Entanglement growth and OTOC in many-body localized (MBL) systems

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Quantum chaos

- Classical chaos

$$\frac{\partial x(t)}{\partial x(0)} \sim e^{\lambda_L t}$$

λ_L, Lyapunov exponent

- Quantum chaos

→ ‘Semiclassical billiards’

Larkin & Ovchinikov (1969)

- Semiclassical limit

$$\frac{\partial x(t)}{\partial x(0)} = \{x(t), p\} \;\Rightarrow\; \frac{1}{i\hbar}[x(t), p(0)]$$

- Chaos correlator $C(t) = -\langle [x(t), p(0)]^2 \rangle$

$$C(t) \sim \hbar^2\, e^{2\lambda_L t}$$

‘Srambling’ or
‘Ehrenfest’ time

$$t_* \sim \frac{1}{\lambda_L} \ln\left(\frac{1}{\hbar}\right)$$

$$C(t_*) \sim 1$$

Large window, $\frac{1}{\lambda_L} < t < t_*$, of exponential growth for $\hbar \to 0$.

Generalize to quantum chaotic (interacting) many-body systems

$$C(t) = -\langle [A(t), B(0)]^2 \rangle$$

$$[A, B] = 0$$

$\langle \cdot \rangle = \frac{1}{Z} Tr\left(e^{-\beta H}\ \cdot\right)$ or $\langle \psi_0 | \cdot | \psi_0 \rangle$

$$A(t) = e^{i\mathcal{H}t} A e^{-i\mathcal{H}t} = A + it[\mathcal{H}, A] - \frac{t^2}{2!}[\mathcal{H}, [\mathcal{H}, A]] - \frac{it^3}{3!}\left[\mathcal{H}, \left[\mathcal{H}, [\mathcal{H}, A]\right]\right] + \cdots$$

Chaotic evolution → For generic interacting $\mathcal{H}$

Local operator will grow in size encompassing the whole system

$$C(t) \sim \epsilon\, e^{\lambda_L t} \qquad \epsilon \to \hbar^2$$

→Information scrambling

→Thermalization (?)

Out-of-time-order (OTO) correlator

Decays exponentially

$$F(t) = \langle A(t) B(0) A(t) B(0) \rangle \sim 1 - \epsilon e^{\lambda_L t}$$

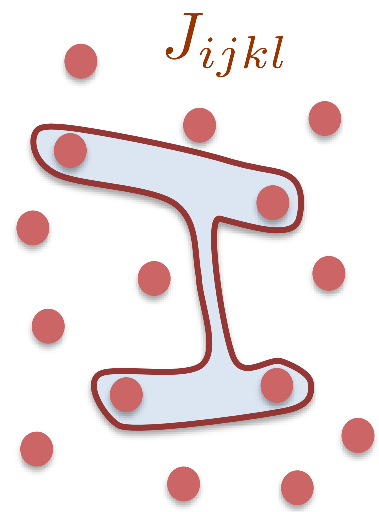
$$F_{MS}(t) = \frac{1}{z} Tr(e^{-\frac{\beta H}{4}} A(t) e^{-\frac{\beta H}{4}} B(0) e^{-\frac{\beta H}{4}} A(t) e^{-\frac{\beta H}{4}} B(0))$$

0-dimensional model: Sachdev-Ye-Kitaev (SYK) model

Sachdev & Ye, PRL (1993)
Kitaev, KITP (2015)
Sachdev, PRX (2015)

$$H_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l - \mu \sum_{i}^{N} c_i^\dagger c_i$$

$$P(J_{ijkl}) \sim e^{-\frac{|J_{ijkl}|^2}{J^2}}$$



N sites

Out-of-time-order correlation

$$\langle\, c_i^\dagger(t) c_j(0) c_i^\dagger(t) c_j(0) \rangle \sim 1 - \left(\frac{\beta J}{N}\right) e^{\lambda_L t}$$

$$\lambda_L = 2\pi T$$

Fastest scrambler! Maximally chaotic.

Upper bound to quantum chaos

Maldacena, Shenker & Stanford (2015)

Srambling time

$$t_* \sim \frac{1}{\lambda_L} \ln N$$

$$N \to \; 1/\hbar$$

Large window for detecting exponential decay for large N

Can the exponential growth window be large enough beyond semiclassical or large-N models to disentangle Lyapunov growth from other effects?

For example, conventional quantum many-body models (Hubbard model, spin models, …) ?

Answer seems to be negative till now (within limited system sizes accessible via numerics).

Are OTOCs good for anything in conventional condensed matter systems?

→ Diagnosing interaction effects in localized systems.

Other quantities related to OTOC, quantum chaos, operator and/or entanglement growth, thermalization

Loschmidt echo

Kurchan (2017)

Fidelity for 'kicked' perturbation

$$F = \mathrm{Tr}[A_{tran} A] = \mathrm{Tr}\left\{ \left[e^{i\frac{t}{\hbar}H} e^{i\frac{\delta}{\hbar}B} e^{-i\frac{t}{\hbar}H} \right] A \left[e^{i\frac{t}{\hbar}H} e^{i\frac{\delta}{\hbar}B} e^{-i\frac{t}{\hbar}H} \right]^{\dagger} A \right\}$$

Choose $\quad B(t) = e^{i\frac{t}{\hbar}H} B e^{-i\frac{t}{\hbar}H}$

$$Tr[A^2] - Tr[A_{tran}A] = -\frac{\delta^2}{2\hbar^2} Tr([B(t), A(0)]^2) + O(\delta^3)$$

1. Loschmidt echo $A = |\psi\rangle\langle\psi| \quad \Rightarrow \quad F = \left| \left\langle \psi \middle| e^{\frac{it}{\hbar}H} e^{\frac{i\delta}{\hbar}B} e^{-\frac{it}{\hbar}H} \middle| \psi \right\rangle \right|^2$

2. OTOC $A \propto e^{-\frac{\beta H}{4}} A e^{-\frac{\beta H}{4}} \quad \Rightarrow \quad Tr([B(t), A(0)]^2) = F_{MS}(t)$

Operator growth,
Lieb-Robinson bound

$$[A(x), B] = 0$$

$$A(t) = e^{i\mathcal{H}t} A e^{-i\mathcal{H}t} = A + it[\mathcal{H}, A] - \frac{t^2}{2!}[\mathcal{H}, [\mathcal{H}, A]] - \frac{it^3}{3!}\big[\mathcal{H}, [\mathcal{H}, [\mathcal{H}, A]]\big] + \cdots$$

$$\| [A(x,t), B(0,0)] \| \leq a \| A \| \| B \| e^{-b(|x| - v_{LR} t)}$$

Light-cone spreading

How to detect the operator growth?

