# Various Aspects of Chiral Anomaly in Lattice QCD with Wilson-like Fermions : Finite Volume and Cutoff Effects

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# Plan of the talk

In lattice simulation understanding the effect of finite volume, finite cutoff and nonzero fermion mass is very crucial. Karsten and Smit (Nucl. Phys. B **183**, 103 (1981)) and Kerler (Phys. Rev. D **23**, 2384 (1981) ) first demonstrated that naive Wilson fermion can reproduce the chiral anomaly on the lattice in infinite volume chiral continuum limit. Lattice calculation of anomaly to one loop perturbation theory provides an excellent laboratory to address some of these issues.

- In this talk I will first describe the calculation of anomaly with Osterwalder-Seiler twisted mass lattice fermion action (OStm WQCD), and compare the approach to continuum chiral limit with naive, $\mathcal{O}(a)$ and $\mathcal{O}(a^2)$ improved Wilson fermion.
- Aoki proposed taking average over $r = \pm 1$ for improving the scaling behaviour of chiral condensate. We test this idea in the context of chiral anomaly.
- Since the emergence of anomaly is intimately tied with the removal of the doublers, the behaviour of the doublers as one varies the parameters of the theory is of interest. We study the doubler contribution as a function of lattice fermion mass ($am$) and $r$.
- Next we compare the finite volume dependence of of the one loop calculation of the anomaly with naive Wilson action and the improved actions with both periodic (P) and anti-periodic(AP) boundary conditions (BC).
- We explicitly show the cancellation between pseudoscalar density (PSD) term and the anomaly term for finite lattice fermion mass with naive Wilson fermion. However, both terms show strong mass dependence. We show that $r$ averaging leads to remarkable $am$ independence.

# Osterwalder-Seiler Chirally Twisted Wilson Term

The chirally twisted Wilson term was first introduced by K. Osterwalder and E. Seiler (Annals Phys. **110**, 440 (1978)). Here I will discuss how twisted mass term can have automatic $\mathcal{O}(a)$ improvement first and then calculation of chiral anomaly with this chirally twisted Wilson term. The Wilson action with twisted mass is,

$$
\begin{aligned}
S_F &= a^4 \sum_{x,y} \overline{\psi}_x \left[\gamma_\mu D_\mu + W + Me^{i\alpha\gamma_5}\right]_{xy} \psi_y \\
&= a^4 \sum_{x,y} \overline{\psi}_x \left[\gamma_\mu D_\mu + W + m_1 + im_2\gamma_5\right]_{xy} \psi_y , \qquad M = \sqrt{m_1^2 + m_2^2} \;\; \text{and} \;\; \tan\alpha = \frac{m_2}{m_1}
\end{aligned}
$$

Now the propagator:

$$
\begin{aligned}
aG(p) &= \frac{1}{\left(\frac{i}{a}\sum_\sigma \gamma_\sigma \sin(ap_\sigma) + \sum_\sigma \frac{r}{a}[1-\cos(ap_\sigma)] + m_1 + im_2\gamma_5\right)} \\
&= \frac{1}{\left(\frac{i}{a}\sum_\sigma \gamma_\sigma \sin(ap_\sigma) + M_0(p) + im_2\gamma_5\right)} , \qquad M_0(p) = \frac{r}{a}[1-\cos(ap_\sigma)] + m_1 \\
&= \frac{\left(-\frac{i}{a}\sum_\sigma \gamma_\sigma \sin(ap_\sigma) + M_0(p) - im_2\gamma_5\right)}{\left(\frac{1}{a^2}\sum_\sigma \sin^2(ap_\sigma) + M_0(p)^2 + m_2{}^2\right)} \\
&= \left(m_1 - im_2\gamma_5 - i\sum_\sigma \gamma_\sigma p_\sigma + \sum_\sigma \frac{r}{2} ap_\sigma{}^2 + O(a^2)\right)\left(\sum_\sigma p_\sigma{}^2 + m_1^2 + \sum_\sigma ram_1 p_\sigma{}^2 + m_2{}^2 + O(a^2)\right)^{-1} \\
&= \frac{(m_1 - im_2\gamma_5 - i\sum_\sigma \gamma_\sigma p_\sigma + \sum_\sigma \frac{r}{2} ap_\sigma{}^2 + O(a^2))}{(\sum_\sigma p_\sigma{}^2 + m_1^2 + m_2^2)}\left(1 - \frac{\sum_\sigma ram_1 p_\sigma{}^2}{\sum_\sigma p_\sigma{}^2 + m_1^2 + m_2^2} + O(a^2)\right)
\end{aligned}
$$

