\Delta_\pm = \frac{d}{2} \pm \nu$$

Sufficient to keep the leading term in each class

$$\Delta_+ + \Delta_- = d$$
$$\Delta_+ - \Delta_- = 2\nu$$

$$\left(\partial_z \phi\right)_\epsilon = \frac{A(k)\,\Delta_-\,\epsilon^{\Delta_- -1} + B(k)\,\Delta_+\,\epsilon^{\Delta_+ -1}}{A(k)\,\epsilon^{\Delta_-} + B(k)\,\epsilon^{\Delta_+}}\;\phi_0(k)$$

$$\epsilon^{1-d}\left(\phi\,\partial_z \phi\right)_\epsilon = \frac{A(k)\,\Delta_-\,\epsilon^{\Delta_- -1} + B(k)\,\Delta_+\,\epsilon^{\Delta_+ -1}}{A(k)\,\epsilon^{\Delta_-}}\left(1 + \frac{B(k)}{A(k)}\,\epsilon^{\Delta_+ - \Delta_-}\right)^{-1}\phi_0(k)\,\phi_0(-k)\,\epsilon^{1-d}$$

$$= \left(\frac{\Delta_-}{\epsilon} + \frac{B(k)}{A(k)}\,\Delta_+\,\epsilon^{\Delta_+ - \Delta_- -1}\right)\left(1 + \frac{B(k)}{A(k)}\,\epsilon^{\Delta_+ - \Delta_-}\right)^{-1}\epsilon^{1-d}$$

$$= \left[\frac{\Delta_-}{\epsilon^d} + \frac{B(k)}{A(k)}\left(\Delta_+ - \Delta_-\right)\epsilon^{2\nu - d}\right]\phi_0(k)\,\phi_0(-k)$$

This gives $\quad (\Delta_+ - \Delta_- = 2\nu)$

$$\boxed{S_0 = \frac{1}{2}\frac{L^{d-1}}{G_N}\int \frac{d^d k}{(2\pi)^d}\;\phi_0(k)\,\phi_0(-k)\left\{\frac{\Delta_-}{\epsilon^d} + \epsilon^{-2\Delta_-} H(k)\right\}}$$

$$\boxed{H(k) = 2\nu\,\frac{\Gamma(-\nu)}{\Gamma(\nu)}\,2^{-2\nu}\,k^{2\nu}}$$

GKPW prescription:

$$\left\langle \exp\left[\int d^dk \; \phi_0(k)\left(\frac{L^{d-1}}{G_N}\right)\epsilon^{-\Delta_-}\mathcal{O}(k)\right]\right\rangle_{CFT}$$

$$= e^{-S_{cl}(\phi_0(k))}$$

Holographic Renormalization

The action is divergent - need to add counterterms
- all terms local on boundary

Induced metric is

$$ds_{ind} = \left(\frac{L}{\epsilon}\right)^2 d\vec{x}^2$$

$$\Rightarrow \quad \sqrt{g} = \left(\frac{L}{\epsilon}\right)^d$$

Thus counter-term is

$$\frac{\Delta_-}{2}\frac{1}{G_N L}\int d^dx \sqrt{g}\,\phi^2 \quad \checkmark$$

Renormalized action

$$S_{ren} = \frac{1}{2}\frac{L^{d-1}}{G_N}\int\frac{d^dk}{(2\pi)^d}\,\epsilon^{-2\Delta_-}\phi_0(k)\phi_0(-k)\,H(k)$$

This immediately gives

$$\langle \mathcal{O}(k)\,\mathcal{O}(-k)\rangle = H(k) \sim k^{2\nu}$$

In position space

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta_+}}$$

$\mathcal{O}$ has conformal dimension Δ_+

OTHER BACKGROUNDS

For backgrounds not purely AdS

- But asymptotically AdS
- Same expansion near boundary
- Now $B(k)$, $A(k)$ related by smoothness condition in the interior.