Standard two-point correlation function

$\langle [A(x,t), B] \rangle$ does not capture the growth for “physical” states $|\psi\rangle$ or ρ_{th}

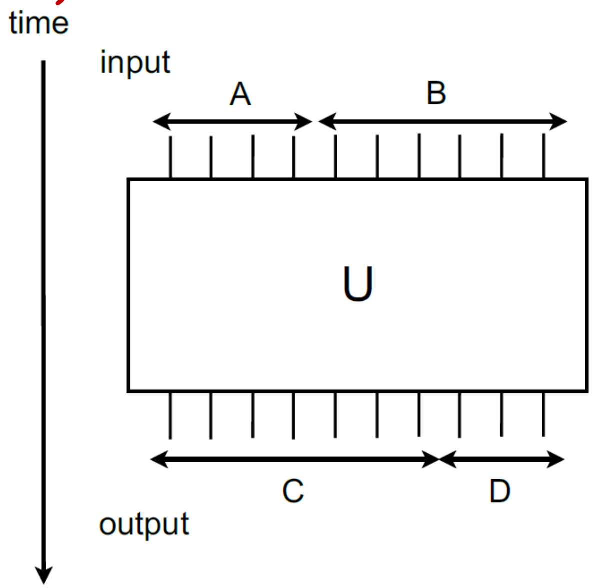
* Can use eigenvector of $i[A(x,t_0), B]$ with largest eigenvalue, but not very physical state.

OTOC, $\langle [A(x,t), B]^2 \rangle$ can capture the growth

$\sim \frac{1}{N} e^{\lambda_L \left(t - \frac{x}{v_B}\right)}$ for chain of SYK dots.

Entanglement growth

Hosur et al. (2016)



$$|\langle \mathcal{O}_D(t)\, \mathcal{O}_A\, \mathcal{O}_D(t)\, \mathcal{O}_A \rangle| \;\sim\; 2^{-S^{(2)}_{AC}}$$

Unitary quantum channel

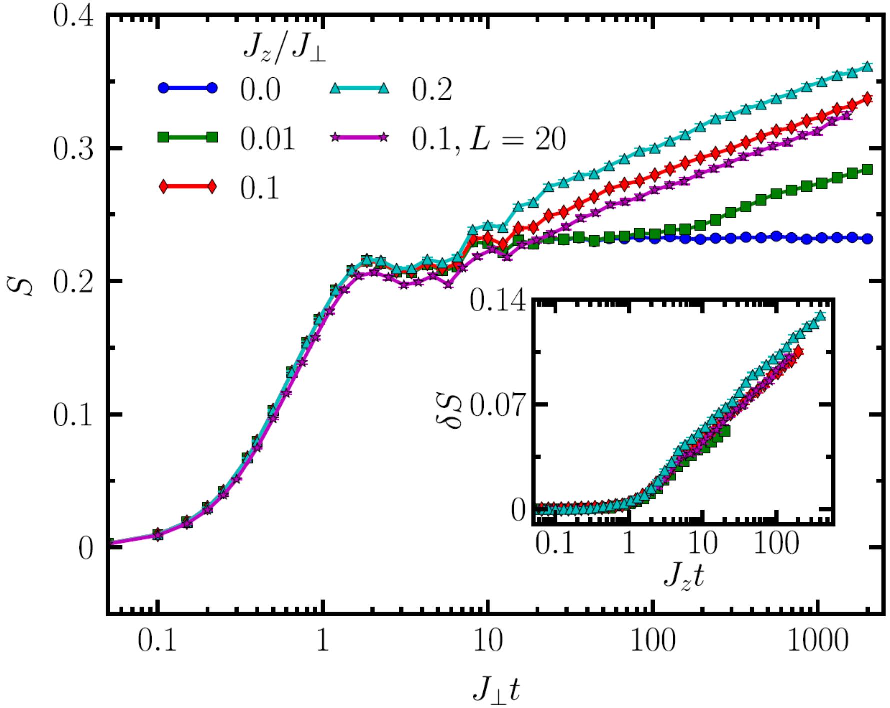
$$|U(t)\rangle$$

Second Renyi entropy

$$S^{(2)}_{AC}(t) \;\leftarrow\; |U(t)\rangle$$

Growth of entanglement in interacting disordered systems and characterizing MBL

Bardarson et al. (2012)



How to distinguish MBL and (non-interacting) Anderson localized phases?

→ Growth of subsystem entanglement entropy.

→ $S_{ent}(t \to \infty) \sim L$ (subthermal volume law)

$$S_{ent} \sim \xi \log(J_z t)$$

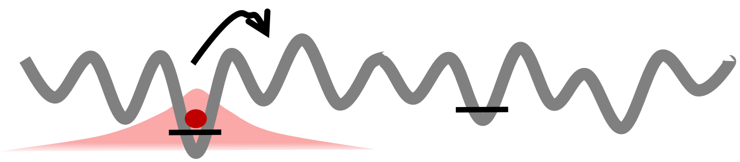
→ OTOC growth.

