Lecture 1-4 : BSSP 2020 : Introduction to Topology in statistical physics

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1 Lecture-1 : Topology in statistical physics : What and why ?

1.1 Introduction and mode of these lectures

In this set of lectures we shall try to understand properties of many-body systems which require invoking ideas from a branch of mathematics called topology. This turns out to be a very interesting topic with wide ranging applications.

However, for some reason it was/is not included in our mathematical physics text books or courses at the UG/Masters level. May be we did not appreciate the importance of these ideas completely. May be it is too diverse– I do not know. But rightly the organisers think that this should be remedied and so here I am....

Our course work in mathematical physics should tell us that a mathematical structures have two characteristic features :

- Have a basic set of mathematical ideas that are independent of physics (may be inspired, but logically independent and abstract). This abstraction is the power of mathematics and also the reason mathematical proofs are very rigorous and abstract.

- This structure is then applied to different physics/non physics settings.

We would not be able to do this here for want of time because :

- I will assume we are practicing physicists so we have some background knowledge at quite an advanced level on which we can build upon. This is the role advanced schools and short courses like this one.

- We should aim to get the basic ideas right in concrete contexts that we are familiar with si that we can build upon them according to our research needs/interests.

So the strategy that we are going to take (and I believe this is what other short courses so far has taken implicitly or explicitly) :

a) Assume that you are familiar the following mathematical topics :

1. Elements of group theory (finite and Lie groups)
2. Vector spaces
3. Complex analysis
4. partial differential equations
5. Integrals and functionals.

b) Assume that you are familiar with the following physics topics :

1. **Classical Mechanics :** Hamiltonian and Lagrangian formalism
2. **Quantum mechanics :**
   i. Harmonic Oscillator.
   ii. Perturbation theory.
   iii. Landau levels of a free electron gas.
3. **Electrodynamics :** Maxwell's equations
4. **Statistical mechanics :**
   i. Phase transitions,
   ii. Spontaneous symmetry breaking and order parameters,
   iii. Landau-Ginzburg theory.

This does not mean that I am going to assume that you know how to solve every contour integral (which by the way is a great thing to be able to do, I wish I could !). But would expect that if I use the above ideas, you would understand.

What if you do not ?

ASK !!

Ask questions absolutely freely. In the worst case, I shall say that it is not essential for the topic that we are discussing, OR "I do not know the answer".

If I do not, then I shall try to answer in the next class if I can.

Given these assumptions : The aim would be to show how to use certain ideas of topology and familiar physics contexts. In the process show you the tools that you can then start using in context of your research needs/interests.

With this, I intent develop the course as follows :

a) Show you one explicit example of topological quantum number which can be completely worked out on the basis of what we already know

b) List situations where similar considerations may arise.

c) Discuss topological *defects* in context of spontaneously broken symmetry.

d) In this context discuss the mathematical techniques of homotopy theory.

e) Vortices and BKT transition and XY duality

f) Depending on interest and time :

    1. Chern Simons theories, quantum Hall effect.

    2. Defect suppressed transitions and deconfined criticality

The aim would be to try and tell you the story in a way you can start thinking for yourself. **Would strongly encourage you not to treat these lectures as another repository of to-be-learnt knowledge.**

Crucially : I shall assume that you all are interested in this and shall be paying attention even in this internet platform.

Given this unsatisfactory mode of operations, I will mostly stick verbatim to this set of notes. Of course, I will deviate once in a while to the white-board to write things, but will try not to do too much of it.

1.2 Example of topological quantum number and associated ideas :

1.2.1 Classical EM theory

Consider classical electromagnetism given by the Maxwell's Equations :

$$\nabla \cdot \mathbf{E} = \rho \tag{1}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} = \partial \mathbf{E}/\partial t + \mathbf{j} \tag{4}$$

The degrees of freedom are : Electromagnetic fields and electric charges. Maxwell's equation capture the effect of the charges on the field. For the effect of the field on the Charges we have Newton's laws.

From the Maxwell's equation, we can easily derive the Coulomb's law for electrostatics (almost the first thing that we learnt formally about electromagnetism) for a point charge , q, at origin

$$\mathbf{E} = \frac{q}{4\pi r^2}\hat{\mathbf{r}} \tag{5}$$

What happens at $\mathbf{r} = 0$

Answer : This theory ceases to be a valid description. We must look for a more general theory which can address those type of questions and boils down to Maxwell's equation within "appropriate" approximation.

In this sense Maxwell's equation describe a **classical EFFECTIVE field theory**.

We could go ahead and try to ask what is a more "fundamental theory" which applies to even $\mathbf{r} = \mathbf{0}$ and that is a valid pursuit.

But in this course we shall try to stick to the effective theories ans ask what can we learn from them even within the approximation.

How do we go about having the $\mathbf{r} = 0$ singularity in the **Measurable/Observable** electric field, $\mathbf{E}$ and a point charge within the Maxwell theory ?

Ultra-violet regulation : Define a small length, a and do all the calculations. Then we shall find two types of quantities :

- Quantities which does not blow up when we set $a = 0$. Great !. Go ahead and do it !

- Quantities for which an observable blows up if we naively set $a = 0$. Have to be careful. Have to learn to set $a = 0$ more carefully. This is the territory of *Renormalisation group theory.*

The upshot is that to define the Maxwell theory consistently a-priori we need a UV regulator.

$$\boxed{\textbf{Generalise} : \textbf{Field theories are effective and needs a UV regulator}}$$
$$\boxed{\textbf{and field theories are ill} - \textbf{defined without UV regulators}.}$$

Sometimes it is easy to see what is a natural regular (lattice scale for condensed matter problems) and sometimes it is more hard (field theories of particle physics).

1.2.2 Symmetries

Charge Conservation

$$\int_{all\ space} \rho(\mathbf{r})\ d^d\mathbf{r} = Q = \text{constant} \tag{6}$$

Similarly there are other conservations as listed in Table 1

Generalise : Symmetries are characteristic ingredients of any physical theory. To specify the theory we need to specify the list of symmetries.

#	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$
electric charge, e	-	+	+
magnetic charge, m	-	-	-
electric current, $\mathbf{j}_e$	-	-	-
magnetic current, $\mathbf{j}_m$	-	+	+
electric field, $\mathbf{E}$	-	-	+
magnetic field, $\mathbf{B}$	-	+	-

Table 1: The transformation of EM charges and fields under $\mathcal{CPT}$.

1.2.3 Magnetic Monopole

Now we modify the Maxwell's equation to incorporate the magnetic monopole. This may come from more high energy theories which we would not discuss, but we are interested in studying the fallouts.

We further assume that the magnetic monopole is very heavy (large mass) and it does not carry any electric charge or electric multipole moment. In the rest frame, the Maxwell's equation is now given by

$$\nabla \cdot \mathbf{E} = \rho_e, \qquad \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t - \mathbf{j}_m \tag{7}$$

$$\nabla \cdot \mathbf{B} = \rho_m \qquad \nabla \times \mathbf{B} = \partial \mathbf{E}/\partial t + \mathbf{j}_e \tag{8}$$

In the magnetostatic limit with the monopole at the origin, *i.e.* $\rho_m = m\delta(\mathbf{r})$, we get in three dimensions :

$$\mathbf{B} = \frac{m}{4\pi r^2}\hat{\mathbf{r}} \tag{9}$$

It has a singularity at the origin, but now we know this can be regulated.

Now consider the motion of an electric charge of strength q. It is much lighter than the monopole. Therefore the monopole is static and the electric charge is moving. We just need to solve :

$$\frac{d\mathbf{P}}{dt} = q\mathbf{v} \times \mathbf{B} \tag{10}$$

And there is no problem as long as the electric charge is away from the origin.

Assignment 1 : Solve the above problem for the equation of motion for the classical electric charge.

Now consider that we need to solve the above problem when the electric charge is quantum mechanical while the monopole can still be treated within classical mechanics (*i.e.*, it just remains static). Now we need to solve :

$$H = \frac{1}{2m}(\mathbf{P} - q\mathbf{A}(\mathbf{r}))^2 \tag{11}$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential. What is it ? Note that

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{12}$$

Note : This is an example of local relation, *i.e.* we need to know the value of $\mathbf{A}(\mathbf{r})$ in the vicinity of $\mathbf{r}$ to calculate $\mathbf{B}(\mathbf{r})$, for example

$$B_z(\mathbf{r}) = \lim_{\Delta_x, \Delta_y \to 0} \left[\frac{A_y(\mathbf{r} + \Delta_x) - A_y(\mathbf{r})}{\Delta_x} - \frac{A_x(\mathbf{r} + \Delta_x) - A_x(\mathbf{r})}{\Delta_y} \right] \tag{13}$$

SINCE $\mathbf{A}(\mathbf{r})$ is not ab observable and we need it to be defined only locally to define $\mathbf{B}(\mathbf{r})$, therefore we do not need the vector

potential to be globally uniquely defined. Then we can define the vector potential for the monopole, as

$$A_\phi(\mathbf{r}) = -\frac{m}{4\pi r}\cot(\theta/2)\hat{\phi} \tag{14}$$

everywhere except on the half-line $\theta = 0$ and

$$A_\phi(\mathbf{r}) = \frac{m}{4\pi r}\tan(\theta/2)\hat{\phi} \tag{15}$$

everywhere except on the half-line $\theta = \pi$. In the common region the difference of the two choices is a pure gauge.