LORENTZIAN SIGNATURE

Now there can be many different Green's functions

With smoothness condition

$H(k) \rightarrow$ Retarded GF

Note

$$\phi(z,k) = z^{\Delta_-}(\phi_0 \epsilon^{-\Delta_-}) + \cdots$$

$$+ z^{\Delta_+}\left(\frac{B}{A}\phi_0 \epsilon^{-\Delta_-}\right) + \cdots$$

$$\langle O \rangle = \frac{\delta \ln Z}{\delta \phi_0 \epsilon^{-\Delta_-}} = H(k)\,(\phi_0 \epsilon^{-\Delta_-})$$

$\Rightarrow \quad \phi_0 \epsilon^{-\Delta_-} \rightarrow$ Source

$H(k) \rightarrow$ Green's function

as in linear response

HOLOGRAPHIC RG

Standard quantization

$$\left\langle \exp \int d^d x \; Z(\epsilon)\, \phi_0(x)\, \mathcal{O}_+(x) \right\rangle_{st} = \Psi[\phi_0(x)]$$

$$\left\langle \exp \int d^d x \; J(x)\, \mathcal{O}_-(x) \right\rangle_{alt}$$

$$= \int \mathcal{D}\phi_0 \left\langle \exp\left[\int \phi_0 Z \mathcal{O}_+ \right] \right\rangle_{st} \exp\left[Z \int \frac{J}{2\nu} \phi_0 \right]$$

Consider the bulk quantity

$$\Psi[\phi_0, \epsilon] = \int_{\phi_b = \phi_0} \mathcal{D}\phi \; e^{-S[\phi]} .$$

Introduce a screen at $z = l$

$$\Psi[\phi_0, \epsilon] = \int \mathcal{D}\tilde{\phi} \; \Psi_{IR}(\tilde{\phi}, l)\, \Psi_{UV}(\tilde{\phi}, \phi_0)$$

$$\Psi_{UV} = \int_{\substack{\phi(\epsilon) = \phi_0 \\ \phi(l) = \tilde{\phi}}} \mathcal{D}\phi \; e^{-\int_\epsilon^l dz\, \mathcal{L}} .$$

Ψ_{UV} satisfies a Schrödinger eqn

$$G_N \frac{\partial}{\partial l} \Psi_{UV} = -H \Psi_{UV} .$$

For field config ind. of $\vec{x}, t$

$$H = \frac{1}{2}\int d^d x \Big[- G_N^2 \left(\frac{\ell}{R}\right)^{d-1} \frac{\delta^2}{\delta\phi^2} + \left(\frac{R}{\ell}\right)^{d-1} (\nabla\phi)^2 + \left(\frac{R}{\ell}\right)^{d+1} m^2\phi^2 \Big]$$

WKB approximation to Schrödinger eqn

$$\psi = e^{K/G_N} \quad \to \text{Ignore } \nabla\phi$$

$$\psi' = \frac{1}{G_N} K' e^K$$

$$\psi'' = \frac{1}{G_N^2}(K')^2 e^K + \frac{1}{G_N} K''$$

$\Rightarrow$

$$\frac{\partial K}{\partial \ell} = + G_N^2 \left(\frac{\ell}{R}\right)^{d-1} \left\{ + \frac{1}{2G_N^2}\left(\frac{\delta K}{\delta\phi}\right)^2 + \frac{1}{2G_N}\left(\frac{\delta^2 K}{\delta\phi^2}\right) \right\} - \frac{1}{2}\left(\frac{R}{\ell}\right)^{d+1} m^2\phi^2$$

$G_N \to 0$ limit

$$\boxed{\frac{\partial K}{\partial \ell} = \frac{1}{2}\left(\frac{\ell}{R}\right)^{d-1}\left(\frac{\delta K}{\delta\phi}\right)^2 - \frac{1}{2}\left(\frac{R}{\ell}\right)^{d+1} m^2\phi^2}$$

Ansatz for K

$$K = \left(\frac{R}{\ell}\right)^d \left[-\frac{1}{2R} g(\ell)\phi^2 + h(\ell)\phi + c(\ell) \right]$$

$$\frac{\delta K}{\delta \phi} = \left(\frac{R}{\ell}\right)^d \left\{ -\frac{1}{R} g(\ell)\phi + h(\ell) \right\}$$

$$\frac{\partial K}{\partial \ell} = -\frac{d}{\ell}\left(\frac{R}{\ell}\right)^d \left[-\frac{1}{2R} g\phi^2 + h\phi + c \right]$$

$$+ \left(\frac{R}{\ell}\right)^d \left[-\frac{1}{2R}(\partial_\ell g)\phi^2 + (\partial_\ell h) + \partial_\ell c \right]$$

$$\frac{\partial K}{\partial \ell} = \left(\frac{R}{\ell}\right)^d \left\{ -\frac{d}{\ell}\left(-\frac{1}{2R} g\phi^2 + h\phi + c \right) + \left(-\frac{1}{2R}(\partial_\ell g)\phi^2 + (\partial_\ell h)\phi + \partial_\ell c \right) \right\}$$