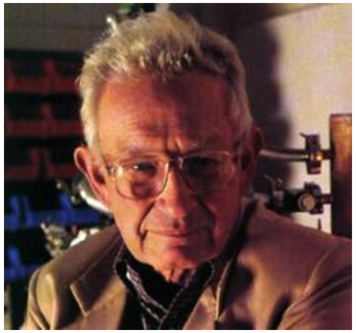
$$\| [A(x,t), B(0,0)] \| \leq C e^{-(|x| - v_{LR} \log(t))/\xi}$$

Logarithmic light-cone spreading in MBL

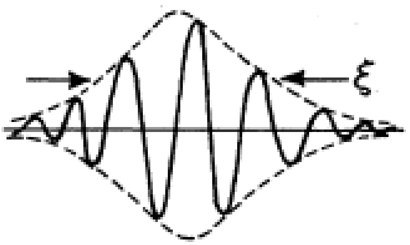
Anderson localization

Anderson (single-particle) localization (1958)





Localized



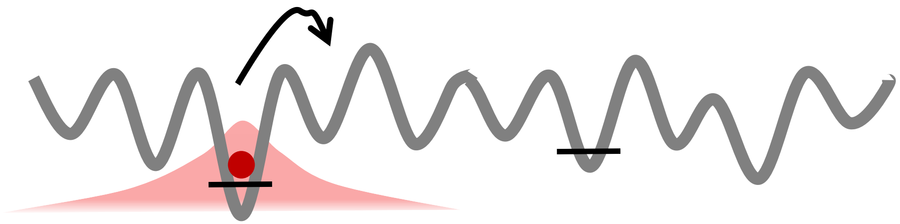
$$|\psi_\alpha(r)|^2 \sim e^{-\frac{|r-r_\alpha|}{\xi}}$$

Extended

$$\sim \frac{1}{L^{d/2}}$$

Abrahams et al. Scaling theory of localization (1979)
Lee & Ramakrishnan (1985), …

Many-body localization (MBL)



All states localized

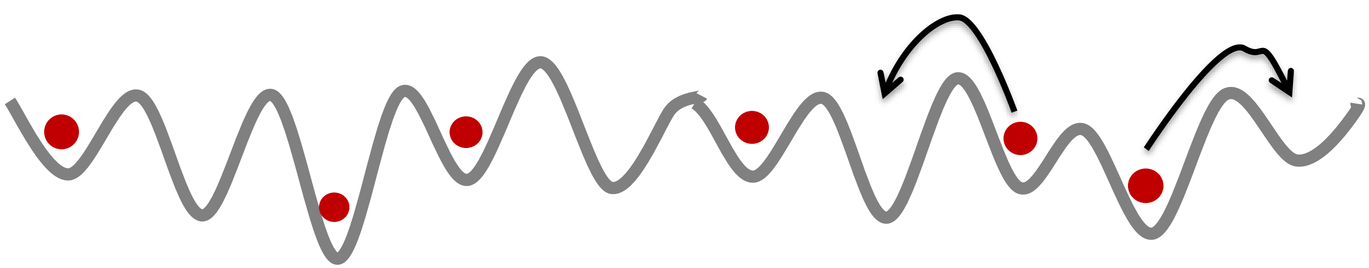
$$|\psi_\alpha(r)|^2 \sim \frac{e^{-\frac{|r-r_\alpha|}{\xi}}}{\xi^d}$$

$$\mathcal{H} = -t\sum_{<ij>}\left(c_i^\dagger c_j + h.c.\right) - \sum_i \varepsilon_i n_i \quad + V\sum_{\langle ij\rangle} n_i n_j$$

$$\varepsilon_i \in [-W, W]$$

$$= \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha \quad + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$

Add interaction



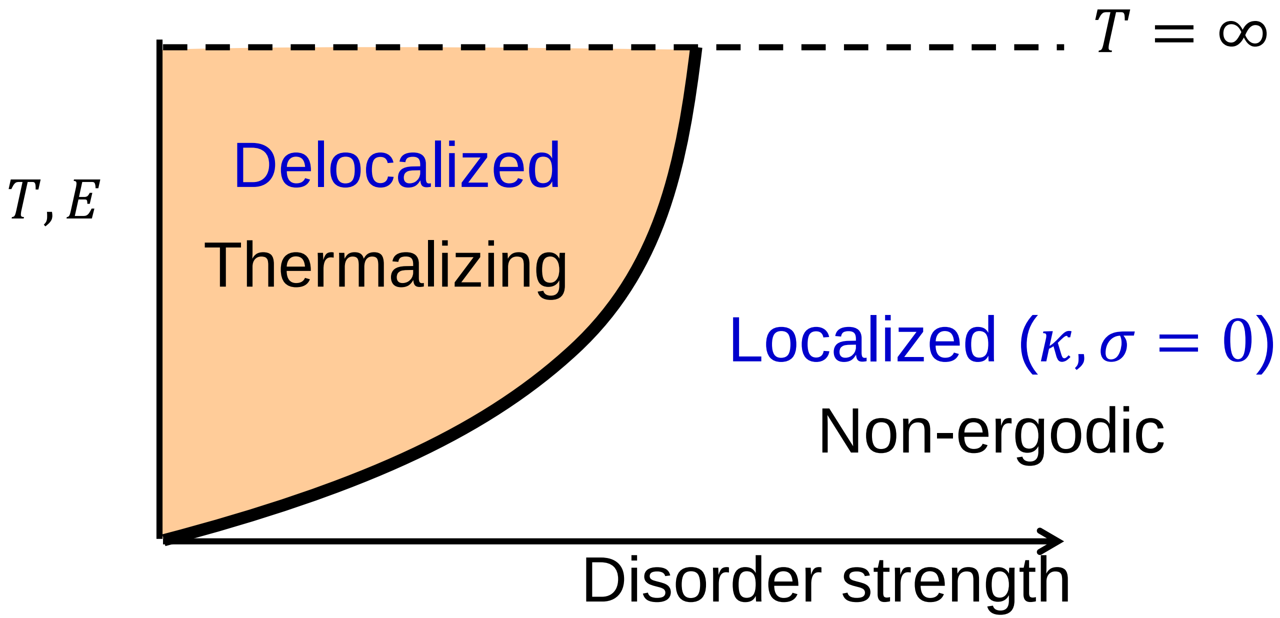
At high energies interaction connects between $\sim \exp(L^d)$ localized states ! Can localization survive?

Many-body localization (MBL)

Yes!

For sufficiently strong disorder

Basko, Aleiner, Altshuler (2005); Gornyi, Mirlin, Polyakov (2005)



Oganesyan and Huse (2007), Pal and Huse (2010), ...