Note : Other choices are also possible. All of them has two types of singularity

- $\mathbf{r} = 0$ singularity : We already know how to regularise it and it is a singularity of an observable. Consequence $\rightarrow$ magnetic monopole similar to the above singularity in the coulomb electric field.

- **Line singularity (Dirac String) :** This singularity is not in a physical observable. Its consequence are more subtle. First pointed out by Dirac.

Before we go on to understand the consequence of the Dirac strong, we note the following :

- The **Un-observable** line singularity of **A** ends at the monopole (**regulated singularity** of the observable, **B**).
- The two singularities cannot be detached from each other.
- The magnetic charge is conserved within the above theory. So It is a *good quantum number* of the system. However this is unlike the electric charge quantum number which does not come with the attached with its Dirac string.

General Forecast : Two types of Quantum Numbers.

1. **Those without "string" : Symmetry related quantum numbers (like the electric charge)**

2. **Those with "string" : Topological quantum numbers (like the magnetic monopole).**

Because of the string, the topological quantum numbers have extra robustness that make them immune to "details".

There would be important modification to the above naive idea, but the central message would not change.

1.2.4 Path Integral formulation of Quantum Mechanics

The quantum mechanics of a quantum electric charge in presence of a magnetic monopole is given by the solution of

$$H = \frac{1}{2m}(\mathbf{P} - q\mathbf{A}(\mathbf{r}))^2 \tag{16}$$

where $\mathbf{A}(\mathbf{r})$ is the monopole potential.

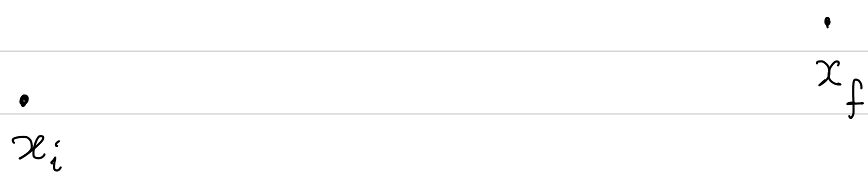
Assignment 2 : Solve the above problem for the equation of motion for the quantum electric charge.

However we want to have a general technique that would be applicable to a large number of situations. To this end we now recap the path-integral form of the above Hamiltonian and understand the physics of Aharonov-Bohm phase.

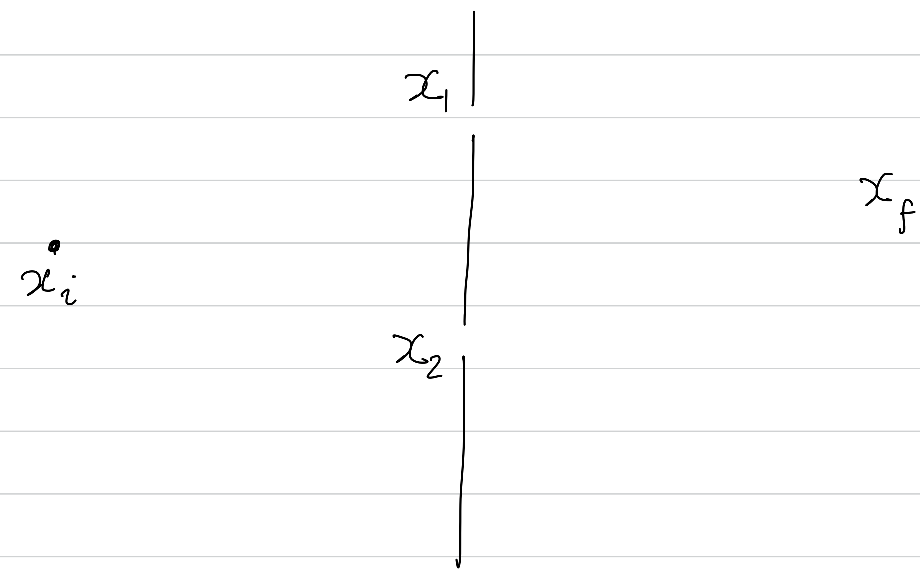
Path Integral formulation of Single-particle quantum Mechanics, (A. Zee: QFT book)

Suppose we wish to understand the amplitude of a particle starting from $|x_i\rangle$ and ending at $|x_f\rangle$ after time t. This is given by

$$A_{fi}(t) = \langle x_f | e^{-iHt/\hbar} | x_i \rangle$$



Now suppose there is a double slit Screen in between.



$$\therefore \; A_{fi}(t) = \Big[\langle x_f | e^{-iHt_1/\hbar} | x_1 \rangle \langle x_1 | e^{-iH(t-t_1)/\hbar} | x_i \rangle$$

$$+ \langle x_f | e^{-iHt_2/\hbar} | x_2 \rangle \langle x_2 | e^{-iH(t-t_2)/\hbar} | x_i \rangle \Big]$$

$$= \sum_{p=1}^{2} \langle x_f | e^{-iHt_p/\hbar} | x_p \rangle \langle x_p | e^{-iH(t-t_p)/\hbar} | x_i \rangle$$

$$= \langle x_f | \, e^{-iHt_p/\hbar} \underbrace{\left(\sum_p |x_p\rangle\langle x_p| \right)}_{\text{Identity}} e^{-iH(t-t_p)/\hbar} |x_i\rangle$$

Therefore this can be extended to many slits.

$$A_{fi}(t) = \sum_{p=1}^{N} \langle x_f | \, e^{-iHt_p/\hbar} |x_p\rangle\langle x_p| \, e^{-iH(t-t_p)/\hbar} |x_i\rangle$$

And many screens.

$$A_{fi}(t) = \sum_{P_1=0}^{N_1} \sum_{P_2=1}^{N_2} \cdots \sum_{P_n=1}^{N_n} \langle x_f | x_n\rangle\langle x_n| e^{-iHt_{p_n}} |x_{p_{n-1}}\rangle\langle x_{p_{n-1}}| e^{-iHt_{p_{n-1}}} |x_{n-2}\rangle$$

$$\langle x_{p_{n-2}}| \cdots\cdots |x_{p_1}\rangle\langle x_{p_1}| e^{-iHt_{p_n}} |x_0\rangle$$

$\langle x_0 | x_i \rangle$

where $t_{p_1} + t_{p_2} + \cdots t_{p_n} = t$

Now Free Space $\equiv$ $N \to \infty$, $n \to \infty$

$$A_{fi}(t) = \int dx_0\, dx_1 \cdots\cdots dx_n \langle x_f | x_n \rangle \langle x_n | e^{-it_0 H/\hbar} | x_{n-1} \rangle \cdots\cdots \langle x_1 | e^{-it_0 H/\hbar} | x_0 \rangle \langle x_0 | x_i \rangle$$

Now we can take $t_1 = t_2 = \cdots t_n = \Delta t$

Such that $\Delta t = \frac{t}{n}$

Therefore $n \to \infty$ limit

$$A_{fi}(t) = \int \underbrace{\left[\lim_{n\to\infty} \prod_{p=0}^{n} dx_p \right]}_{\equiv \mathcal{D}X} \langle x_f | x_n \rangle \langle x_0 | x_i \rangle$$

$$\langle x_n | e^{-iH\Delta t/\hbar} | x_{n-1} \rangle \langle x_{n-1} | e^{-iH\Delta t/\hbar} | x_{n-2} \rangle$$

$$\cdots \langle x_1 | e^{-iH\Delta t/\hbar} | x_0 \rangle$$

Let us look at each time-slice

$$M_{\tau,\tau-1} = \langle x_\tau | e^{-iH\Delta t/\hbar} | x_{\tau-1} \rangle$$

For free particles. $H = \frac{\hat{p}^2}{2m}$

$$M_{\tau,\tau-1} = \langle x_\tau | \exp\left[-\frac{i\Delta t}{\hbar} \frac{p^2}{2m} \right] | x_{\tau-1} \rangle$$

$$M_{\tau,\tau-1} = \int dP_\tau \, \langle x_\tau | P_\tau \rangle \langle P_\tau | \exp\left[-\frac{i\Delta t}{\hbar} \frac{\hat{P}^2}{2m} \right] | x_{\tau-1} \rangle$$

$$= \int dP_\tau \, \langle x_\tau | P_\tau \rangle \langle P_\tau | x_{\tau-1} \rangle \exp\left[-\frac{i\Delta t}{\hbar} \frac{P_\tau^2}{2m} \right]$$

$$\langle x | p \rangle = \# \, e^{-\frac{ixp}{\hbar}}$$

$$M_{\tau,\tau-1} = \# \int dP_\tau \, e^{-i(X_\tau - X_{\tau-1}) P_\tau - \frac{i\Delta t}{\hbar} \frac{P_\tau^2}{2m}}$$

$$= \# \int dP_\tau \, e^{-\frac{i\Delta t}{2\hbar} \left[2\dot{X}_\tau P_\tau + \frac{P_\tau^2}{m} \right]}$$

$$= \# \int dP_\tau \, e^{-\frac{i\Delta t}{2\hbar} \left[\left(\frac{P_\tau}{\sqrt{m}} + \sqrt{m}\,\dot{X}_\tau \right)^2 - m\dot{X}_\tau^2 \right]}$$

Integrate out P (Trick → Contour Integral by rotating to imaginary axis or imaginary time)

$$M_{\tau,\tau-1} = \# \; e^{\frac{i\Delta t}{\hbar} \frac{m\dot{X}_\tau^2}{2}}$$

$$\therefore \quad A_{fi}(t) = \# \int [\mathcal{D}X] \langle x_f | x_n \rangle \langle x_0 | x_i \rangle \, e^{\frac{i\Delta t}{\hbar} \sum_{p=0}^{n} \frac{m\dot{X}_p^2}{2}}$$

Take $\Delta t \to 0$

$$A_{fi}(t) = \# \int [\mathcal{D}X] \langle x_f | x_n \rangle \langle x_0 | x_i \rangle \, e^{\frac{iS}{\hbar}}$$

$S = \int dt \, \frac{m\dot{X}^2}{2}$ is the Action

$$\mathcal{L} = \frac{m\dot{x}^2}{2}$$ is the Lagrangian.