Eqn is

$$\left(\frac{R}{\ell}\right)^d \left\{ + \frac{d}{2R\ell} g\phi^2 - \frac{dh}{\ell}\phi - \frac{cd}{\ell} - \frac{1}{2R}(\partial_\ell g)\phi^2 + (\partial_\ell h)\phi + \partial_\ell c \right\}$$

RHS is

$$\frac{1}{2}\left(\frac{\ell}{R}\right)^d \left(\frac{R}{\ell}\right)\left(\frac{R}{\ell}\right)^{2d} \left(\frac{1}{R} g^2\phi^2 + h^2 - \frac{2gh}{R}\phi \right)$$

$$- \left(\frac{R}{\ell}\right)^{d+1} (m^2\phi^2)$$

$$\left(\frac{R}{\ell}\right)^d \left\{ \left(\frac{d}{2R\ell} g - \frac{1}{2R}\partial_\ell g\right)\phi^2 + \left(\partial_\ell h - \frac{dh}{\ell}\right)\phi + \left(\partial_\ell c - \frac{cd}{\ell}\right)\right\}$$

$$= \left(\frac{R}{\ell}\right)^{d+1} \frac{1}{2}\left\{ \frac{1}{R} g^2\phi^2 + h^2 - \frac{2gh}{R}\phi - m^2\phi^2 \right\}$$

Equate powers of ϕ

$$\frac{d}{2R\ell} g - \frac{1}{2R}\partial_\ell g = \frac{1}{2}\left(\frac{R}{\ell}\right)\left(\frac{1}{R} g^2 - m^2\right)$$

Set $R=1$

$$\ell\partial_\ell g = -g^2 + dg + m^2$$

Fixed point $\quad g_*^2 - dg_* - m^2 = 0$

$$g_* = \frac{1}{2}\left\{ d \pm \sqrt{d^2+4m^2}\right\}$$

$$g = g_* + \delta g$$

$$\ell\partial_\ell \delta g = -(g_* + \delta g)^2 + d(g_* + \delta g) + m^2$$

$$= -(\delta g)^2 - 2g_*\delta g + d\delta g$$

$$d - 2g_* = d - \left(d \pm \sqrt{d^2+4m^2}\right)$$
$$= \mp 2\nu$$

$$\Rightarrow \boxed{-\ell\partial_\ell \delta g = (\delta g)^2 - 2\nu\,\delta g}.$$

CENTRAL CHARGE $\rightarrow$ ANOMALY

On a curved space we can have $\Theta \neq 0$ for an action which is classically Weyl invariant
In 2d

$$\Theta = a_1 R$$

by dimension counting
As tensor

$$T_{z\bar{z}} = \frac{a_1}{2} g_{z\bar{z}} R$$

$$\nabla^{\bar{z}} T_{\bar{z}z} = \frac{a_1}{2} \nabla^{\bar{z}} (g_{z\bar{z}} R) = \frac{a_1}{2} g_{z\bar{z}} \nabla^{\bar{z}} R$$
$$= \frac{a_1}{2} \nabla_z R = \frac{a_1}{2} \partial_z R$$

By conservation

$$\nabla^{\bar{z}} T_{\bar{z}z} + \nabla^z T_{zz} = 0$$

Thus

$$\nabla^z T_{zz} = -\frac{a_1}{2} \partial_z R$$

Now consider a Weyl transformation on both sides by

$$g_{ab} \to e^{2\omega} g_{ab} \sim (1 + 2\omega) g_{ab}$$

$$\text{ie} \quad \delta g_{ab} = 2\omega g_{ab}$$

Under Weyl

$$\sqrt{g'} R' = \sqrt{g} (R - 2\nabla^2 \omega)$$

If g' is flat $\quad R = 2\nabla^2 \omega$

~~Now, under a transf~~

$$\cancel{z \to z + v(z)} \quad \text{~~infinitesimal~~}$$

$$\cancel{\delta T(z) = -\frac{c}{12}\partial_z^3 v - 2(\partial_z v)T - v\partial_z T}$$