Models for MBL

- Disordered interacting fermions

$$\mathcal{H} = -t \sum_{<ij>} (c_i^\dagger c_j + h.c.) - \sum_i \varepsilon_i n_i + V \sum_{\langle ij \rangle} n_i n_j$$

$\varepsilon_i \in [-W, W]$

Hilbert space dimension

$D = 2^N$

- Disordered spin chains

$$\mathcal{H} = J \sum_i (S_i^+ S_{i+1}^- + h.c.) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

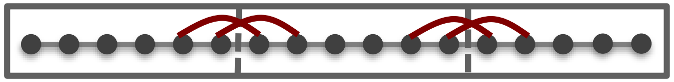
h_i, random

- Disordered Hubbard model, transverse field Ising model, ...

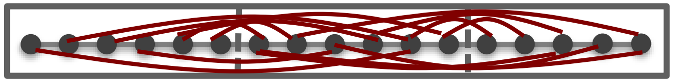
Many-body eigenstates $\mathcal{H}|\Psi_n\rangle = E_n|\Psi_n\rangle$ Bounded spectrum

Localization and thermalization in eigenstates

MBL



Dynamical
transition



MBL | Thermal

E, T, disorder, U

1. Memory of local initial Condition persists to $t \to \infty$

2. Quantum dynamics

3. Area law, $S_A \sim L^{d-1}$, ETH violated.

4. Discrete local spectra

5. Logarithmic entanglement growth to subthermal volume law.

6. OTOC → Logarithmic light-cone spreading

1. Memory of local initial condition is lost

2. Classical hydrodynamics

3. Volume law, $S_A \sim L^d$, ETH satisfied.

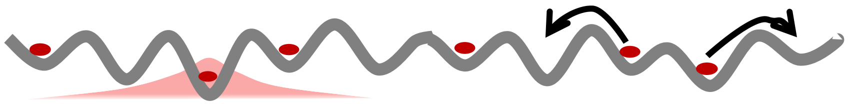
4. Continuous local spectra

5. Power-law entanglement growth to thermal volume law.

6. Linear light-cone spreading.

Effective model of MBL

$$H = \sum_{\alpha} \epsilon_\alpha c_\alpha^\dagger c_\alpha + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$



$$c_\alpha^\dagger = \sum_i \psi_\alpha(i) c_i^\dagger$$

MBL fixed point → Emergent integrability
→ Quasi local integrals of motion
(LIOMs or "l-bits")

Huse & Oganesyan (2013)
Serbyn et al. (2013)

$$\tilde{n}_\alpha = \tilde{c}_\alpha^\dagger \tilde{c}_\alpha = 0,1$$

$$H = \sum_\alpha \epsilon_\alpha \tilde{n}_\alpha + \sum_{\alpha\beta} J_{\alpha\beta}\, \tilde{n}_\alpha \tilde{n}_\beta + \cdots$$

$$J_{\alpha\beta} = J e^{-|r_\alpha - r_\beta|/\xi}$$

$\tilde{c}_\alpha^\dagger = U c_\alpha^\dagger U^\dagger$, Quasi local unitary transformation U

$$\tilde{c}_\alpha^\dagger \simeq c_\alpha^\dagger + \sum_{\beta\gamma\delta} \frac{V_{\delta\gamma\beta\alpha}}{\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta} c_\delta^\dagger c_\gamma^\dagger c_\beta + \cdots$$

Basko, Aleiner, Altshuler (2005)

$$[\tilde{n}_\alpha, H] = [\tilde{n}_\alpha, \tilde{n}_\beta] = 0$$

MBL eigenstates, $|E, \{\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N\}\rangle$

Spin model

$$H = J\sum_i (S_i^+ S_{i+1}^- + h.c.) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

h_i, random

LIOMs $\{\tau_i^z\}$

$\rightarrow$ $H = \sum_i \tilde{h}_i \, \tau_i^z + \sum_{ij} J_{ij} \tau_i^z \tau_j^z + \cdots$

$J_{ij} \sim e^{-|r_i - r_j|/\xi}$

$$S_i^\alpha = U^\dagger \tau_i^\alpha U = \sum_{j\beta} c_{j\beta} \, \tau_j^\beta + \sum_{jk\beta\gamma} c_{jk\beta\gamma} \tau_j^\beta \tau_k^\gamma + \cdots$$

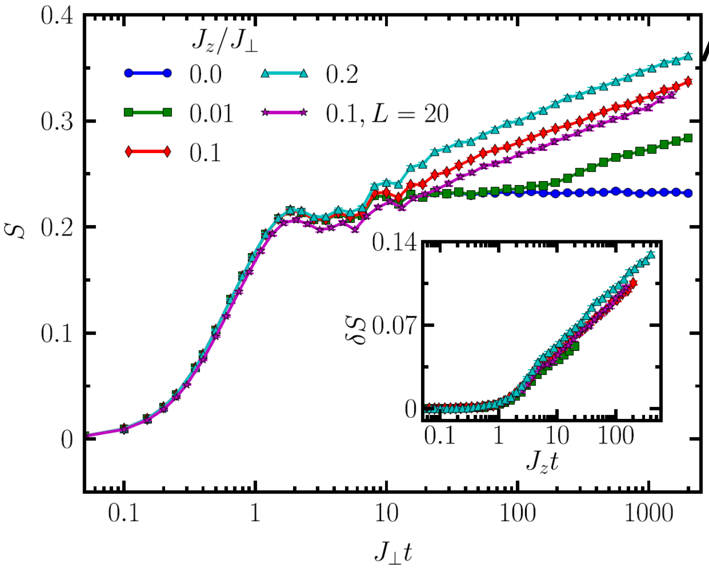
MBL eigenstates, $|E, \{\tau_1^z, \tau_2^z, \ldots, \tau_N^z\}\rangle$

Effective Hamiltonian corresponding to Dasgupta-Ma real-space strong-disorder RG fixed point

Vosk & Altman (2012,2014)

How to understand $log(t)$ entanglement growth in MBL phase?

