Geometry and Integrability of Mechanics on Curved Spaces

8 th August, 2018

Sumanto Chanda
S.N. Bose National Center for Basic Sciences
Department of Theoretical Sciences

Introduction

The geodesic equation describing dynamics on curved spaces is a non-linear differential equation.

$$\frac{d^2\, x^i}{d\,\tau^2} + \Gamma^i_{jk}\ \dot{x}^j\ \dot{x}^k\ =\ \vec{F}\ (\vec{x}).$$

In the theory of integrable systems, we are interested in studying non-linear differential equations that are ideally analytically solvable, for first integrals or constants of motion that are in involution with each other. Having first integrals, or constants of motion reduces the number of equations of motion to solve, simplifying the problem.

A dynamical system is integrable if enough first integrals are known so that it can be completely solved, and is “superintegrable" if the number of known first integrals exceeds the number of degrees of freedom.

Contents

- Reduction of geodesics via Jacobi-Maupertuis formulation

- Generalized Darboux-Halphen system Reduction of Yang-Mills Theory

- Relativistic mechanics on curved spaces

Reduction of geodesics via Jacobi-Maupertuis formulation

Dynamics in a gravitational field is distinct from that due to conventional forces by being a consequence of the curvature of spacetime, rather than a physical force.

There are 2 known ways to construct a metric such that dynamics due to curvature alone imitates the effects of a physical force:

1. Jacobi metric formulation
2. Eisenhart lifts

Here, we will see how the Jacobi metric formulation also acts like a reduction process for dimensions of geodesics on a curved space, due to the existence of a constant of motion.

Preliminaries

- If the action S of a mechanical system with Lagrangian L along a curve from point 1 to 2 is given as

$$S = \int_1^2 d\tau \; L \; , \qquad L = \sqrt{g_{\mu\nu}(\vec{x})\,\dot{x}^\mu \dot{x}^\nu} + A_\alpha(\vec{x})\dot{x}^\alpha .$$

- Then the action in Maupertuis form can be given as:

$$S = \int_1^2 p_\mu \, dx^\mu \; , \qquad \Rightarrow \qquad L = p_\mu \, \dot{x}^\mu \;\Rightarrow\; p_\mu \, \dot{x}^\mu - L = 0$$

- If we have a cyclical variable x^0, then we have according to Euler-Lagrange equation:

$$\frac{d}{d\,\tau}\left(\frac{\partial\, L}{\partial\, \dot{x}^i}\right) = \frac{\partial\, L}{\partial\, x^i} = 0 \qquad \Rightarrow \qquad p_0 = \frac{\partial\, L}{\partial\, \dot{x}^0} = -E = const$$

then for parametrization wrt x^0, we will have a conserved quantity according to Legendre's theorem

$$-\, p_0 = p_i \, \dot{x}^i - L$$

and the effective action:

$$\delta S = \delta \int_1^2 dt \, (p_i \, \dot{x}^i + p_0) \equiv \delta \int_1^2 dt \;\, p_i \dot{x}^i = 0 \; , \qquad \Rightarrow \qquad S_{eff} = \int_1^2 dt \, p_i \, \dot{x}^i$$

- However, for Lagrangians of the form below, the Legendre rule will apply

$$L = g_{\mu\nu}(\vec{x})\,\dot{x}^\mu \dot{x}^\nu + A_\alpha(\vec{x})\dot{x}^\alpha . \qquad\qquad H = p_\mu \, \dot{x}^\mu - L$$

Jacobi metric: a projection to a surface

- For time-independent systems, the Hamiltonian is defined by the Legendre transform as:

$$H(\boldsymbol{x},\boldsymbol{p}) = \frac{1}{2} g^{ij}(\boldsymbol{x})\, p_i\, p_j + U(\boldsymbol{x}) \equiv T + U = E$$

- The Maupertuis form of the metric described by Jacobi is written as:

$$S = \int p_\mu\, d\,x^\mu = \int dt\left(p_i \dot{x}^i - E\right) \qquad p_0 = -E$$

$$S_{eff} = \int dt\, p_i \dot{x}^i = \int dt\, g_{ij}\, \dot{x}^i \dot{x}^j = \int dt\, 2\,T, \qquad p_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij}\, \dot{x}^j$$

- Thus, the effective action can be written as a geodesic:

$$dS = \sqrt{dl^2} = \int dt\, 2\,T = \int dt\, \sqrt{2m(E-U)g_{ij}\, \dot{x}^i \dot{x}^j}$$

$$\Rightarrow \qquad \tilde{g}_{ij}(\boldsymbol{x}) = 2m\big(E - U(\boldsymbol{x})\big)\, g_{ij}(\boldsymbol{x})$$

$$\Rightarrow \qquad dl^2 = 2m(E-U)g_{ij}\, dx^i dx^j$$

- The Jacobi Hamiltonian in this case is given by taking the inverse of the metric, showing that it confines the momentum vector to a unit sphere

$$\widetilde{H}(\boldsymbol{x},\boldsymbol{p}) = \frac{1}{2m}\frac{g^{ij}(\boldsymbol{x})p_i p_j}{E - U(\boldsymbol{x})} \equiv \frac{T}{E - U(\boldsymbol{x})} = 1$$