~~Since~~

$$R' = e^{-2\omega}\left(R - 2\nabla^2\omega\right)$$

For small ω

$$R' = (1-2\omega)(R - 2\nabla^2\omega)$$

$$= R - 2\omega R - 2\nabla^2\omega$$

$$\delta R = -\,2\omega R - 2\nabla^2\omega$$

Consider a metric close to flat space Then $\delta R \sim -2\nabla^2\omega$

Thus

$$\delta\left(-\frac{a_1}{2}\partial_z R\right) \sim a_1\,\partial_z\nabla^2\omega$$

But $\nabla^2 \approx g^{z\bar z}\partial_z\partial_{\bar z}\omega = 4\partial_z\partial_{\bar z}\omega$

$$\delta\left(-\frac{a_1}{2}\partial_z R\right) = 4a_1\,\partial_{\bar z}\partial_z^2\omega$$

Under a conformal transf

$$\delta T(z) = -\frac{c}{12}\partial_z^3 v - 2(\partial_z v)T - v\partial_z T$$

This changes the metric to

$$(1 + \partial_z v)\, dz\, d\bar{z}$$

$\Rightarrow$ equivalent to a coord. transf. + Weyl transf by

$$\omega = \frac{1}{2}\partial_z v \qquad \partial_z v = 2\omega$$

Thus

$$\delta_W T_{zz} = -\frac{c}{6}\partial_z^2 \omega$$

$\Rightarrow$

$$\delta_W \nabla^2 T_{zz} \sim \delta_W \partial^2 T_{zz}$$

$$= \partial^2 \delta_W T_{zz} = g^{z\bar{z}} \delta_W \partial_{\bar{z}} T_{zz}$$

$$= \underline{2}\, \partial_{\bar{z}} \delta_W T_{zz} = -\frac{c}{3}\partial_{\bar{z}}\partial_{zz}\omega$$

Thus $\quad -\frac{c}{3} = 4a_1 \quad \Rightarrow \quad a_1 = -\frac{c}{12}$

$$\Rightarrow \quad \boxed{\Theta = -\frac{c}{12} R}$$

<u>a - theorem in $d = 4$</u>

$$\langle T^\mu_\mu \rangle = -aE_4 + cW^2$$

$$E_4 = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \quad \text{Euler}$$

$$W^2 = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

$(\text{Weyl})^2$

Conjecture: $\exists$ a a-theorem.

<u>f theorem in 3 dim</u>

$\Rightarrow$ Finite part of free energy on S^3 decreases.

KUBO FORMULA - VISCOSITY

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu}$$

where

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \Big[\eta \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha \right) + \left(\zeta - \tfrac{2}{3}\eta \right) g_{\alpha\beta} \nabla . u \Big]$$

and

$$P^{\mu\alpha} = g^{\mu\nu} + u^\mu u^\nu$$

$\longrightarrow$ projection operator onto directions perp to u^μ $\quad (u^\mu u_\mu = -1)$

$$\Big[\; u_\mu P^{\mu\nu} u_\nu = (g^{\mu\nu} + u^\mu u^\nu) u_\mu u_\nu = -1 + 1 = 0 \Big]$$

Consider fluid in a metric

$$g_{ij} = \delta_{ij} + h_{ij}(t)$$

$$g_{00} = -1 \qquad g_{0i} = 0$$

Then for fluid at rest

$$u^\mu = (1, 0, 0, 0)$$

$$T^{xy} = -\sigma^{xy}$$

Compute

$$\sigma^{xy} = 2\eta \, \Gamma^0_{xy} = \eta \, \partial_0 h_{xy}$$

Momentum space

$$T^{xy}(\omega) = -\eta (i\omega) h_{xy}$$

However linear response requires

$$\langle O \rangle (\omega) = - J(\omega)\, G_R(\omega)$$

Here source is $h_{xy}(\omega)$

$$\sigma_{xy}(\omega) = - h_{xy}(\omega)\, G_R(\omega)$$
$$- i\eta\omega = - G_R(\omega)$$

Generally can be

$$- i\eta\omega + O(\omega^2) = G_R(\omega)$$

$$\Rightarrow \quad \eta = -\frac{1}{\omega} \operatorname{Im} G_R \Big|_{\omega \to 0}$$