A volume law $S_{ent}(t \to \infty) \propto L$



Correlation functions (Two-point)

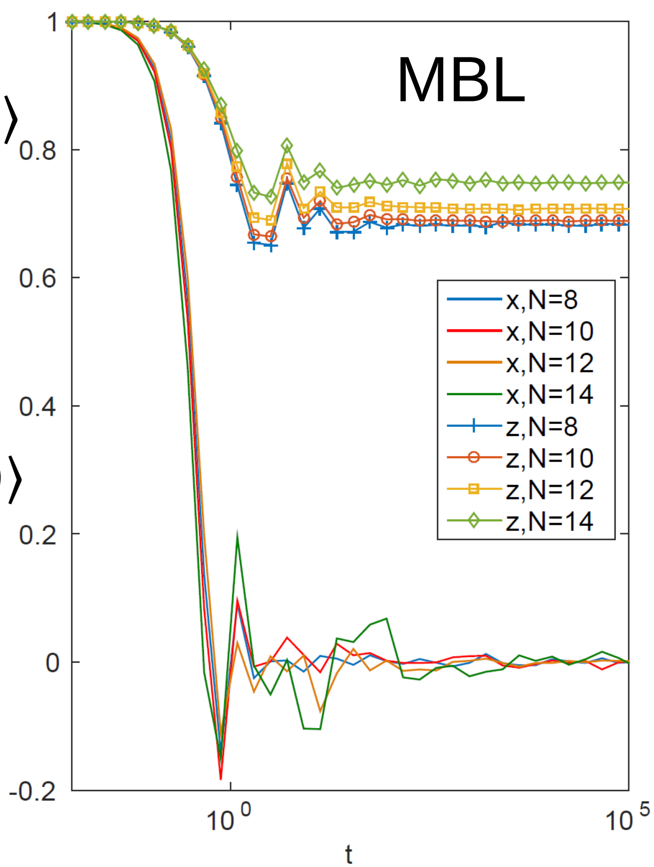
Infinite temperature
$\beta = 0$

$\langle \tau_i^z(t)\tau_j^z(0)\rangle =$ constant $\qquad \langle \tau_i^x(t)\tau_j^x(0)\rangle$

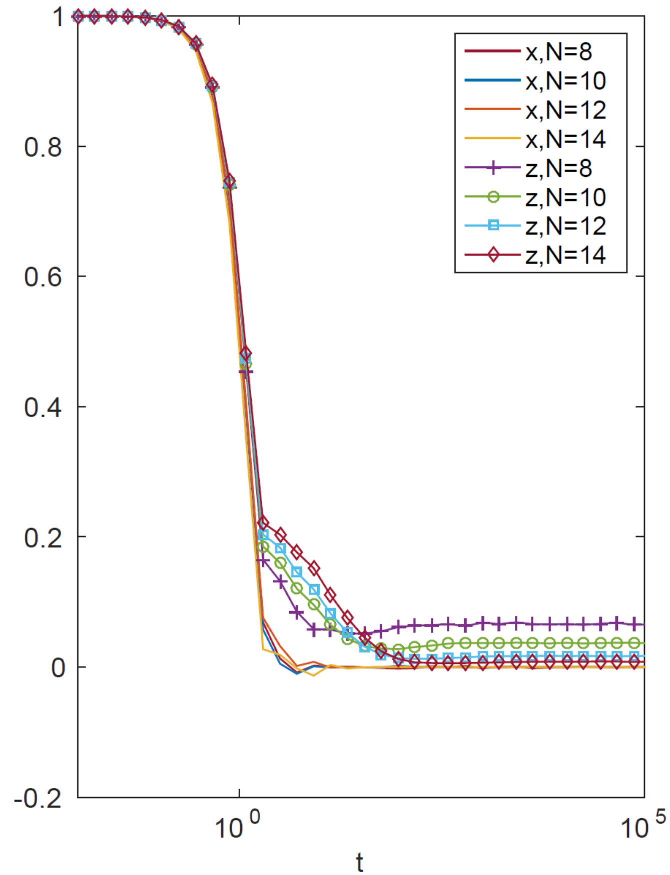
$\langle S_i^z(t)S_j^z(0)\rangle \qquad S_i^z = c_i\tau_i^z + c_{ix}\,\tau_i^x + \cdots$

$\langle S_i^z(t)S_j^z(0)\rangle$

$\langle S_i^x(t)S_j^x(0)\rangle$



Thermal



OTOC

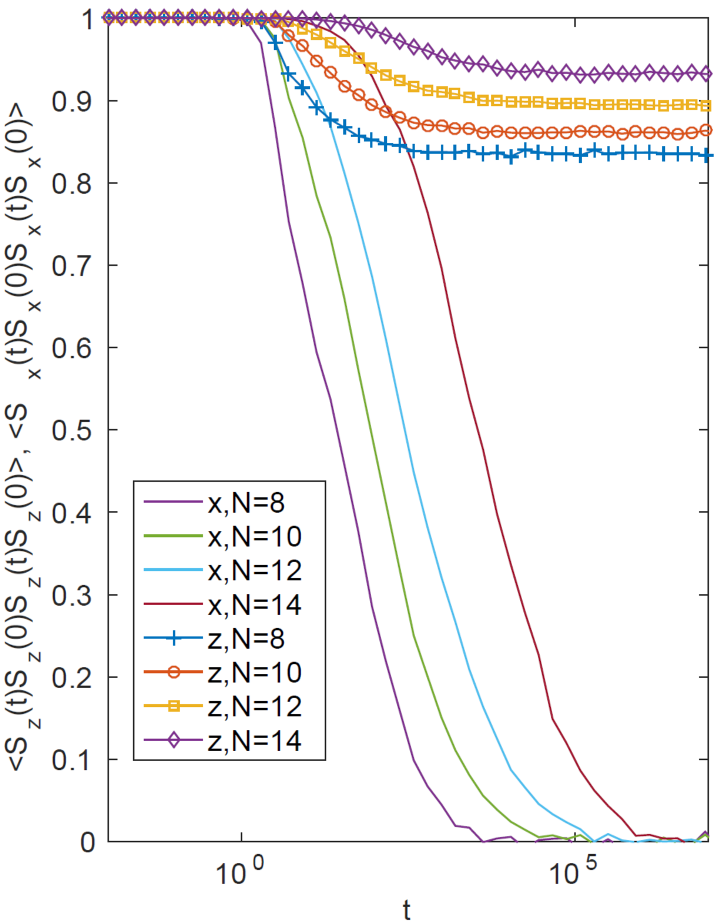
$$\langle \tau_i^z(t)\tau_j^z(0)\tau_i^z(t)\tau_j^z(0)\rangle = \text{constant}$$