- It essentially converts the Euler-Lagrange equation into a geodesic equation

$$\ddot{x}^i = -\sum_{jk} \Gamma^i_{jk}\, \dot{x}^j \dot{x}^k - \sum_{l} g^{il}(\boldsymbol{x})\, \partial_l\, U(\boldsymbol{x}) \quad \Longrightarrow \quad \frac{d^2 x^i}{d\,\sigma^2} = -\tilde{\Gamma}^i_{jk} \frac{dx^j}{d\,\sigma}\frac{d\,x^k}{d\,\sigma}$$

- All other conserved quantities are preserved under the formulation, implying preservation of symmetries.

S. Chanda, G. Gibbons and P. Guha, *Jacobi-Maupertuis-Eisenhart metric and geodesic flows*, arXiv:1612.00375 [math-ph]. J. Math. Phys. **58** (2017) 032503.

Jacobi metric: relativistic

- The Jacobi metric formulation from the Maupertuis formulation can be demonstrated for a static spacetime metric

$$ds^2 = V^2(\boldsymbol{x})c^2dt^2 - g_{ij}(\boldsymbol{x})dx^i dx^j \qquad V^2(\mathbf{x}) = 1 + \frac{2U(\boldsymbol{x})}{mc^2}$$

- Here the effective Maupertuis Lagrangian leads us to Gibbons' Jacobi metric

$$\mathcal{E} = -p_0 = -\frac{c^2V^2(\boldsymbol{x})}{\sqrt{g_{\mu\nu}(\boldsymbol{x})\dot{x}^\mu\dot{x}^\nu}} \quad \Rightarrow \quad \frac{\mathcal{E}^2 - c^2V^2(\boldsymbol{x})}{c^2V^2(\boldsymbol{x})} = \sqrt{\frac{g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}{g_{\mu\nu}(\boldsymbol{x})\dot{x}^\mu\dot{x}^\nu}}$$

$$L_{eff} = p_i\dot{x}^i = \frac{g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}{\sqrt{g_{\mu\nu}(\boldsymbol{x})\dot{x}^\mu\dot{x}^\nu}} = \sqrt{\left(\frac{\mathcal{E}^2 - c^2V^2(\boldsymbol{x})}{c^2V^2(\boldsymbol{x})}\right)g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j} \quad \Rightarrow \quad J_{ij}(\boldsymbol{x}) = \left(\frac{\mathcal{E}^2 - c^2V^2(\boldsymbol{x})}{c^2V^2(\boldsymbol{x})}\right)g_{ij}(\boldsymbol{x}),$$

- Consider the following stationary spacetime metric:

$$ds^2 = \left(dt + A_i(\boldsymbol{x})dx^i\right)^2 c^2V^2(\boldsymbol{x}) - g_{ij}(\boldsymbol{x})dx^i dx^j \qquad V^2(\mathbf{x}) = 1 + \frac{2U(\boldsymbol{x})}{mc^2}$$

- The relativistic Jacobi metric for this spacetime, which under non-relativistic limits, returns the classical version is:

$$J_{ij}(\boldsymbol{x}) = \frac{\mathcal{E}^2 - m^2c^4V^2(\boldsymbol{x})}{c^2V^2(\boldsymbol{x})}\, g_{ij}(\boldsymbol{x}) \qquad \mathcal{E}^2 = |\boldsymbol{\Pi}|^2c^2 + m^2c^4$$

$$2\mathcal{E} \ll mc^2, \quad g^{ij}\Pi_i\Pi_j \ll m^2c^2 \qquad \mathcal{E} \approx mc^2 + T + U = mc^2 + E$$

$$\Rightarrow \frac{\mathcal{E}^2 - m^2c^4V^2(\mathbf{x})}{c^2V^2(\mathbf{x})}\, g_{ij}(\mathbf{x}) \approx 2m(E-U)g_{ij}$$

S. Chanda, G. Gibbons and P. Guha, *Jacobi-Maupertuis-Eisenhart metric and geodesic flows*, arXiv:1612.00375 [math-ph]. J. Math. Phys. **58** (2017) 032503.