The $O(a)$ artifact is:

$$\frac{\sum_\sigma \frac{r}{2} a {p_\sigma}^2}{(\sum_\sigma {p_\sigma}^2 + m_1^2 + m_2^2)} + \frac{(\sum_\sigma r a m_1 {p_\sigma}^2)(-m_1 + i m_2 \gamma_5 + i \sum_\sigma \gamma_\sigma p_\sigma)}{(\sum_\sigma {p_\sigma}^2 + m_1^2 + m_2^2)^2}$$

So, by setting $m_1 = 0$, $O(a)$ effect can partially be removed in tree level.
Now, $m_1 = 0 \Rightarrow \alpha = \frac{\pi}{2}$.
The action becomes,

$$\begin{aligned} S_F &= a^4 \sum_{x,y} \overline{\psi}_x \left[ \gamma_\mu D_\mu + W + m_2 e^{i \frac{\pi}{2} \gamma_5} \right]_{xy} \psi_y \\ &\equiv a^4 \sum_{x,y} \overline{\psi}_x \left[ \gamma_\mu D_\mu - i \gamma_5 W + m_2 \right]_{xy} \psi_y \end{aligned}$$

These are related by a chiral rotation $\psi \rightarrow e^{-i \frac{\pi}{2} \gamma_5} \psi \quad , \quad \overline{\psi} \rightarrow \overline{\psi} e^{-i \frac{\pi}{2} \gamma_5}$

We have used same action with $m_2 = m$ to calculate chiral anomaly. So, $W \to R = -i\gamma_5 W$.
This leads to the Flavor Singlet Axial Ward Identity

$$\begin{aligned} \langle \Delta^b_\mu J_{\mu 5}(x) \rangle &= -2m \text{ Trace } \gamma_5 G^{os} - \langle \chi^{os}(x) \rangle \\ &= -2m \text{ Trace } \gamma_5 G^{os} - \text{Trace } \gamma_5 (G^{os} R + R G^{os}) \end{aligned}$$

with

$$\begin{aligned} G^{os} &= \frac{1}{\not{D} + m - i\gamma_5 W} \\ &= (\not{D} - m - i\gamma_5 W) \frac{1}{(\mathscr{G}^{os})^{-1} - V} \end{aligned}$$

where

$$(\mathscr{G}^{os})^{-1} = D^2 - m^2 - W^2 \text{ and } V = V_1 + V_2^{os} = \frac{i}{2} \sigma_{\mu\nu} [D_\mu, D_\nu] - i\gamma_5 [\not{D}, W] \; .$$

$$\begin{aligned}\langle \chi^{os}(x)\rangle &= -2m \text{ Trace } \gamma_5 G^{os} R \\ &= (\not{D} - m - i\gamma_5 W)(\mathcal{G}^{os} + \mathcal{G}^{os} V \mathcal{G}^{os} + \mathcal{G}^{os} V \mathcal{G}^{os} V \mathcal{G}^{os} + \cdots)R\end{aligned}$$

There are two kinds of contributions upto $\mathcal{O}(g^2)$, they are pseudoscalar and scalar. First consider the pseudoscalar terms.

$$\text{Trace } \gamma_5 W \mathcal{G}^{os} V_1 \mathcal{G}^{os} V_1 \mathcal{G}^{os} W$$

and

$$\text{Trace } \gamma_5 \not{D} \mathcal{G}^{os} V_1 \mathcal{G}^{os} [\not{D}, W] \mathcal{G}^{os} W \ .$$

Explicitly, the contribution to axial vector Ward identity from the Osterwalder-Seiler Wilson term background gauge field,

$$\begin{aligned}\langle \chi_x^{os} \rangle &= 2\, g^2\, \varepsilon_{\mu\nu\rho\lambda}\, F_{\mu\nu}(x) F_{\rho\lambda}(x)\, \frac{1}{(2\pi)^4} \sum_p \cos(p_\mu a)\cos(p_\nu a)\cos(p_\rho a) \\ &\times\; W_0(p)\Big[\cos(p_\lambda a) W_0(p) - 4\sin^2(p_\lambda a)\Big] (\mathscr{G}_0^{os}(p))^3 \,. \\ &= \frac{g^2}{16\pi^2}\, \varepsilon_{\mu\nu\rho\lambda}\, \mathrm{trace}\; F_{\mu\nu}(x) F_{\rho\lambda}(x) I(am, r, L)\end{aligned}$$