Now let us start with

$$H = \frac{\hat{P}^2}{2m} + V(\hat{x})$$

Assignment-3: Show that Lagrangian

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x)$$

Now let us concentrate on our

$$H = \frac{1}{2m}\left(\hat{P} - qA(\hat{x})\right)^2$$

$$\mathcal{H} = \frac{1}{2m}\left[\hat{P}^2 - q\hat{A}\cdot\hat{P} - q\hat{P}\cdot\hat{A} + q^2\hat{A}^2\right]$$

Now, $\left[\hat{A}_\alpha, \hat{P}_\beta\right] = i\hbar \dfrac{\partial A_\alpha}{\partial x_\beta}$

$$\therefore \quad \hat{A}\cdot\hat{P} = A_\alpha P_\alpha = i\hbar\frac{\partial A_\alpha}{\partial x_\alpha} + \hat{P}\cdot\hat{A} = i\hbar\,\nabla\cdot\hat{A}$$

$$\therefore \quad \mathcal{H} = \frac{1}{2m}\left[\hat{P}^2 - 2q\hat{P}\cdot\hat{A} - i\hbar q\,\nabla\cdot\hat{A} + q^2\hat{A}^2\right]$$

Now each time-slice is given by.

$$M_{\tau,\tau-1} = \langle x_\tau | \exp\left[-\frac{i\Delta\tau}{2\hbar m}\left[\hat{P}^2 - 2q\hat{P}\cdot\hat{A} - i\hbar q\nabla\cdot\hat{A} + q^2\hat{A}^2\right]\right] |x_{\tau-1}\rangle$$

(for small $\Delta\tau$)

$$\approx \langle x_\tau | \exp\left[-\frac{i\Delta\tau}{2\hbar m}\left[\hat{P}^2 - 2q\hat{P}\cdot\vec{A}(x_{\tau-1})\right]\right] |x_{\tau-1}\rangle$$

$$\times \exp\left[-\frac{i\Delta\tau}{2\hbar m}\left(-i\hbar q\vec{\nabla}\cdot\vec{A}(x_{\tau-1}) + q^2 A^2(x_{\tau-1})\right)\right]$$

$$= \# \int dP_\tau \quad \langle x_\tau | P_\tau \rangle \langle P_\tau | x_{\tau-1} \rangle$$

$$\exp\left[-\frac{i\Delta\tau}{2\hbar m}\left(P_\tau^2 - 2q\vec{P}_\tau\cdot\vec{A}_{\tau-1} - i\hbar q\vec{\nabla}\cdot\vec{A}_{\tau-1} + q^2 A^2_{\tau-1}\right)\right]$$

$$M_{\tau,\tau-1} = \# \int dP_\tau \; e^{-\frac{i\Delta\tau}{\hbar}\left[\dot{\vec{X}}_\tau \cdot \vec{P}_\tau + \frac{1}{2m}\left(P_\tau^2 - 2q\vec{P}_\tau\cdot\vec{A}_{\tau-1} + q^2 A_{\tau-1}^2\right) - i\hbar q \vec{\nabla}\cdot\vec{A}_{\tau-1}\right]}$$

$$= \# \int dP_\tau \; e^{-\frac{i\Delta\tau}{2\hbar m}\left[2m\dot{\vec{X}}_\tau\cdot\vec{P}_\tau + \left(\vec{P}_\tau - q\vec{A}_{\tau-1}\right)^2\right]} \; e^{-\frac{\Delta\tau}{2m} q\vec{\nabla}\cdot\vec{A}_{\tau-1}}$$

$$= \# \int dP_\tau \; e^{-\frac{i\Delta\tau}{2\hbar m}\left[2m\dot{\vec{X}}_\tau\cdot\left(\vec{P}_\tau - e\vec{A}_{\tau-1}\right) + \left(\vec{P}_\tau - e\vec{A}_{\tau-1}\right)^2 + 2m\dot{\vec{X}}_\tau\cdot e\vec{A}_{\tau-1}\right]} \times e^{-\frac{\Delta\tau}{2m}\vec{\nabla}\cdot\vec{A}_{\tau-1}}$$

$$= \# \; e^{i\Delta\tau\left(\frac{1}{2}m\dot{x}_\tau^2 + q\dot{\vec{X}}_\tau\cdot\vec{A}_{\tau-1}\right)}$$

In the Limit $\Delta\tau \to 0$

$$M_{\tau,\tau-1} = \# \, e^{i\Delta\tau \left(\frac{1}{2} m \dot{x}_\tau^2 + q \dot{\vec{X}}_\tau \cdot \vec{A}_\tau\right)}$$

$\therefore$ The Action is given by

$$S = \int dt \left[\frac{1}{2} m \dot{x}^2\right] + q \int dt \, \dot{\vec{X}} \cdot \vec{A}(\vec{x})$$

$$= S_{\text{free particle}} + q \int \vec{A}(\vec{x}) \cdot d\vec{x}$$

$$\therefore \; e^{\frac{iS}{\hbar}} = e^{\frac{iS_0}{\hbar}} \, e^{\frac{iq}{\hbar} \int \vec{A}(\vec{x}) \cdot d\vec{x}}$$

2 Lecture-2 :

2.1 Continuation of the example from electromagnetism

Recap of Lecture 1 :

- Theories are effective in nature. There is a UV regulator.
- We need to specify the symmetries and how the degrees of freedom transform under these symmetries
- We can extend Maxwell's Equation to include monopoles. The vector potential is now piecewise defined and has a line singularity ending on a monopole.

- In quantum mechanics we need this vector potential to solve for the motion of the electric charge.

- We derived the path integral for such a system and we found

$$\mathcal{S} = \mathcal{S}_0 + \int d\mathbf{r} \cdot \mathbf{A}$$

2.1.1 Berry Phase

Now

$$\mathcal{S} = \int dt \, \frac{1}{2} m \dot{\mathbf{r}}^2 + q \int \mathbf{A} \cdot d\mathbf{r} = \mathcal{S}_0 + \mathcal{S}_{Berry} \tag{17}$$

while the first term depends on the details of the path from i to f, the **second term is "path independent"**.

Also for $i = f$ (return amplitude) the second term is given by

$$\mathcal{S}_{Berry} = q \oint_c \mathbf{A} \cdot d\mathbf{r} \tag{18}$$

Naively we would expect that if this is path independent since the initial and final points are same then this integral is zero and we should be able to neglect it.

However this is not true. In certain circumstances, as we shall see below, the above Berry phase depends on the **Topology** of the **manifold** on which the closed loop is drawn. In particular, it is sensitive to the **singularities**.

We shall see two examples.

2.1.2 Quantisation of electric charge

Consider the magnetic monopole and consider the Dirac string along $\theta = 0$ such that