Jacobi metric for time-dependent metrics

- A sample time-dependent spacetime metric:

$$ds^2 = c^2\, V^2(\boldsymbol{x},t)d\,t^2 - \; g_{ij}\,(\boldsymbol{x},t)\;dx^i\;dx^j \qquad V^2(\boldsymbol{x},t) \;=\; 2m\,W(\boldsymbol{x},t)$$

- Under Eisenhart-Duval lift, we can lift the spacetime by adding a cyclical co-ordinate in manner that does not alter the dynamics:

$$ds^2 = c^2\,V^2(\boldsymbol{x},t)\;dt^2 + 2c\;d\sigma\;dt \;-\; g_{ij}\,(\boldsymbol{x},t)\;dx^i\;dx^j$$

this allows us to define a conserved momentum conjugate to the cyclical co-ordinate

$$p_\sigma = \frac{\partial\; L}{\partial\;\dot{x}^\sigma} \;=\; const$$

- Corresponding relativistic Jacobi metric:

$$J_{ij}\,(\boldsymbol{x},t) = [2\{\,p_\sigma\;p_t - {p_\sigma}^2 W\;(\boldsymbol{x},t)\} \;-\; m^2\;c^2]\;g_{ij}\,(\boldsymbol{x},t)$$

$$2c\;p_t\;p_\sigma \;\;=\;\; c^2\;g^{ij}\;\;p_i\;p_j \;+\; c^2\,V^2\;(\boldsymbol{x},t)\;p_\sigma^2 \;+\; m^2\;c^4 \;\;=\;\; Q^2$$

- Classical Jacobi metric:

$$Q \;\approx\; mc^2 + \mathcal{E}(t) \qquad\qquad 2\;q\;p_t \;\approx\; m^2\;c^2 \;+\; 2\;m\;\mathcal{E}(t)$$

$$\Rightarrow J_{ij}\,(\boldsymbol{x},t) = 2m\;[\mathcal{E}(t) - q^2\;W(\boldsymbol{x},t)]\;g_{ij}\,(\boldsymbol{x},t)$$

S. Chanda, G. Gibbons and P. Guha, *Jacobi-Maupertuis-Eisenhart metric and geodesic flows*, arXiv:1612.00375 [math-ph]. J. Math. Phys. **58** (2017) 032503.

Jacobi metric formulation as a Bohlin's transformation

- Suppose the co-ordinates are written as a complex variable

$$r = q_1 + i\,q_2 \qquad\qquad z = x + i\,y$$

for which Bohlin's canonical transformation is given by the conformal transformation

$$z = \frac{r^2}{2} = \frac{q_1^2 - q_2^2}{2} + i\,(q_1 q_2) = x + i\,y$$

$$\frac{p_1 + i p_2}{q_1 - i q_2} = p_x + i\,p_y$$

- If the oscillator Hamiltonian is given by:

$$H\,(q_i, p_i) = \frac{1}{2}\,(p_1^2 + p_2^2) + \frac{a^2}{2}\,(q_1^2 + q_2^2) + b = E$$

- Then under co-ordinate map, the Jacobi Hamiltonian for *E*=0 is

$$\widetilde{H}(x,y) = \frac{H(x,y)}{|E - V\,(\boldsymbol{x})|} = \frac{p_x^2 + p_y^2}{a^2} + \frac{b}{2\,a^2\sqrt{x^2 + y^2}} + 1$$

which is roughly the Kepler Hamiltonian

S. Chanda G. Gibbons and P. Guha, *Jacobi-Maupertius metric and Kepler equation*. Int. J. Geom. Methods Mod. Phys. **14** (2017) 1730002.

Houri's analysis of the Kepler system

- The canonical transformation T. Houri is

$$\tilde{x}^{\mathrm{i}} = p_{\mathrm{i}} \qquad \tilde{p}_{\mathrm{i}} = x^{\mathrm{i}}$$

- On application to the Kepler Hamiltonian, we can get a new Hamiltonian

$$\widetilde{H} = \left[E - \frac{\tilde{r}^2}{2}\right]^2 (\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2) = \alpha^2$$

- On taking the inverse of the above metric and setting the energy to $\mathrm{E} = -\frac{\mathrm{k}^2}{2}$ we get the corresponding metric

$$ds^2 = 4(k^2 + |\boldsymbol{x}|^2)^{-2}\, d\,\boldsymbol{x}^2$$

- Applying Milnor's momentum inversion

$$\left(2\widetilde{H}^{\frac{1}{2}}\,|\boldsymbol{p}|\right)^{-1} X_{\widetilde{H}} = \frac{p^i}{|\boldsymbol{p}|^3}\frac{\partial}{\partial\, x^i} - x_i\frac{\partial}{\partial\, p_i}$$

we find that the geodesic flow is preserved.

S. Chanda G. Gibbons and P. Guha, *Jacobi-Maupertius metric and Kepler equation*. Int. J. Geom. Methods Mod. Phys. **14** (2017) 1730002.