In the evaluation of $\langle \chi^{os}(x) \rangle$, now we encounter some parity violating terms which do not contribute to the anomaly, appear to contribute to the Axial Ward Identity. Specifically, up to $\mathscr{O}(g^2)$ that we are interested in,

$$\begin{aligned}&\mathscr{O}\,(g^0): && -\; 2im\ \mathrm{Trace}\, W \mathscr{G}^{os} \\ &\mathscr{O}\,(g^1): && \text{vanishes because of Dirac Trace} \\ &\mathscr{O}\,(g^2): && -\; 2im\ \mathrm{Trace}\, W \mathscr{G}^{os} V_1 \mathscr{G}^{os} V_1 \mathscr{G}^{os} - 2im\ \mathrm{Trace}\, W \mathscr{G}^{os} V_2^{os} \mathscr{G}^{os} V_2^{os} \mathscr{G}^{os}\end{aligned}$$

In the Axial Ward Identity, the contribution from the mass term is

$$-2m\mathrm{Trace}\gamma_5 G^{os} = -2m\mathrm{Trace}\gamma_5(\not{D} - m - i\gamma_5 W)(\mathscr{G}^{os} + \mathscr{G}^{os}V\mathscr{G}^{os} + \mathscr{G}^{os}V\mathscr{G}^{os}V\mathscr{G}^{os} + \cdots)$$

up to $\mathscr{O}(g^2)$ that we are interested in, the parity violating contributions are

$$\begin{aligned}
\mathscr{O}\ (g^0):&\quad + \quad 2im\ \mathrm{Trace}W\mathscr{G}^{os}\\
\mathscr{O}\ (g^1):&\quad - \quad 2im\ \mathrm{Trace}\not{D}\mathscr{G}^{os}[\not{D}, W]\mathscr{G}^{os}\\
\mathscr{O}\ (g^2):&\quad + \quad 2im\ \mathrm{Trace}W\mathscr{G}^{os}V_1\mathscr{G}^{os}V_1\mathscr{G}^{os} + 2im\ \mathrm{Trace}W\mathscr{G}^{os}V_2^{os}\mathscr{G}^{os}V_2^{os}\mathscr{G}^{os}\ .
\end{aligned}$$

$\mathscr{O}(g^0)$ and $\mathscr{O}(g^2)$), parity violating terms cancel between mass term and anomaly term. The $\mathscr{O}(g^1)$ term

$$\begin{aligned}
&\sum(\not{D}_0)_{xx_1}(\mathscr{G}_0)_{x_1x_2}([\not{D}, W]])_{x_2x_3}(\mathscr{G}_0)_{x_3x}\\
\Longrightarrow \sin(ap_\mu)F_{\mu\rho}&\Big[(\cos(ap_\mu) + i\sin(ap_\mu))(\cos(ap_\rho) + i\sin(ap_\rho))\\
&-(\cos(ap_\mu) - i\sin(ap_\mu))(\cos(ap_\rho) - i\sin(ap_\rho))\\
&-(\cos(ap_\mu) + i\sin(ap_\mu))(\cos(ap_\rho) - i\sin(ap_\rho))\\
&+(\cos(ap_\mu) - i\sin(ap_\mu))(\cos(ap_\rho) + i\sin(ap_\rho))\Big]\\
&\Longrightarrow 0 \text{ on integration}
\end{aligned}$$

Thus we explicitly verify the cancellation of parity violating terms in the flavor singlet Axial Ward Identity up to $\mathscr{O}(g^2)$.

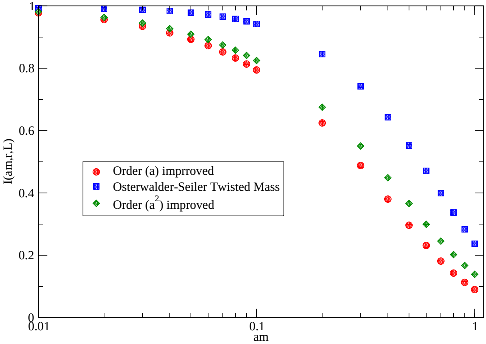



Figure: Comparison of $I(am,r,L)$ for $r = 1.0$ for $\mathcal{O}(a)$ improved and $\mathcal{O}(a^2)$ improved (H. W. Hamber and C. M. Wu, Phys. Lett. B 133, 351 (1983)) and twisted mass Wilson fermions at $L = 40$.