$$A_\phi(\mathbf{r}) = -\frac{m}{4\pi r}\cot(\theta/2)\hat{\phi} \tag{19}$$

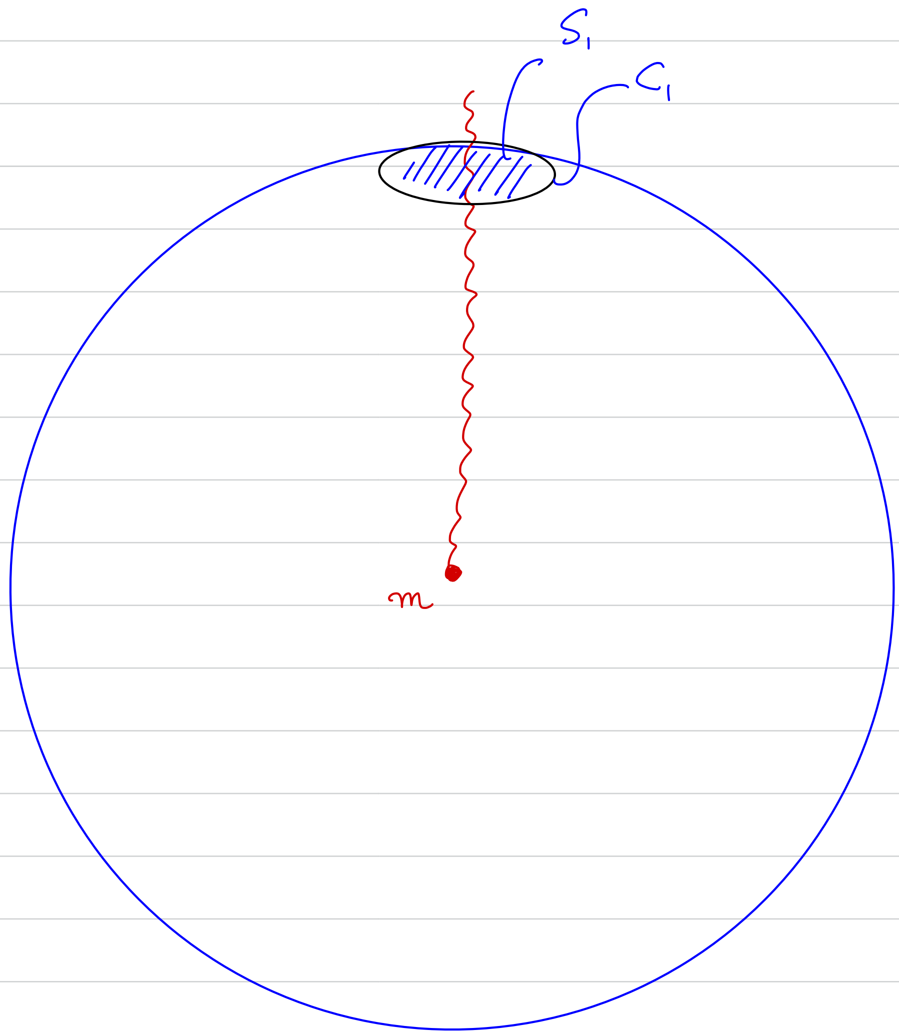
Now consider the electric charge going around the Dirac string as shown in the figure in the next page. Then the Berry phase gained is

$$\Phi = \frac{q}{\hbar}\oint_{c_1} \mathbf{A}\cdot d\mathbf{r} = \frac{q}{\hbar}\oiint_{s_1} \mathbf{B}\cdot d\mathbf{\Omega} = \frac{q}{\hbar}\oiint_{\bar{s}_1} \mathbf{B}\cdot d\mathbf{\Omega} = \frac{qm}{\hbar} \tag{20}$$

Since this is the phase gained by going around the string should not lead to any observable consequence therefore

$$\Phi = 2n\pi \quad \Rightarrow \qquad qm = nh \tag{21}$$

where n is an integer.



S_1
C_1
m

2.1.3 Angular momentum quantization

Charge quantisation can also be obtained, as done by M. N. Saha by quantising the angular momentum of a system containing a monopole and an electric charge.

If there is a monopole-charge system with a very heavy monopole (so that we can work in the frame where the centre of mass is at the monopole, origin) then the Lorentz force on the charge is:

$$\mu\ddot{\mathbf{r}} = \frac{qm}{4\pi r^3}\dot{\mathbf{r}} \times \mathbf{r} \tag{22}$$

Using the velocity in spherical polar coordinates, we can prove that the rate of change of the angular momentum is given by

$$\dot{\mathbf{L}} = \frac{em}{4\pi}\dot{\hat{\mathbf{r}}} \tag{23}$$

so that

$$\mathcal{L} = \mathbf{L} - \frac{em}{4\pi}\hat{\mathbf{r}} \tag{24}$$

is conserved. Now using the Dirac quantization condition, we get:

$$\mathcal{L} = \mathbf{L} - \frac{n\hbar}{2}\hat{\mathbf{r}} \tag{25}$$

which says that the system may have half integer angular momentum even if the origin particles have spin zero. Such a composite of a monopole and a charge can be looked up as a Dyon.

Hence such a static Dyon already has an angular momentum of $n\hbar/2$ even if the spins of the monopole and the charge are zero. Particularly consider the case of $n = 1$ which is a composite of $(e, \hbar/e)$. The angular momentum is then $-\hbar/2\hat{\mathbf{r}}$ and pointing from the charge to the monopole.

2.1.4 Dyons

Dyons can be considered as composites of electric charge and magnetic monopole and are denoted by (e, m). Now Dirac quantization condition says that $em = 2\pi n$ $(\hbar = 1)$. This is generalized for two dyons (e_1, m_1) and (e_2, m_2). Consider the equation of motion for the first dyon due to the second (again we think the second to be heavy and hence at the centre of mass. origin)

$$\mu\ddot{\mathbf{r}} = e_1(\mathbf{E}_2 + \dot{\mathbf{r}} \times \mathbf{B}_2) + m_1(\mathbf{B}_2 - \dot{\mathbf{r}} \times \mathbf{E}_1) \tag{26}$$

Proceeding as before we get that the angular momentum is given by:

$$\mathcal{L} = \mathbf{L} - \frac{e_1 m_2 - e_2 m_1}{4\pi}\hat{\mathbf{r}} \tag{27}$$

If we want to quantize the angular momentum along the line joining the two dyons and demand that it be either an integer or a half odd integer then we get

$$e_1 m_2 - e_2 m_1 = 2\pi n \tag{28}$$

In a world with electric charges $(e, 0)$, the lowest monopole charge is $2\pi/e$. The electric charge of the monopole is however not constrained because it never enters into the quantization condition. If we have two such monopoles with same minimal magnetic charge but different electric charges, $(q, 2\pi/e)$ and $(q', 2\pi/e)$. Then the above condition gives $q - q' = ne$, i.e. the difference between the charges have to be integer times the electric charge. However there is no constrain on the individual charges. (If however the magnetic charges are also different, say $(q, 2p\pi/e)$ and $(q', 2p'\pi/e)$ then we have $p'q - pq' = ne$).

Consider the transformation of the different elements under $\mathcal{CPT}$ as given in Table 1

Then if we want a theory which is invariant under $\mathcal{CP}$ then it must allow dyons of the type $(q, 2\pi/e)$ and $(-q, 2\pi/e)$. Then the above condition states that

$$\frac{q}{e} = \frac{n}{2} \tag{29}$$

Therefore the charges will be either integer or half integer.

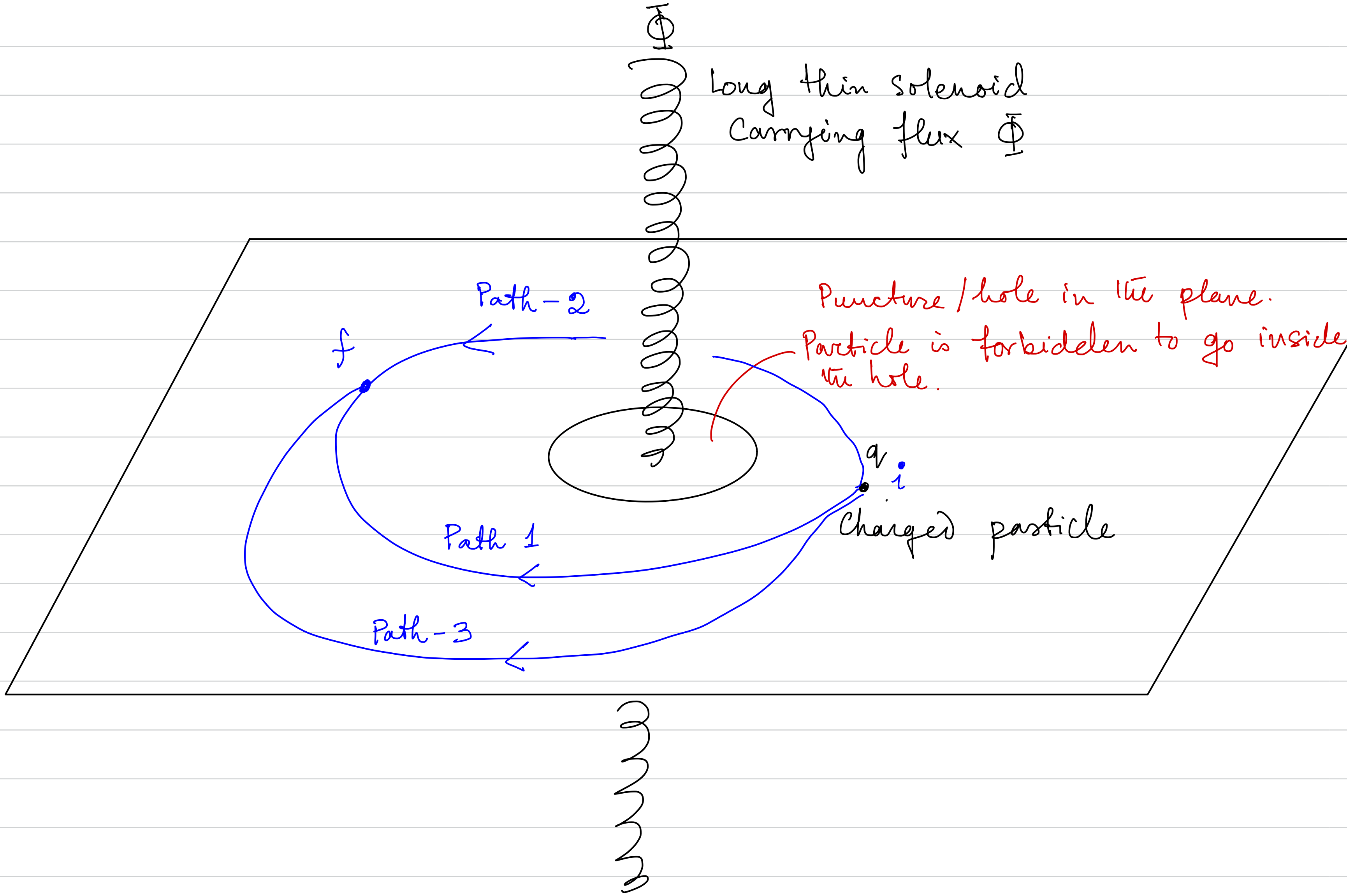
2.1.5 Aharonov-Bohm Phase as an example of Berry phase

$$A_{fi} = \sum_{paths} e^{i\mathcal{S}/\hbar} \quad \Rightarrow \quad A_{ii} = \sum_{closed\ paths} e^{i\mathcal{S}/\hbar} = \sum_{closed\ paths} e^{i\mathcal{S}_0/\hbar} e^{i\mathcal{S}_{Berry}/\hbar} \tag{30}$$