The Lienard mechanical system (2nd kind)

- Consider the Lienard mechanical system of the 2nd kind:

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$$

- Due to the quadratic dependence on velocity, we can say that this is a variable mass mechanical system, whose Lagrangian L and geodesic equation are :

$$L = \frac{1}{2}M(x)\dot{x}^2 - U(x) \quad \Rightarrow \quad \ddot{x} + \frac{d}{dx}\left(\ln\sqrt{M}\right)\dot{x}^2 + \frac{1}{M(x)}\frac{dU}{dx} = 0$$

$$2\left(E - U(x)\right) = M(x)\dot{x}^2 \quad \Rightarrow \quad \ddot{x} + \frac{d}{dx}\left(\ln\sqrt{M}\right)\dot{x}^2 + \frac{1}{2(E - U(x))}\frac{dU}{dx}\dot{x}^2 = 0$$

- Thus, we can say that the mechanical system described is time-independent, and thus, the Jacobi metric can be appropriately formulated for this case:

$$d\,\tilde{s}^2 = 2\,M(x)\left(E - U\,(x)\right)dx^2$$

and the geodesic equation for this Jacobi metric is given as:

$$\frac{d^2x}{d\sigma^2} + \frac{1}{4}\left[\frac{U'(x)}{2(E - U(x))} - \frac{d}{dx}\left(\sqrt{M(x)}\right)\right]\left(\frac{dx}{d\sigma}\right)^2 = 0$$

S. Chanda, A. Ghose-Choudhury, and P. Guha, *Jacobi-Maupertuis metric of Liernard type equations and Jacobi Last Multiplier, Electr. J. Differ. Equ. Vol. 2018 (2018), No.120, pp. 1-9.*

Generalized Darboux-Halphen system: Reduction of Yang-Mills Theory

R.S. Ward's conjectured that perhaps all solvable or integrable partial differential equations are merely various reductions of the Self-Dual Yang-Mills (SDYM) equations.

SDYM systems have numerous applications, appearing in gauge theory, classical general relativity, and the analysis of 4-manifolds.

One such reduction produces the generalized Darboux-Halphen system.

The Bianchi-IX metric

This is a general setup of metrics on Euclidean spaces whose self-dual (hodge dual) cases can be studied to describe gravitational instantons on such spaces.

$$ds^2 = (c_0)^2 dr^2 + \sum_{i=1}^{3} (c_i)^2 (\sigma^i)^2, \quad (c_0)^2 = \Omega_1 \Omega_2 \Omega_3, \quad (c_i)^2 = \frac{\Omega_j \Omega_k}{\Omega_i}, \quad \sigma^i = -\frac{1}{r^2} \eta^i_{\mu\,\nu}\, x^\mu\, dx^\nu$$

Bianchi IX systems provide a general framework to study various other self dual metrics, like Gibbons hawking or Taub NUT or Eguchi Hanson spaces.

Two versions of self-duality

• Connection-wise self-duality	• Curvature-wise self-duality
$\omega_{ab} = \pm \frac{1}{2} \varepsilon_{ab}{}^{cd}\, \omega_{cd}$	$R_{\mu\nu\gamma\lambda} = \pm \frac{1}{2} \varepsilon_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma\gamma\lambda}$
• Defines the Lagrange system a.k.a. the Euler-top	• Defines the classical Darboux-Halphen system
$\dot{\Omega}_i = \mp \Omega_j \Omega_k$	$\dot{\Omega}_i = \Omega_j \Omega_k - \Omega_i (\Omega_j + \Omega_k)$

- **Self-dual spaces are Ricci-flat, and arguably imply integrability.**

$$R_{aa} = R_{abab} + R_{acac} + R_{adad} = \pm(R_{abcd} + R_{acdb} + R_{adbc}) = 0$$

- Decomposition of the general Riemann curvature tensor into anti/self-dual parts:

$$R_{abcd} = G^{(+)}_{ij}(x)\, \eta^{(+)i}_{ab} \eta^{(+)j}_{cd} + G^{(-)}_{ij}(x)\, \eta^{(-)i}_{ab} \eta^{(-)j}_{cd}$$

$$\mathrm{R}_{ab} = \sum_{i=1}^{3} \varepsilon_{ijk} \frac{(c_i^2 + c_j^2 + c_k^2)(c_j^2 + c_k^2) - 2 c_i^4 - 4 c_j^2 c_k^2}{2 c_0^2} \bar{\eta}^i_{ab} \bar{\eta}^i_{cd}\, e^c \wedge e^d$$

S. Chanda, P. Guha, R. Roychowdhury, *Bianchi-IX, Darboux-Halphen and Chazy-Ramanujan*, Int. J. Geom. Methods Mod. Phys. 13 (2016), no. 4, 1650042

Yang-Mills fields and Nahm's equation

- The Yang-Mills gauge field strength and the SDYM are defined by hodge duality:

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b] \qquad F_{ab} = \pm\frac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}$$

$$A_a = A_a^i T_i \qquad [T_i, T_j] = f_{ij}^k T_k$$

A reduction of the SDYM fields will gave us the generalized Darboux-Halphen system, via Nahm's equations produced by dependence on a single variable.