$$\langle \tau_i^x(t)\tau_j^x(0)\tau_i^x(t)\tau_j^x(0)\rangle$$

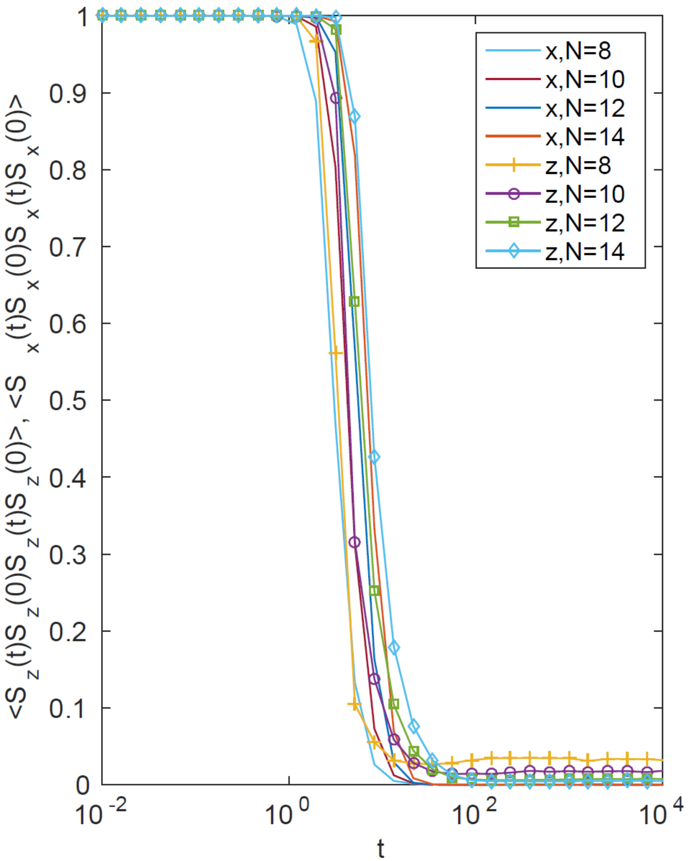
$$\langle S_i^z(t)S_j^z(0)S_i^z(t)S_j^z(0)\rangle$$

$$S_i^z = c_i\tau_i^z + c_{ix}\,\tau_i^x + \cdots$$

MBL



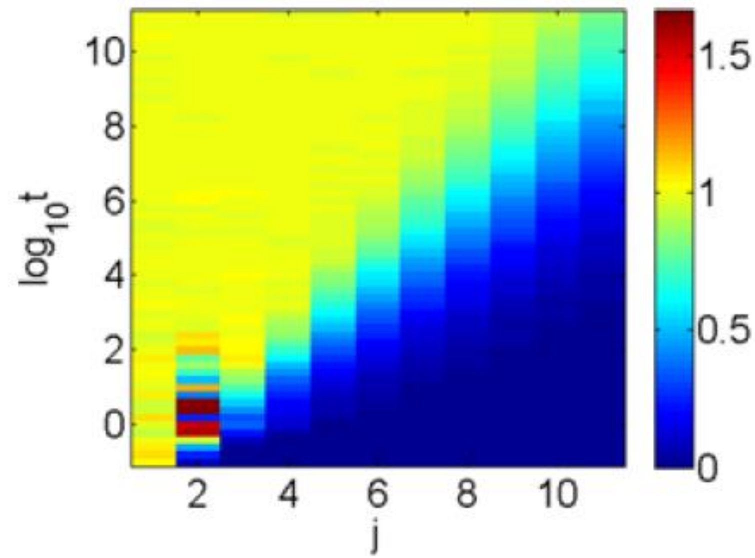
Thermal



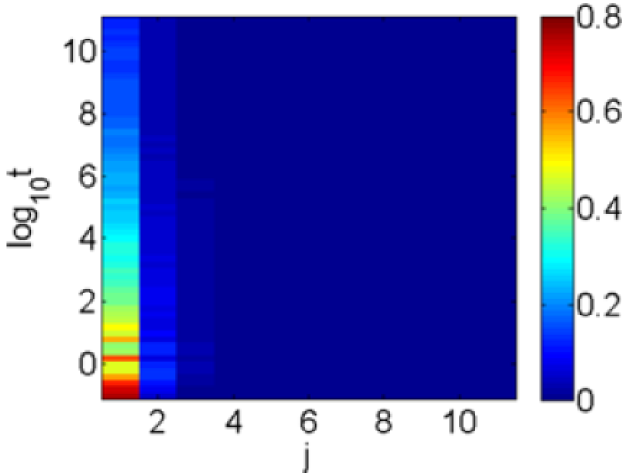
Logarithmic light-cone spreading in OTOC in MBL phase

Huang et al., Ann. Phys. (2017)

MBL

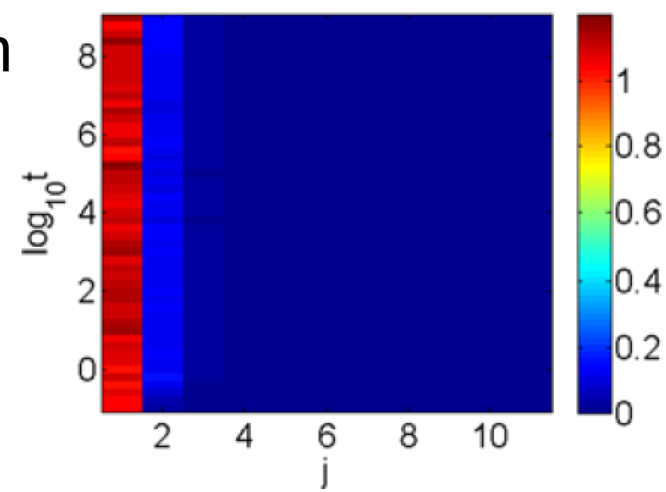


$$1 - \mathrm{Re}\langle \sigma_1^x \sigma_j^x(t) \sigma_1^x \sigma_j^x(t) \rangle_{\mathrm{th}}$$

