Thermalization in Quantum Systems

Problem Set 1

1. Show that the ergodic condition

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f\left[\phi_t(x)\right] dt = \frac{1}{\int_\Omega dx}\int_\Omega f(x)dx,$$

implies that for a system starting in a micro state x, the fraction of time spent in any region of phase space is equal to the fraction of the volume it occupies. Which f will you choose to prove this?

2. Consider the one dimensional harmonic oscillator whose Hamiltonian is given by

$$H = \frac{\omega}{2}\left(p^2 + q^2\right).$$

The constant energy surfaces are circles in phase space and can be parametrized by an angle θ, i.e. each microstate at a given energy corresponds to a fixed value of θ.

   (a) Show that the system is ergodic.

   (b) Show that it is not chaotic. This means that an initial compact phase space density $\rho(\theta)$ will not densely cover all of the constant energy surface no matter how long you wait.

3. The density matrix of a quantum mechanical system is given by

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|,$$

where p_α is the probability that the system is in the state ψ_α. $\sum_\alpha p_\alpha = 1$ and the states $|\psi_\alpha\rangle$ are distinct but not necessarily orthogonal.

   (a) Show that $\mathrm{Tr}(\rho) = 1$ and every eigenvalue of $\rho \in [0,1]$

   (b) The von Neumann entropy for ρ is given by

$$S = -\mathrm{Tr}\left(\rho \log \rho\right).$$

   What is the form of the density matrix that maximizes S in a Hilbert space of dimension N? What is the form that minimizes S?

4. For a classical Hamiltonian system, show that any quantity given by an integral of the form

$$\int_\Omega f[\rho(t)]dx,$$

is independent of time. Here f is any smooth function of the instantaneous phase space density $\rho(t)$ and the integral is evaluated over a constant energy surface Ω in phase space. What does this imply for the time dependence of the Boltzmann entropy of the system?

5. Now, prove the analog of the above for a quantum system with a time independent Hamiltonian, i.e. show that $\mathrm{Tr}[\mathrm{f}(\rho(\mathrm{t})]$ is independent of time.

6. Show that for a quantum system with a time independent Hamiltonian, there is no equivalent of the statement made in (1) for a classical system. In other words show that if the system starts out in an initial state $|\psi\rangle$, the fraction of time it spends in any region of Hilbert space is in general not equal to the fraction of the volume that region occupies. For this, define "a region of Hilbert space" to mean any given subspace and the fraction of the volume the region occupies to be the ratio of the dimension of the subspace to the dimension of the full Hilbert space.