$$F_{0i} = \dot{A}_i \qquad F_{ij} = -[A_i, A_j] \quad \Rightarrow \quad \dot{A}_i = \frac{1}{2}\varepsilon_i^{jk}[A_j, A_k]$$

$$A_i = -M_{ij}(t) O_{jk} X_k \qquad M = M_s + M_a$$

- Depending on the SO(3) matrix P, we will get different matrices, and corresponding generalized Darboux-Halphen systems

$$M = P^{-1} M_0 P, \qquad M_0 = d + a$$

- This generalized system was formulated by M.J. Ablowitz, S.Chakravarty and R.G. Halburd. For $P \neq \mathrm{I}$, we will get what is called the DH-IX system

$$M_s = \begin{pmatrix} \Omega_1 & \sigma_1 & \sigma_2 \\ \sigma_1 & \Omega_2 & \sigma_3 \\ \sigma_2 & \sigma_3 & \Omega_3 \end{pmatrix} \qquad M_a = \begin{pmatrix} 0 & \tau_3 & -\tau_2 \\ -\tau_3 & 0 & \tau_1 \\ \tau_2 & -\tau_1 & 0 \end{pmatrix}$$

$$\dot{\Omega}_i = \Omega_j \Omega_k - \Omega_i(\Omega_j + \Omega_k) - \theta_i \phi_i + \theta_j^2 + \phi_k^2$$

$$\dot{\theta}_i = -(\theta_i + \phi_i)\,\Omega_i - (\theta_i - \phi_i)\,\Omega_k + \theta_k\,(\theta_j + \phi_j)$$

$$\dot{\phi}_i = -(\theta_i + \phi_i)\,\Omega_i + (\theta_i - \phi_i)\,\Omega_j + \phi_j\,(\theta_k + \phi_k)$$

$$\theta_i = \sigma_i + \tau_i, \qquad \phi_i = \sigma_i - \tau_i$$

S. Chanda, S. Charavarty, and P. Guha *On a reduction of the generalized Darboux-Halphen system, Phys. Lett. A* **382** Iss. 7 (2018) 455-460

Generalized Darboux-Halphen system

- If we apply the condition

$$\theta = \theta_3, \phi = \phi_3, \qquad \theta_1 = \theta_2 = \phi_1 = \phi_2 = 0$$

  then we will have the DH-5 system studied by Ablowitz et al, where P describes an SO(2) rotation about one axis

$$\dot{\Omega}_1 = \Omega_2 \Omega_3 - \Omega_1 (\Omega_2 + \Omega_3) - \phi^2$$
$$\dot{\Omega}_2 = \Omega_3 \Omega_1 - \Omega_2 (\Omega_3 + \Omega_1) + \theta^2$$
$$\dot{\Omega}_3 = \Omega_1 \Omega_2 - \Omega_3 (\Omega_1 + \Omega_2) + \theta\phi$$

$$\dot{\theta} = -(\theta + \phi) \Omega_3 - (\theta - \phi) \Omega_2$$
$$\dot{\phi} = -(\theta + \phi) \Omega_3 + (\theta - \phi) \Omega_1$$

- For P = I, the symmetric part is diagonalized:

$$P = I \quad \Rightarrow \quad M = M_0 = d + a$$

$$M_s = \begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & \Omega_2 & 0 \\ 0 & 0 & \Omega_3 \end{pmatrix} \qquad M_a = \begin{pmatrix} 0 & \tau_3 & -\tau_2 \\ -\tau_3 & 0 & \tau_1 \\ \tau_2 & -\tau_1 & 0 \end{pmatrix}$$

- This produces the familiar generalized Darboux-Halphen (DH-6) system

$$\dot{\Omega}_i = \Omega_j \Omega_k - \Omega_i (\Omega_j + \Omega_k) + \tau^2 \qquad (\tau_i)^2 = \alpha_i^2 \, x_j x_k, \quad \tau^2 = \sum_{i=1} (\tau_i)^2,$$

$$\dot{\tau}_i = -\tau_i (\Omega_j + \Omega_k) \qquad x_i = \Omega_j - \Omega_k$$

- Both, classical and generalized DH-6 systems, are characterised by a constraint given by:

$$Q = \frac{\Omega_1^2}{x_2 x_3} + \frac{\Omega_2^2}{x_3 x_1} + \frac{\Omega_3^2}{x_1 x_2} = -1$$

S. Chanda, S. Charavarty, and P. Guha *On a reduction of the generalized Darboux-Halphen system, Phys. Lett. A* **382** Iss. 7 (2018) 455-460