If $\mathcal{S}_0$ does not vary to a good approximation over the paths much (and there are such situations as we shall soon see), then

$$A_{ii} \approx \# \sum_{\text{closed paths}} e^{i\mathcal{S}_{Berry}/\hbar} \tag{31}$$

Charges in a multiply connected plane : Consider a charge moving in an infinite plane with one hole. The hole is threaded by a long thin solenoid carrying a magnetic flux Φ. (Next page)



Φ
Long thin solenoid
Carrying flux Φ
Path - 2
f
Puncture / hole in the plane.
Particle is forbidden to go inside the hole.
q
i
Charged particle
Path 1
Path - 3

Let us only consider three paths. We can arrange the potentials such that $_0$ is same for the three paths. Therefore

$$A_{fi} \propto \exp\left[\frac{iq}{\hbar}\int_{path-1}\mathbf{A}d\mathbf{r}\cdot\right] + \exp\left[\frac{iq}{\hbar}\int_{path-2}\mathbf{A}d\mathbf{r}\cdot\right] + \exp\left[\frac{iq}{\hbar}\int_{path-3}\mathbf{A}d\mathbf{r}\cdot\right] \tag{32}$$

$$= \exp\left[\frac{iq}{\hbar}\int_{path-1}\mathbf{A}d\mathbf{r}\cdot\right]\left[1 + \exp\left[\frac{iq}{\hbar}\int_{path-(2-1)}\mathbf{A}d\mathbf{r}\cdot\right] + \exp\left[\frac{iq}{\hbar}\int_{path-(3-1)}\mathbf{A}d\mathbf{r}\cdot\right]\right] \tag{33}$$

Path (2-1) and path (3-1) are closed loops as indicated in the figure. Therefore

$$\exp\left[\frac{iq}{\hbar}\oint_{loop-(2-1)}\mathbf{A}d\mathbf{r}\cdot\right] = \exp\left[\frac{iq}{\hbar}\oiint_{loop-(2-1)}\mathbf{B}d\sigma\cdot\right] = \exp[iq\Phi/\hbar] \tag{34}$$

$$\exp\left[\frac{iq}{\hbar}\oint_{loop-(3-1)}\mathbf{A}d\mathbf{r}\cdot\right] = \exp\left[\frac{iq}{\hbar}\oiint_{loop-(3-1)}\mathbf{B}d\sigma\cdot\right] = 1 \quad (35)$$

Therefore

$$A_{fi} \propto [2 + \exp[iq\Phi/\hbar]] \quad (36)$$

In fact all loops that circle the hole contributes

$$\exp[iq\Phi/\hbar] \quad (37)$$

while if it does not it contributes 1.

So the contribution is **ONLY** sensitive to the holes and nothing else– not any fine features. Further :

Paths 1 and 3 can be continuously deformed into each other. **How ?**

- Consider a parameter $\tau \in [0,1]$. Now consider three **continuous** different functions :

$$f_1(\tau), f_2(\tau), f_3(\tau)$$

- such that

$$f_1(0) = f_2(0) = f_3(0) = \text{initial position}, i$$

$$f_1(1) = f_2(1) = f_3(1) = \text{final position}, f$$

- Now define the functions that as τ is varied between $[0,1]$ they go along three paths respectively and at each point the function is continuous.

- Now consider another parameter $\eta \in [0,1]$. Now consider the function

$$f_{13} = \eta f_1(\tau) + (1-\eta) f_3(\tau)$$

- Clearly as we vary η we can draw an infinite set of curves whose limiting sequence are path-1 and path-3 (one can find other parameterization).
- But if we try to define similar deformation between 1 or 3 and 2. Then the interpolating curve does not stay in the area restricted for the particle and hence does not represent a continuous path for the particle.
- To change path 1 or 3 to path 2 we need to **CUT** the path and **REJOIN**.
- This is because **Holes** are gross features. To create them we need to create discontinuities– topological features (as we shall see).
- With each class of deformable paths we associate a phase :

  Path-class 1,3 : Phase =0

  path-class 2 : Phase= Φ.

path-class $\bar{2}$ (path 2 but in opposite direction) : Phase$= -\Phi$.

- Therefore all loops starting at i and passing through f can be divided into three mutually exclusive sub-sets (equivalence classes) characterised by the phase. There two such classes in this case.

Can we see the phase ?

The probability should show oscillations with frequency

$$\omega = q\Phi/\hbar \tag{38}$$

as a function of Φ. This is called Ahoronov-Bohm phase.

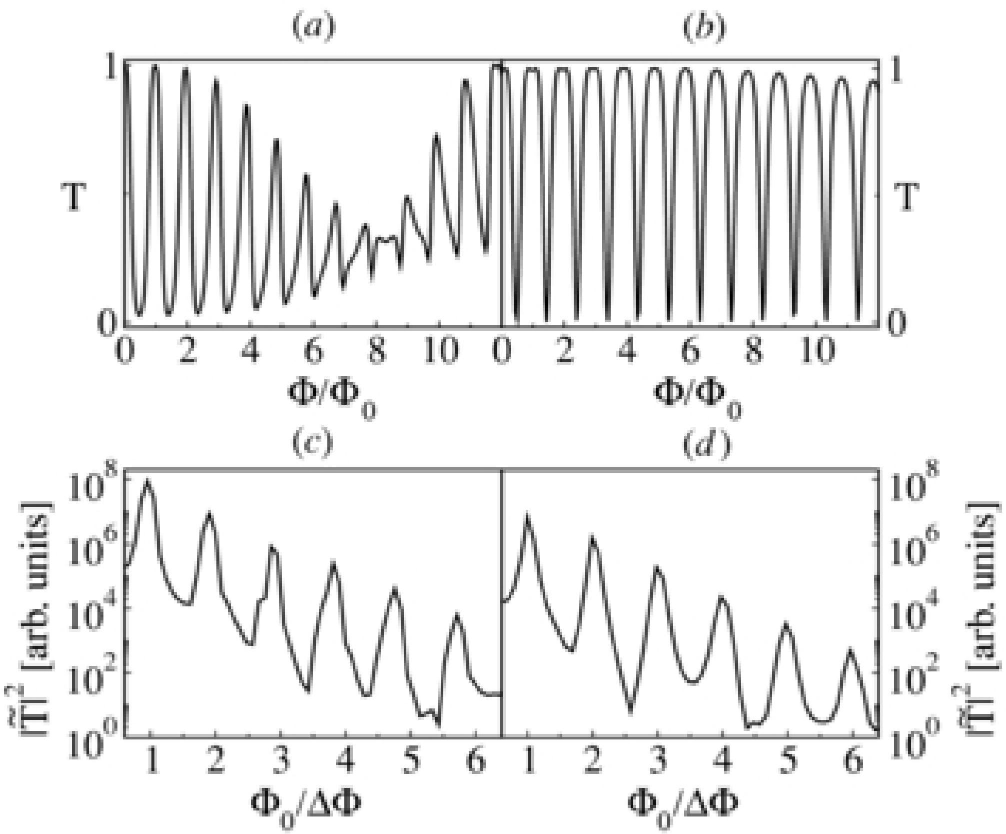


Figure 1: AB effect in Graphene rings. Reference : Wurm et. al. Semiconductor Science and Technology, 2010

3 Lecture-3

3.1 Classification of closed curves

At this point it is useful to generalise some of the above concepts and put them on generalised mathematical grounds. In context of the above problem, we want to :

- Classify closed curves
- Associate a robust invariant with each class.

To this end :

1. We need to fix one point on the curve to compare them meaningfully. Let us call this the **base point**.

2. Consider all set of curves passing through a particular point i (**base point**) in the above figure. Let us denote them by

$$f_n(\tau)$$

with $\tau \in [0, 1]$ such that

$$f_n(0) = i \ \ \forall n$$

3. The above parameterization assigns direction to each curve. **Directed curves :** oppositely directed curves are counted as different curves.

4. We will be interested in closed curves. So consider closed curves defined as

$$f_n(0) = f_n(1) = i \ \ \forall n$$

5. Let us start with individual curves at the base point i. The set of curves are closed under following definition of addition (curve composition).

$$f_1(\tau) \circ f_2(\tau) = \begin{cases} f_1(2\tau) & \tau \in [0, 1/2] \\ f_2(2\tau - 1) & \tau \in [1/2, 1] \end{cases} \tag{39}$$

But this is bad because this means that at $\tau = 1/2$ the loop is at i. This is certainly a sub-class of all the loops passing through i.