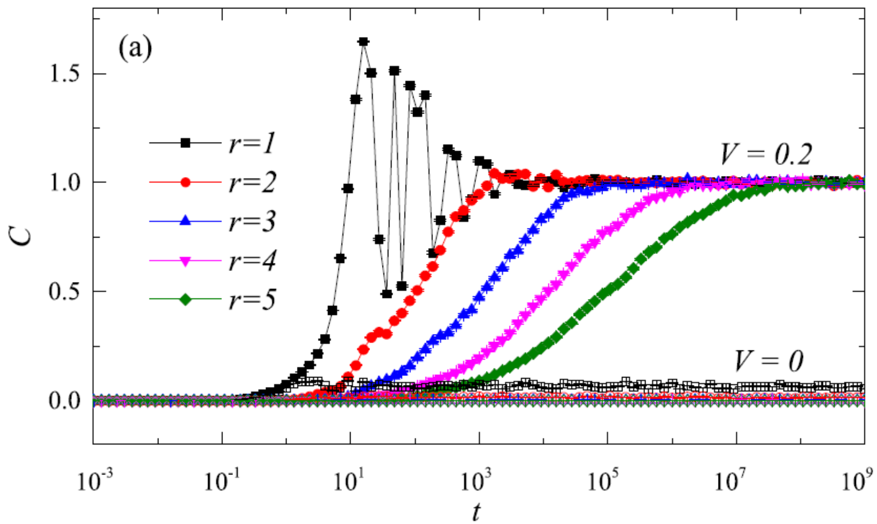


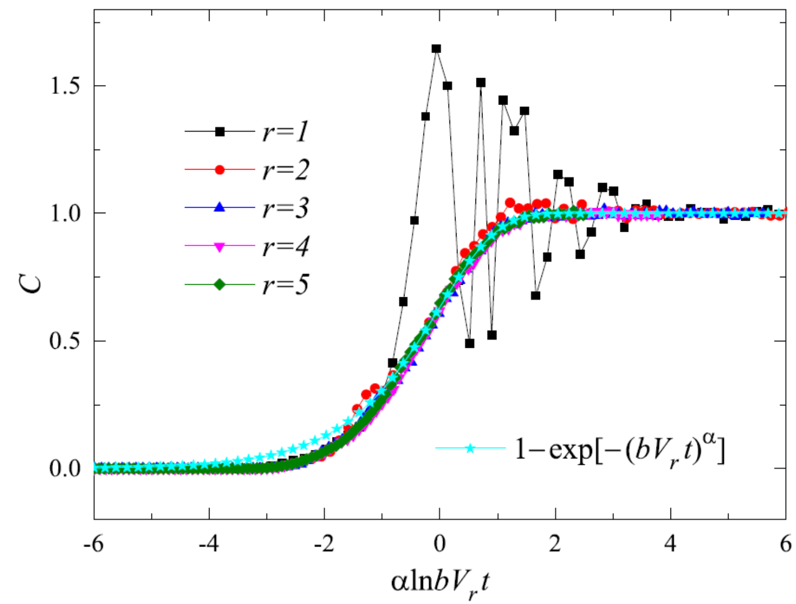
$$|\langle i[\sigma_1^x, \sigma_j^x(t)] \rangle_{\mathrm{th}}|$$

Anderson



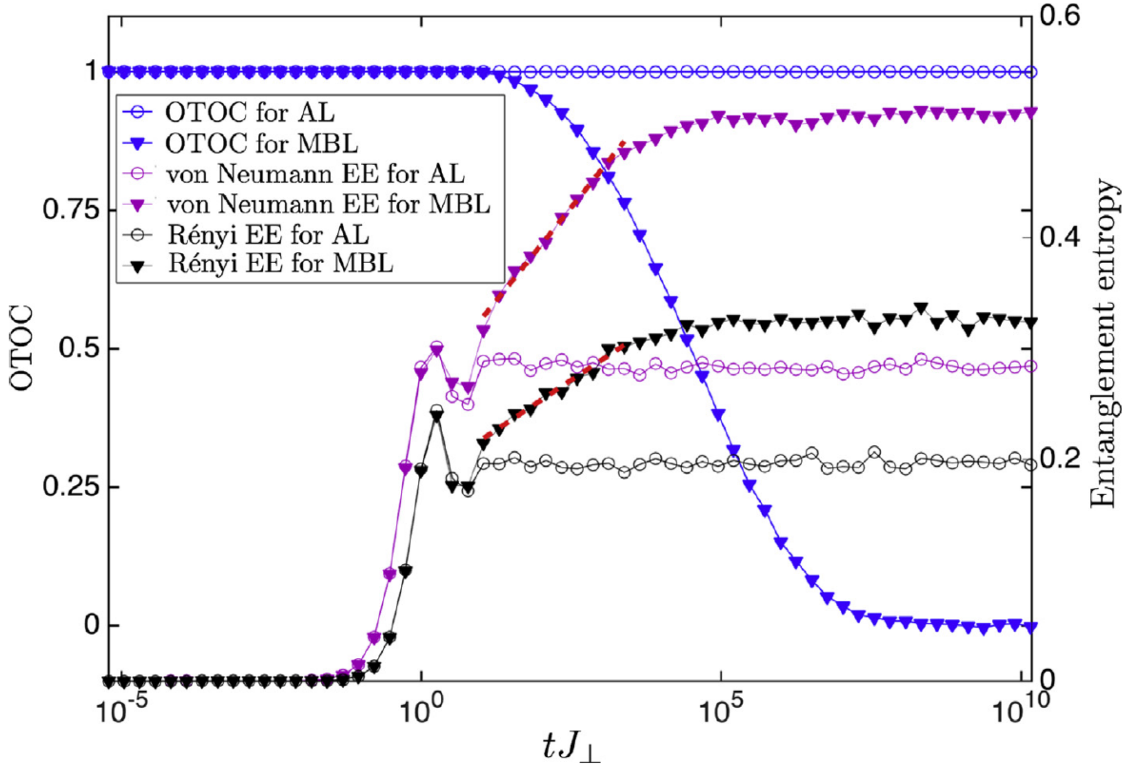
He & Lu., PRB (2017)