Solutions of DH System

- If we define a variable s such that:

$$s = \frac{\Omega_3 - \Omega_2}{\Omega_1 - \Omega_2} = -\frac{x_1}{x_3}$$

then we can write solutions to the DH equations as functions of s:

$$x_1 = -\frac{1}{2}\frac{\dot{s}}{s-1} \qquad \Omega_1 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{s-1}\right)\right]$$

$$x_2 = -\frac{1}{2}\frac{\dot{s}}{s} \qquad \Omega_2 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{s}\right)\right]$$

$$x_3 = -\frac{1}{2}\frac{\dot{s}}{\sqrt{s(s-1)}} \qquad \Omega_3 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{\sqrt{s(s-1)}}\right)\right]$$

$$\dot{x}_i = -2\Omega_i x_i$$

where s obeys the Schwarzian equation

$$\frac{\dddot{s}}{s} + \frac{3}{2}\left(\frac{\ddot{s}}{s}\right)^2 + \frac{\dot{s}^2}{2}\left[\frac{1-\alpha_1^2}{s^2} + \frac{1-\alpha_1^2}{(s-1)^2} + \frac{\alpha_1^2+\alpha_2^2-\alpha_3^2-1}{s(s-1)}\right] = 0$$

which can be linearized in terms of hypergeometric equations

$$\chi'' + \left(\frac{1-\alpha_1}{s} + \frac{1-\alpha_2}{s-1}\right)\chi' + \left(\frac{(\alpha_1+\alpha_2-1)^2-\alpha_3^2}{4s(s-1)}\right)\chi = 0, \qquad t(s) = \frac{\chi_2(s)}{\chi_1(s)}$$

where X_1 and X_2 are solutions of the hypergeometric equation.

S. Chanda, S. Charavarty, and P. Guha *On a reduction of the generalized Darboux-Halphen system, Phys. Lett. A* **382** Iss. 7 (2018) 455-460

Lax Pair for the generalized DH system

- N=4 SDYM theories are exactly solvable class of theories that excite string theorists. Here, we argue that self-duality implies integrability.
- Integrability implies the existence of conserved quantities. This is not Louiville integrability, which exists on finite dimensional spaces.
- In integrable systems, a Lax-Pair is a pair of time-dependent matrices or operators that satisfy a differential equation called the Lax equation.

The Lax-Pair for Darboux-Halphen systems can be written as

$$\dot{L} = -[L, A] \qquad W_i = \frac{\Omega_i}{x_j x_k} \qquad A_1 = 0$$

$$A_2 = x_3 W_2 + \frac{\tau^2}{W_1\sqrt{x_1 x_2}}$$

$$L = \begin{pmatrix} 0 & W_3 & W_2 \\ W_3 & 0 & W_1 \\ W_2 & -W_1 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & A_3 & A_2 \\ A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix}$$

$$A_3 = x_2 W_3 + \frac{\tau^2}{W_1\sqrt{x_1 x_3}}$$

Alternatively, with a variable transformation, the Lax-Pair can be written as

$$\widehat{W}_1 = \frac{W_1}{2} \qquad \widehat{W}_2 = \frac{W_2}{2i} \qquad \widehat{W}_3 = \frac{W_3}{2i} \qquad f(s) = \frac{\alpha_1^2(s-1) - \alpha_2^2 s - \alpha_3^2 s(s-1)}{4s(s-1)}$$

$$\widehat{W}_1' = \frac{\widehat{W}_2\widehat{W}_3}{s-1} + \frac{f(s)}{\sqrt{s-1}} \qquad \widehat{W}_2' = \frac{\widehat{W}_3\widehat{W}_1}{s} + \frac{f(s)}{\sqrt{s}} \qquad \widehat{W}_3' = \frac{\widehat{W}_1\widehat{W}_2}{s(s-1)} - \frac{f(s)}{\sqrt{s(s-1)}}$$

$$U' = -[U, V] \qquad v_1 = 0$$

$$v_2 = \frac{\widehat{W}_2}{s(s-1)} + \frac{f(s)}{\widehat{W}_1\sqrt{s(s-1)}}$$

$$U = \begin{pmatrix} 0 & \widehat{W}_3 & \widehat{W}_2 \\ \widehat{W}_3 & 0 & \widehat{W}_1 \\ \widehat{W}_2 & -\widehat{W}_1 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & \hat{v}_3 & \hat{v}_2 \\ \hat{v}_3 & 0 & \hat{v}_1 \\ \hat{v}_2 & -\hat{v}_1 & 0 \end{pmatrix}$$

$$v_3 = \frac{\widehat{W}_3}{s} + \frac{f(s)}{\widehat{W}_1\sqrt{s}}$$

Hamiltonian formulation for the gDH system

- The fundamental Poisson Bracket relations are:

$$\{\widehat{W}_1, \widehat{W}_2\} = \widehat{W}_3 \qquad \{\widehat{W}_2, \widehat{W}_3\} = -\widehat{W}_1 \qquad \{\widehat{W}_3, \widehat{W}_1\} = \widehat{W}_2$$