6. If we wish to keep the above composition for three curves

$$(f_1 \circ f_2) \circ f_3 = \begin{cases} f_1(4\tau) & \tau \in [0, 1/4] \\ f_2(4\tau - 1) & \tau \in [1/4, 1/2] \\ f_3(2\tau - 1) & \tau \in [1/2, 1] \end{cases} \tag{40}$$

This is even worse as the path needs to be at the base point i at $\tau = 1/4, 1/2$. So this would not work. We need a different strategy.

7. We note that we do not wish to track each curve. But we want to know about clubbing of curves that can be deformed together continuously since they have the same phase.

8. **Homotopic class of curves at base point i :** Consider a sub-set of the above curves which can be continuously deformed into each other.

$$[f]$$

This sub-set contains all curves with base-point, i, that can be deformed into each other continuously using a parameter such as η as defined before.

Each Homotopic family forms an **equivalence class**.

Equivalence Relation : A relation($\sim$) defined on the elements of a set X is called an equivalence relation if

(a) reflextive: $a \sim a$ holds for all $a \in X$.

(b) symmetric: If $a \sim b$ then $b \sim a$ for all $a, b \in X$.

(c) Transitive: If $a \sim b$ and $b \sim c$, then $a \sim c$ for all $a, b, c \in X$.

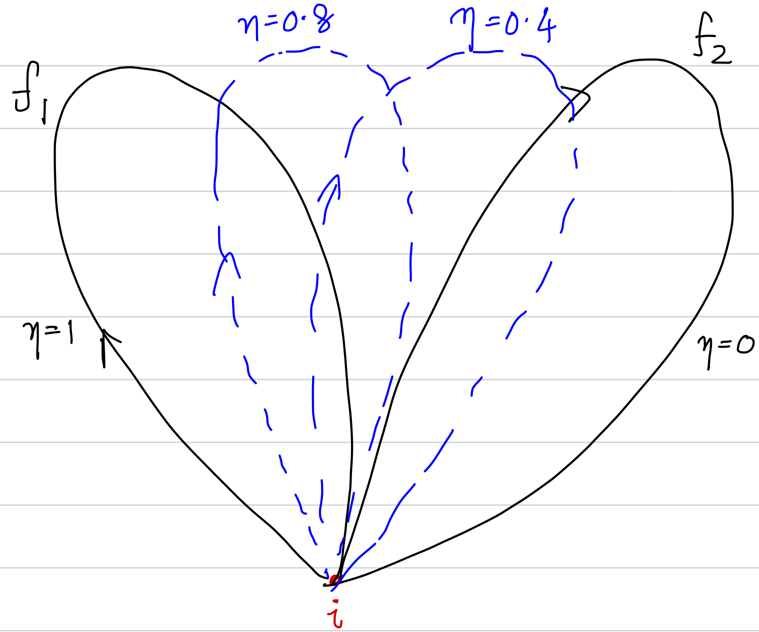
An equivalence relation partitions (mutually disjoint subsets) the set into classes called **equivalence classes**. These are

$$[a] \equiv \{x \in X | x \sim a\} \tag{41}$$

- $[a] \neq \Phi$, since it contains atleast $a (a \sim a)$.

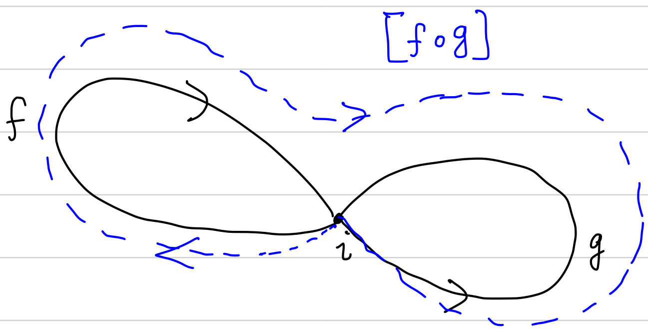
- If $[a] \cap [b] \neq \Phi$, then $[a] = [b]$.
- Thus equivalence classes are represented by a representative element. The set of these representative element is called the **quotient set**. This is denoted by $X/\sim$.

$$[f] = \eta f_1(\tau) + (1-\eta) f_2(\tau)$$



Now

$[f] \circ [g]$



9. Now we can define a composition law

$$[f] \circ [g]$$

10. Now the composition is associative

$$([f] \circ [g]) \circ [h] = [f] \circ ([g] \circ [h])$$

11. Contains an identity

$$[I]$$

Such that they are continuously deformable to $f_{ID}(\tau) = i \;\; \forall \; \tau \in [0, 1]$

12. Contains an Inverse : Contains curves with opposite directions.

$$[f] \circ [f^{-1}] = [I]$$

13. **Therefore the set of closed loops homotopic with a base-point, i, form a group.**

14. This is called the **Fundamental homotopy group** or **first homotopy group** denoted by

$$\pi_1(\mathcal{M}; i)$$

where $\mathcal{M}$ is the space in which the base-point, i, lives.

In the case that we are concerned with $\mathcal{M}$ is a **Manifold**.

A manifold is a space which locally looks like $\mathbb{R}$ or $\mathbb{C}$. Hence is it locally continuous and smooth (infinitely differentiable).

In physics we need to study properties of various such manifolds. Depending on the kind of property we need to study about the manifold, we employ different techniques. The two extreme view points are that of vector spaces and topology :

- **Vector spaces:** At every point of the manifold we define a tangent plane. A vector space may be defined on this tangent plane to describe the manifold locally. (To draw the tangent plane. first set up an orthogonal coordinate system of the "directions" within the manifold. The unit vectors along these directions define the tangent plane.)
- **Topology:** On the other hand, when one tries to study the global properties of the manifold, it falls under topology. One defines differ-

ent kinds of "measures" to study and accordingly, we call algebraic or geometric topology.

In our case

$$\mathcal{M} = R^2 - \{\mathbf{r}_0\}$$

where $\mathbf{r}_0$ denotes the position of the hole and the area around it which is forbidden.

15. In this case

$$\pi_1(R^2 - \{\mathbf{r}_0\}, i)$$

contains the countably infinite set of elements elements.

$$\{0, \pm\Phi, \pm 2\Phi, \cdots\} = \{n\}\Phi \qquad n \text{ is an integer}$$

$$n = n_{anti\ clock-wise} - n_{clock-wise}$$

where $n_{anti\ clock-wise\ (clock-wise)}$ is the anti clock-wise (clock-wise) number of winding around the hole. Thus this is an abelian group under addition.

Abstract Fundamental group $\pi_1(\mathcal{M})$ is obtained by relating fundamental groups at two different base-points through **path isomorphisms**.

For each homotopy class in π_1 we have the following relation :

$$[f] \to n\Phi$$

This means that with every class of closed curves based at i, we can associate a phase $n\Phi$. This is an example of a **Map**.

Let $\{x\}$ and $\{y\}$ are two sets, such that there is a rule f which assigns to every element in $\{x\}$, an element in $\{y\}$. This is denote by

$$f : x \rightarrow y = f(x) \tag{42}$$

$\{x\}$ is call the **domain** and $\{y\}$ is called the **range** and y is called the **image** of x under f. One can also try to define an inverse image. However, it is not guaranteed that such inverse image exists and even if it exists it is unique (e.g. $y = x^2$).

Important points regarding mapping

1. A map $f : x \rightarrow y$, such that if $x \neq x'$ then $f(x) \neq f(x')$, then the mapping is called **one to one** or **injective**.

2. A map $f : x \to y$, such that for all y there is an x, then the mapping is called **onto** or **surjective**.

3. If a mapping is both injective and surjective then it is called **bijective**. If a map is bijective, then an inverse map exist.

4. There exists a trivial bijective map that maps $x \to x$. This is called the **Identity map**.

5. The rules of composition (addition/multiplication) of elements of a set defines an **Algebra**. If a mapping $f : x \to y$ preserves the algebra, then it is called **homomorphism**. Further, if the mapping is bijective as well as homomorphic, then it is called **Isomorphism**. In that case it is said that x is isomorphic to y and is denoted as $x \cong y$.

Zero holes : $\pi_1(R^2) = 0$, *i.e.*, there is only one element– Identity. **This is the simply connected plane**.

One hole : $\pi_1(R^2 - \mathbf{r}_0) = \mathbb{Z}$. It requires **one** integer. **Multiply connected plane, Multiplicity=1**.

n **holes :** each carrying flux Φ_i, we should have

$$A_{fi} \propto \sum_{[f]} e^{iq\Phi_{[f]}/\hbar} \tag{43}$$

where $[f]$ is the homotopy class and

$$\Phi_{[f]} = \sum_{i=1}^{n} \Phi_i n_i$$

where n_i is the net number of winding around the ith hole. Thus we can define an integer vector

$$\mathbf{n} = \{n_1, n_2, \cdots, n_n\}.$$

Thus, this forms a group under vector addition of a n dimensional lattice.

(For $a, b \in X$, All ordered pair $\{a, b\}$ is called a cartesian product.)