OTOC and entanglement growth

Fan et al., Sci. Bulletin (2017)

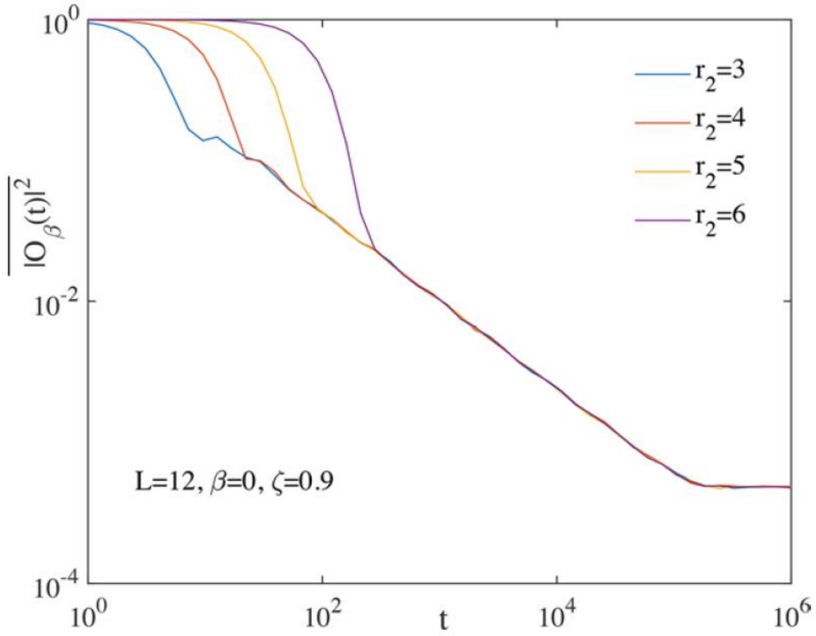


How OTOC behaves in short, intermediate and long times.

Numerics in effective model

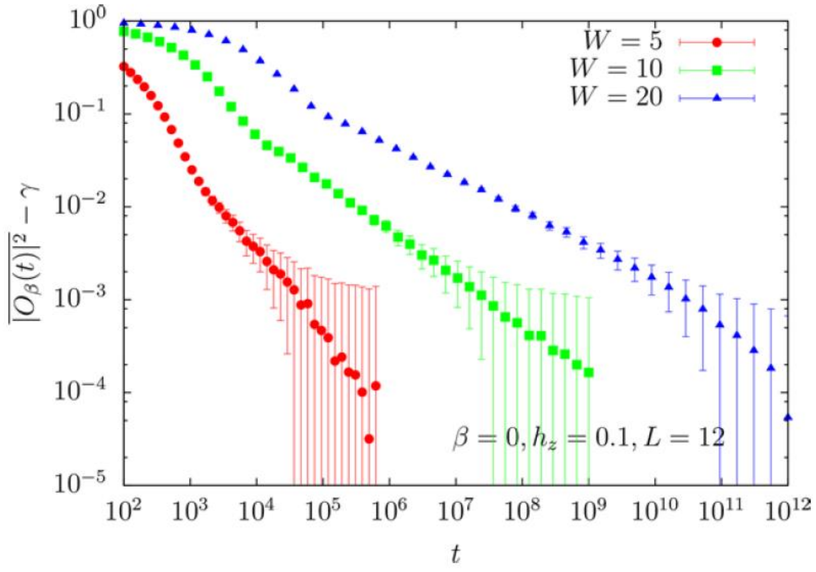
$\rightarrow$ $H = \sum_i \tilde{h}_i \, \tau_i^z + \sum_{ij} J_{ij} \tau_i^z \tau_j^z + \cdots$

$J_{ij} \sim e^{-|r_i - r_j|/\xi}$

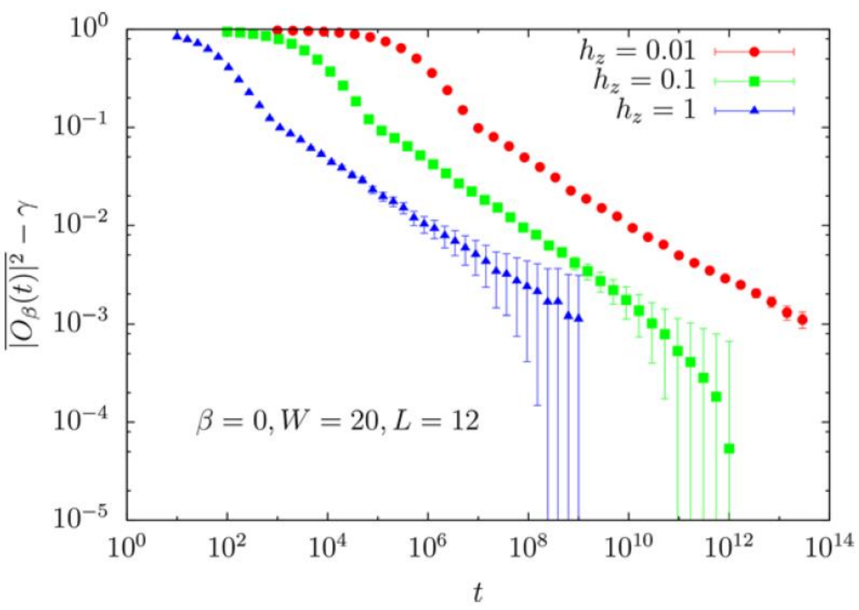


Chen et al., Ann. Phys. (2017)

Disorder and eigenstate averaged OTOC



Chen et al., Ann. Phys. (2017)



Without disorder and eigenstate average, OTOC