- A Casimir function acting as a constraint, constructed from the Lax Matrix U is:

$$C = -\frac{1}{2}\,Tr\,U^2 = \widehat{W}_1^2 - \widehat{W}_2^2 - \widehat{W}_3^2 = -\frac{1}{4}$$

- Another constraint is given by

$$l = -\frac{1}{2}\,Tr\,US = \frac{\widehat{W}_1}{\sqrt{s-1}} + \frac{\widehat{W}_2}{\sqrt{s}} + \frac{\widehat{W}_3}{\sqrt{s(s-1)}} = 0$$

- The Hamiltonian function on the phase-space is given as

$$H = -\frac{1}{2}\,Tr(UIU - 4cf(s)US) = \left[\frac{\widehat{W}_1^2}{s-1} - \frac{\widehat{W}_2^2}{s} + \frac{\widehat{W}_3^2}{s(s-1)}\right] - 4c\,f(s)l$$

- Thus, we have

$$\dot{W}_i = \{\widehat{W}_i, H\} \qquad \frac{dC}{ds} = \{C, H\} \qquad \frac{dl}{ds} = \frac{\partial l}{\partial s} + \{l, H\} = 0$$

S. Chanda, S. Charavarty, and P. Guha *On a reduction of the generalized Darboux-Halphen system, Phys. Lett. A* **382** Iss. 7 (2018) 455-460

Related Dynamical Systems

Chazy III equation

This is a 3rd order equation with a movable singularity, whose solution is given by the Eisenstein series E_2

$$\dddot{y} - 2y\ddot{y} + 3\,\dot{y}^2 = 0$$

Ramanujan system

This series of equations can reproduce the Chazy equation by solving for one variable. Each variable solutions is given by the Eisenstein series E_2 and E_4.

$$\dot{P} = \frac{i\pi}{6}\,(P^2 - Q)$$

$$\dot{Q} = \frac{2i\pi}{3}\,(PQ - R)$$

$$\dot{R} = i\pi(PR - Q^2)$$

Ramamani system

This series will transform into the classical Darboux-Halphen system under a variable transformation

$$\dot{P} = \frac{i\pi}{2}\,(P^2 - Q)$$

$$\dot{\tilde{P}} = i\pi\left(P\tilde{P} - Q\right)$$

$$\dot{Q} = 2i\pi\left(P - \tilde{P}\right)Q$$

S. Chanda, P. Guha, R. Roychowdhury, *Bianchi-IX, Darboux-Halphen and Chazy-Ramanujan*, Int. J. Geom. Methods Mod. Phys. 13 (2016), no. 4, 1650042

Relativistic mechanics on curved spaces

Relativistic mechanics on curved space

- Here, we formulate relativistic mechanics in presence of a scalar potential as a perturbation of the spacetime metric.

$$ds^2 = V^2(\vec{x})\, c^2\, dt^2 - |d\vec{x}|^2, \qquad V^2(\vec{x}) = 1 + \frac{2U(\vec{x})}{mc^2}$$

- This leads to a new formalism connecting classical and relativistic Lagrangians.

$$\mathcal{L} = -mc\sqrt{\left(\frac{ds}{dt}\right)^2} = -mc^2\sqrt{1 - \frac{2}{mc^2}\left(\frac{m|\vec{v}|^2}{2} - U(\vec{x})\right)} \qquad L_{class} = \frac{m|\vec{v}|^2}{2} - U(\vec{x})$$

$$L_0 = -mc^2$$

$$\therefore\ \mathcal{L} = L_0\sqrt{1 + \frac{2L_{class}}{L_0}} \xrightarrow{L_{\text{class}} \ll L_0} L_0 + L_{class}$$

- From the Euler-Lagrange equations of motion we will get the deformed equation of motion

$$\left[\frac{d}{dt}\left(\frac{\partial L_{class}}{\partial v^i}\right) - \frac{\partial L_{class}}{\partial x^i}\right] = \Gamma^2\left(\frac{\partial L_{class}}{\partial v^i}\right)\frac{dL_{class}}{dt}$$

$$\Gamma = \frac{1}{\sqrt{1 + \frac{2L_{class}}{L_0}}} \xrightarrow{L_{\text{class}} \ll L_0} \gamma = \frac{1}{\sqrt{1 - \left(\frac{|\vec{v}|}{c}\right)^2}}$$

S. Chanda and P. Guha, *Geometrical Formulation of Relativistic Mechanics*, arXiv: 1706.1921
Int. J. Geom. Methods in Mod. Phys. **15** (2018) 1850062

A modified local Lorentz transformation

- The familiar Lorentz transformation and the time-dilation / length contraction that derive from it are given by:

$$c\,d\tilde{t} = \gamma c\,dt - \gamma\beta dx$$

$$d\,\tilde{x} = \gamma\beta c\,dt - \gamma dx$$

$$d\tilde{t} = \gamma\,dt = \frac{dt}{\sqrt{1-\beta^2}}$$

$$d\,\tilde{x} = \gamma^{-1}dx = dx\sqrt{1-\beta^2}$$

- This applies only to special relativity where we consider free particles in flat spaces.