Assignment : Consider the case of two holes : The figure of 8 **group. What is** π_1 **?**

Two important conclusions which we would generalise

- Closed curves on a manifold can be classified.
- With each class we can associate an invariant.

3.2 Generalising the discussions to Many-body systems

How do we describe many-body systems ?

Many-body $\sim 10^{23}$.

There are roughly three ingredients to specify a many-body system :

1. Degrees of freedom : Quantum/classical **degrees of freedom are effective and depend on the energy-scale**
2. Symmetries : (can be effective/approximate).
3. Hamiltonian : Interactions between the degrees of freedom obeying the symmetries

Let us choose classical degrees of freedom :

N-component unit vectors (classical spins), $\mathbf{n}_i$, on a lattice interacting by short range interactions which has global rotational symmetry. Under rotation

$$\mathbf{n}_i \to \mathcal{R} \cdot \mathbf{n}_i \qquad \mathcal{R} \in O(N) \tag{44}$$

- $N = 1$: Ising model Z_2
- $N = 2$: XY model $O(2)$
- $N = 3$: Heisenberg model $O(3)$

There are two length-scales :

1. lattice length-scale a (Also the UV regulator)
2. Linear dimensions of the system, L [**IR scale**] (Such that total number of degrees of freedom= L^d/a^d).

We are generally interested in some average **long-wavelength** behaviour. Then we try to introduce **averaged out degree of freedom**. This particularly makes sense when we are interested in the physics at length-scales, l (experimental probes etc decide this), such that

$$a \ll l \ll L$$

A physically relevant way to ensure that there is a huge regime of such length-scales, l is by taking

$$\frac{a}{L} \to 0 \qquad \Rightarrow \textbf{Scaling Limit}$$

Two ways of getting the scaling limit :

- take $a \to 0$ at fixed L : **Continuum limit**

- take $L \to \infty$ at fixed a : **Thermodynamic limit**

Depending on the system taking one limit may be easier that the other. But both gives rise to **coarse-grained degrees of freedom** and continuum classical/quantum field theories.

Coarse grained degrees of freedom : In our case

$$\mathbf{\Phi}(\mathbf{r}) \sim \frac{1}{l^d} \sum_{i \leq l/2} \mathbf{n}_{i+\mathbf{r}} \tag{45}$$

$\mathbf{\Phi}(\mathbf{r})$ is the continuum field defined on a manifold R^d.

Symmetries of $\mathbf{\Phi}(\mathbf{r})$ are obtained from those of $\mathbf{n}_i$.

$\Rightarrow \mathbf{\Phi}$ is a $O(N)$ vector in our case.

Two manifolds : In fact we have two manifolds :

1. The manifold of $\mathbf{r}$ which is R^d for infinite system with open boundary conditions (T^d for infinite system with periodic boundary conditions). **Let us call this the base manifold/domain manifold**.

2. The manifold in which the field is defined (in our case O(N) which came from the symmetry) **Let us call this the field manifold/range manifold**.

We know the classification of closed paths (also surfaces and higher generalisation on the base manifold.

We can similarly classify closed paths (or surfaces etc) in the field manifold.

How are these classifications related ?

3.3 Contexts

1. Symmetry breaking order parameter field (Unconventional phase transitions)

2. many-body wave functions (topological phases)

3. Hilbert-space of identical particles (Anyons, boson, fermions)

4. Velocity field of fluids (non Hamiltonian extensions) (vortices)

5. Electromagnetism (monopole, Chern Simons Theory, Axion electrodynamics).

6. Various quantum field theories (Solitons, skyrmions.......)

In each of this contexts the answer to the above questions yield a wealth of information. We shall study the first in context of phase transitions.

4 Lecture-4

4.1 Recap of the $O(N)$ spin model

N-component unit vectors (classical spins), $\mathbf{n}_i$, on a lattice interacting by short range interactions which has global rotational symmetry. Under rotation

$$\mathbf{n}_i \to \mathcal{R} \cdot \mathbf{n}_i \qquad \mathcal{R} \in O(N) \tag{46}$$

- $N = 1$: Ising model Z_2
- $N = 2$: XY model $O(2)$
- $N = 3$: Heisenberg model $O(3)$

Coarse grained degrees of freedom :

$$\mathbf{\Phi}(\mathbf{r}) \sim \frac{1}{l^d} \sum_{i \leq l/2} \mathbf{n}_{i+\mathbf{r}} \tag{47}$$

$\mathbf{\Phi}(\mathbf{r})$ is the continuum field defined on a manifold R^d.

Symmetries of $\mathbf{\Phi}(\mathbf{r})$ are obtained from those of $\mathbf{n}_i$.

$\Rightarrow$ $\mathbf{\Phi}$ is a $O(N)$ vector in our case.

Two manifolds : In fact we have two manifolds :

1. The manifold of $\mathbf{r}$ which is R^d for infinite system with open boundary conditions (T^d for infinite system with periodic boundary conditions). **Let us call this the base manifold/domain manifold**.

2. The manifold in which the field is defined (in our case O(N) which came from the symmetry) **Let us call this the Order-parameter field manifold/range manifold**.

We know the classification of closed paths (also surfaces and higher generalisation on the base manifold.

We can similarly classify closed paths (or surfaces etc) in the order-parameter field manifold.

How are these classifications related and what are their physical significance as far as the phases and phase transitions are concerned ?

Let us start by trying to understand the order-parameter manifold a bit better.

4.2 Order-parameter space

Consider $N > 1$ for the moment (shall shortly return to the $N = 1$ Ising case).

The **Global** symmetries are :

- **Space symmetries :**
  1. Lattice Translation
  2. Lattice Rotation
  3. Lattice reflections
  4. $\cdots$
- **Internal Symmetries :**
  1. O(N) Rotations.

Depending on the interactions, the system can choose to break one class of symmetries or a mixture. Example :

1. Ferromagnet : Breaks $O(N)$ but not lattice symmetries

2. Antiferromagnet/spiral : Breaks both $O(N)$ and lattice symmetries

The order parameter needs to be chosen such that they can reflect the kind of symmetry breaking the system wants (**Generally we do not know what the system wants. Sometimes it is easy to see by inspecting the Hamiltonian. Sometimes it is not. In this latter case we need to do many different kinds of calculations to arrive at the right order-parameter field which is the low energy long-wavelength field**).

Then, we write symmetry allowed Landau-Ginzburg action (free energy). For our case (assuming we want O(N) ferromagnet), we can write :

$$\mathcal{S} = \int d^4\mathbf{x} \left[\frac{1}{2}|\nabla\mathbf{\Phi}|^2 + \frac{r}{2}|\mathbf{\Phi}|^2 + \frac{u}{4}|\mathbf{\Phi}|^4\right] \tag{48}$$

- $r > 0$:**Disordered phase** : $\langle\mathbf{\Phi}\rangle = 0$: The order-parameter Correlators are short ranged and exponentially decaying with finite correlation length.

- $r < 0$:**Ordered phase** : $\langle\mathbf{\Phi}\rangle \neq 0$: long-range correlation for the order parameter (unconnected).

  a) **The order-parameter quantifies the magnitude of symmetry breaking**

  b) Gapless spin-wave modes

How to find the manifold in which the order parameter lives ?

Let us ask for the ferromagnet. What is the manifold in which the ferromagnetic order parameter lives ?

Answer : Unit sphere S^{N-1}.

This is easy, but how to find this systematically ?

1. The list of symmetries form a group called **the symmetry group**.

   • If this group is continuous then are called **Continuous groups/ topological groups**.

   • If they are locally smooth in a local neighbourhood of identity, then they are called **Lie groups** which can then be generated from the continuous parameters and generators.

• However the symmetry group can have disjoint pieces. So have to be careful. Will have to encounter this for Ising case $N = 1$.

• For our purpose, in case of $N > 1$, whenever we encounter disconnected pieces, we shall only consider the sub-piece that contains identity. A more general consideration will not alter the result because each of these disconnected pieces have a group structure by themselves (at least in our cases).

2. The order parameter transforms **non-trivially** under a subset of the symmetry transformations and hence can break that symmetry spontaneously in the ordered phase.

$$\mathbf{\Phi} \to \mathbf{\Phi}' = s \cdot \mathbf{\Phi} \qquad s \in \text{Sym. group.}, G$$

3. For a given $\mathbf{\Phi}$, there is a subgroup of transformations in G, belonging to $H_{\mathbf{\Phi}}$ that does not change the order parameter, *i.e.*

$$\mathbf{\Phi} \rightarrow \mathbf{\Phi} = h_{\mathbf{\Phi}} \cdot \mathbf{\Phi} \qquad s \in \text{sub group.}, H_{\mathbf{\Phi}}$$

So these transformations do not generation any "Movement" of the order parameter. Hence only elements of G from which the above transformations have been *factored out* can generate such movements.

Since by tracking all these *movements* of the order parameter, we can estimate the order-parameter manifold, we must use all such factored out transformations of G. This then is given by

$$G/H_{\mathbf{\Phi}}$$

(formally the **cosets** of $H_{\mathbf{\Phi}}$ in G)

Now let us study examples :

4.2.1 XY spins ($N = 2$)

The order parameter is a two component spin

$$\mathbf{\Phi} = \{\Phi_1, \Phi_2\} \Rightarrow \Phi_1 + i\Phi_2 = e^{i\theta}$$

(expect to live on a circle)

Symmetry group : $O(2)$, Piece connected to Identity $SO(2) = G$.