- In General Relativity, we deal with curved spaces caused by gravitational potentials. The modified local Lorentz transformation is given by:

$$\Lambda = \begin{pmatrix} \Gamma\sqrt{g_{00}} & -\beta\Gamma\left(\sqrt{g_{00}}\right)^{-1} \\ -\beta\Gamma\sqrt{g_{00}} & \Gamma\sqrt{g_{00}} \end{pmatrix}$$

$$d\tilde{t} = \Gamma\sqrt{g_{00}}\,dt$$

$$d\,\tilde{x} = \left(\Gamma\sqrt{g_{00}}\right)^{-1}dx$$

S. Chanda and P. Guha, *Geometrical Formulation of Relativistic Mechanics*, arXiv: 1706.1921
Int. J. Geom. Methods in Mod. Phys. **15** (2018) 1850062

Future Directions

- I wish to work on quantum integrable systems and integrable models.

- I am interested in further exploring nature and applications of Eisenhart lifts in various problems on curved spaces, such as Kaluza-Klein reduction, especially the Eisenhart-Duval lift's usefulness for dealing with time-dependent systems like those in the study of gravitational waves.

- I want to apply my formalism for relativistic mechanics on curved spaces, by going beyond the 2-body problem to the relativistic 3-body problem, then proceed to the *n*-body problem on spaces of constant curvature.

  Furthermore, I wish to apply the Eisenhart-Duval lift to study gravitational waves of 3-body problems.

- Based on my work done on Relativistic Mechanics on Curved Spaces, I am interested in working on the topic of relativistic thermodynamics, on the suggestion of Frederic Barbaresco.

List of Publications

1. S. Chanda, P. Guha, R. Roychowdhury, *Bianchi-IX, Darboux-Halphen and Chazy-Ramanujan*, Int. J. Geom. Methods Mod. Phys. 13 (2016), no. 4, 1650042, 25 pp, arXiv:1512.01662 [hep-th].
2. S. Chanda, P. Guha, R. Roychowdhury, *Schwarzschild instanton in emergent gravity*, Int. J. Geom. Methods Mod. Phys. 14 (2017), 1750006, 31 pp , arXiv:1406.6459 [hepth].
3. S. Chanda, P. Guha, and R. Roychowdhury, *Taub-NUT as Bertrand spacetime with magnetic fields*, J. Geom. Symm. Phys. 41 (2016) 33-67, arXiv:1503.08183 [hepth].
4. S. Chanda, G. Gibbons and P. Guha, *Jacobi-Maupertuis-Eisenhart metric and geodesic flows*, arXiv:1612.00375 [math-ph], J. Math. Phys. **58** (2017) 032503.
5. S. Chanda, G. Gibbons and P. Guha, *Jacobi-Maupertius metric and Kepler equation*, Int. J. Geom. Methods Mod. Phys. **14** (2017) 1730002.
6. S. Chanda and P. Guha, *Geometrical Formulation of Relativistic Mechanics*, Int. J. Geom. Methods in Mod. Phys. **15** (2018) 1850062 , arXiv: 1706.1921
7. S. Chanda, S. Charavarty, and P. Guha *On a reduction of the generalized Darboux-Halphen system, Phys. Lett. A* **382** Iss. 7 (2018) 455-460, arXiv: 1710.00158.
8. S. Chanda, A. Ghose-Choudhury, and P. Guha, *Jacobi-Maupertuis metric of Liernard type equations and Jacobi Last Multiplier*, Electr. j. differ. equ. Vol. 2018 (2018), No.120, pp. 1-9, arXiv: 1706.02219 [hep-th].

References

- G.W. Gibbons, *The Jacobi-metric for timelike geodesics in static spacetimes*, Class. Quantum Grav. **33** (2015) 025004, arXiv:1508.06755.
- M.J. Ablowitz, S. Chakravarty and R.G. Halburd, *Integrable systems and reductions of the self-dual Yang-Mills equations* J. Math. Phys. **44** (2003) 3147-72.
- P. Marios Petropoulos and Pierre Vanhove, *Gravity, strings, modular and quasimodular forms*, CPHT-RR005.0211, IPHT-t12/016, IHES/P/12/08 , arXiv:1206.0571 [math-ph],.
- M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope Bianchi IX self-dual Einstein metrics and singular G2 manifolds, Class. Quant. Grav. **20**, No.19 arXiv:0206151 [hep-th],
- K.P. Tod, *Self-dual Einstein metrics from the Painleve VI equation*, Physics Letters A **190** (1994) 221-224.