$H = I$ since no element in $SO(2)$ keep the order parameter unchanged. Hence

$$G/H = SO(2) \rightarrow S^1$$

4.2.2 Heisenberg Spins (N=3)

The order parameter is a three component vector

$$\mathbf{\Phi} = \{\Phi_1, \Phi_2, \Phi_3\}$$

Similarly $G = SO(3)$

The axial rotation around $\mathbf{\Phi}$ are still a well defined symmetry. Therefore,

$$H = SO(2)$$

Thus the order parameter manifold is given by

$$SO(3)/SO(2) = S^2$$

which is basically the different axes along which the vector order parameter can point, *i.e.* a 2-sphere.

4.2.3 A phase of Superfluid Helium (N=3)

In the A phase of superfluid Helium, the order parameter is given by an SO(3) order parameter given by

$$\mathcal{R} = \{\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_1} \times \mathbf{n_2}\}$$

where

$$|\mathbf{n_1}| = |\mathbf{n_2}| = 1, \mathbf{n_1} \cdot \mathbf{n_2} = 0$$

$G = SO(3), H = I$. Therefore,

$$G/H = SO(3) = S^3/Z_2 = RP^3$$

A three sphere with antipodes identified.

4.2.4 Uniaxial Nematic liquid crystals :

Order parameter is a director such that

$$\mathbf{\Phi} = \{\Phi_1, \Phi_2, \Phi_3\}$$

with

$$-\mathbf{\Phi} \equiv \mathbf{\Phi}$$

$$G = SO(3)$$

$$H = SO_{\parallel}(2) \times C_{2\perp} \equiv D_{\infty}$$

Therefore

$$G/H = S^2/Z_2 = RP^2$$

We could go on and on and try to characterise more complex order parameters which break space symmetries such as solids, but let us stop here for the time being. I shall try to return to solids, but if not, strongly urge you to read up.

4.3 Homotopic classification of the order-parameter manifolds

1. **XY Spins**

$$\pi_1(S^1) = \mathbb{Z}, \pi_{n>1}(S^1) = 0$$

2. **Heisenberg Spins**

$$\pi_1(S^2) = 0, \pi_2(S^2) = \mathbb{Z}, \pi_{n>2}(S^2) = 0$$

3. **SO(3) order parameter :**

$$\pi_1(S^3/Z_2) = \mathbb{Z}_{\nvDash}, \pi_2(S^3/Z_2) = 0, \pi_3(S^3/Z_2) = \mathbb{Z}, \pi_{n>3}(S^3/Z_2) = 0$$

4. **Uniaxial nematic :**

$$\pi_1(S^2/Z_2) = \mathbb{Z}_2, \pi_2(S^2/Z_2) = \mathbb{Z}, \pi_{n>2}(S^2/Z_2) = 0$$

4.4 Consequences : Topological defects

Now we have a homotopic classification of both the base manifold (R^d) and the different order parameter manifolds. Now what ?

Let us condentrate on the respective π_1 or the first homotopy groups

1. Consider closed loops with a base-point $\mathbf{x_0}$ in the base-manifold, R^d. So we have functions

$$[f]$$

as defined before.

2. Now consider $\mathbf{\Phi}(\mathbf{r})$ as a map that creates an image of the closed path in the base-manifold in the order-parameter manifold.

3. **The image is also a closed loop with a base-point in the order-parameter space** because of the single-valued-ness of the order parameter field at a given $\mathbf{r}$.

4. But if $\pi_1 \neq 0$ then it means that such closed loops in the order parameter space cannot be continuously deformed to zero.

5. But then the corresponding loops in the base-manifold cannot be continuously be deformed to zero if I have to keep the sanctity of the map $\mathbf{\Phi}(\mathbf{r})$.

6. **How is this possible ?**

   Because $\mathbf{\Phi}(\mathbf{r})$ is singular at isolated points (in two dimensions) and lines (in three dimensions).

7. **How can it be singular ? Is it not a physical observable ?**

   Answer : Remember UV regulator ?

8. Within the effective theory such singularities exist and are called **Topological defects**.

Topological defects are essentially isolated point/lines/surfaces at which the order parameter field is not well defined. This is **OK** because the order parameter field can allow this singularities as it is obtained by some coarse graining from the lattice scale and hence can allow these defects on the lattice scale. Once these defects are present, if they cannot be removed by smooth deformation of the order parameter field at the course grained level, they acquire stability in the thermodynamic limit and these defects are called topological defects. Another way of saying this is as follows. For a system sitting on a lattice (or even for condensed matter systems in continuum with hard core repulsion) the continuum approximation of the real space breaks down when we zoom in to the lattice/atomic scale. Then the space as if it is multiply connected because of the presence of lattice plaquettes. These allow defects to exist.

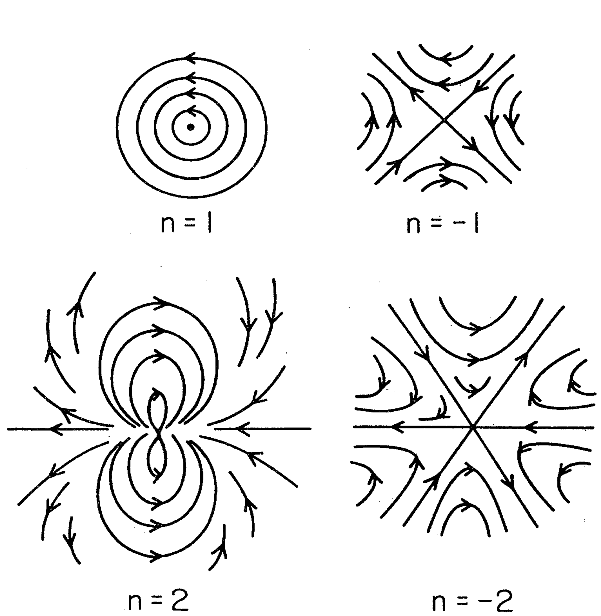
4.4.1 List of Topological Defects

1. XY Model : point (in d=2) and line (d=3) vortices.

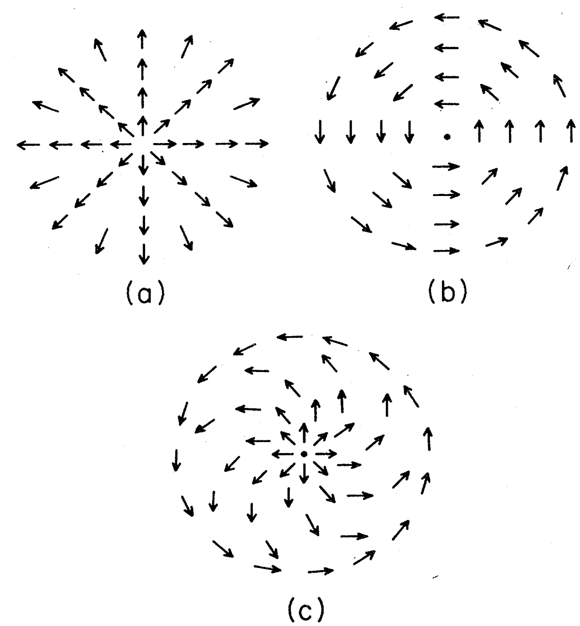
2. Heisenberg : Hedgehogs (d=3)

3. SO(3) : Ising vortices , Shankar Monopoles (d=4)

4. Uniaxial Nematic : Disclinations (d=2,3), Hedgehogs (d=3).

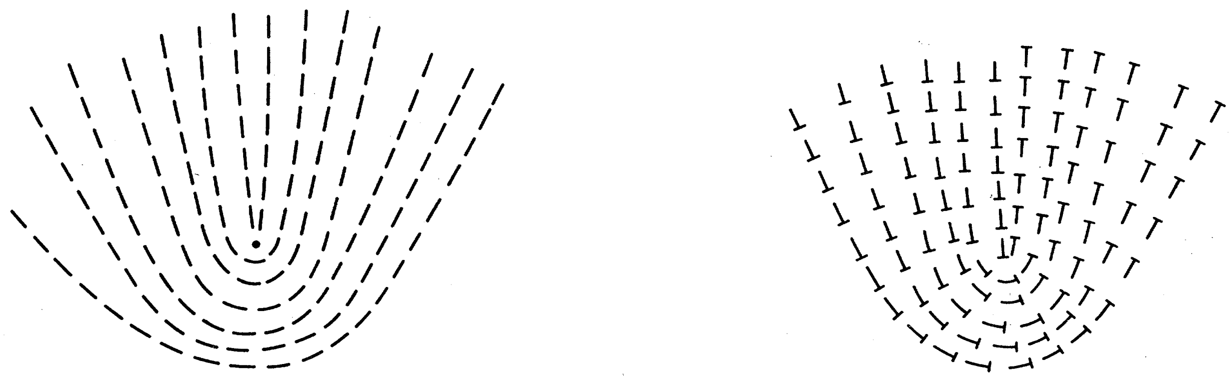


(a) XY vortices



(b) n=1 vortices

Figure 2: Mermin RMP, 1979



(a) (b)

Figure 3: π disclinations of a uniaxial nematic. Mermin RMP, 1979