# Wilson parameter $r$ Averaging

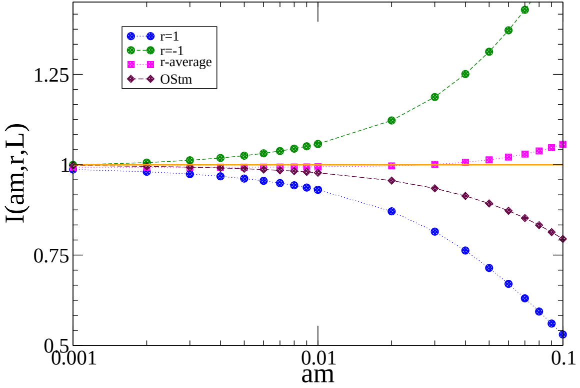



Figure: Anomaly integral for $0.001 < am < 0.1$ for naive Wilson fermions at $L = 40$ for $r = \pm 1$ and the average compared with OStm Wilson fermions with $r = 1$.

# Anomaly: contribution from different regions of the Brillouin zone

- It is instructive to separately consider the contribution to the Anomaly integral from different regions of the Brillouin zone in order to compare and contrast the contribution of the physical fermion and the doublers as a function of the different parameters.
- This has been studied in the infinite volume chiral continuum limit by Hamber and Wu (H. W. Hamber and C. M. Wu, Phys. Lett. B **136**, 255 (1984)). However, for practical simulations the behavior at non-zero quark masses is more relevant.

- Following Karsten and Smit, the limits on the momentum sum are changed from $(-\pi/a, +\pi/a)$ to $(-\pi/(2a), 3\pi/(2a))$
- Then momentum sum hypercube is divided into 16 smaller hypercubes corresponding to $(-\pi/(2a), +\pi/(2a))$ and $(+\pi/(2a), +3\pi/(2a))$ for each $p_\mu, \mu = 1, 2, 3, 4$.
- Thus the total anomaly contribution is decomposed into the contributions from the five species and the Anomaly Integral $I = I_0 - 4I_1 + 6I_2 - 4I_3 + I_4$.

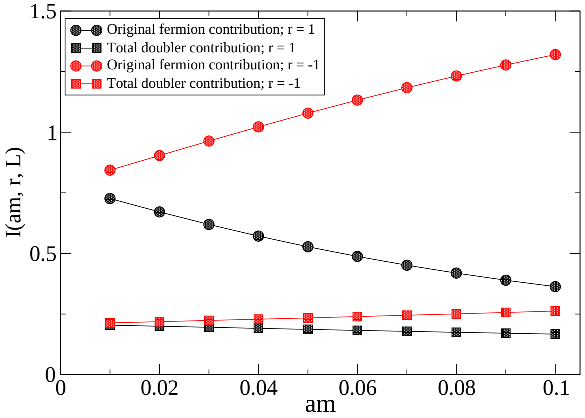



Figure: Comparison of total doubler contribution and the physical fermion contribution for $0.01 < am < 0.1$ with naive Wilson fermions at $L = 40$ and $r = \pm 1$.

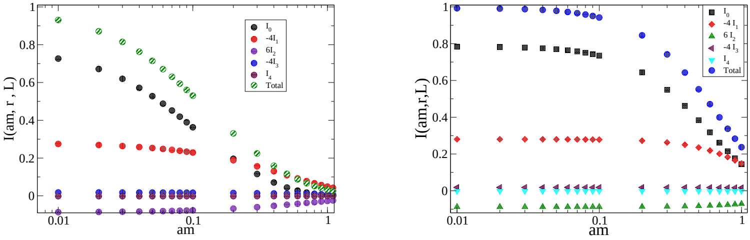



Figure: Contributions from different regions of the Brillouin zone to the anomaly integral for $r = 1$ and $0.01 < am < 1.0$ for naive Wilson and OStm Wilson fermions at $L = 40$ for $r = 1$ compared with total contribution.

- As expected, as the physical fermion gets lighter, its contribution to the anomaly is dominant compared to the doublers.
- To verify that the doublers do decouple when their masses become very heavy so that the sole contribution to the anomaly comes from the physical fermion, we study the behavior of $I_i, i = 1, 2, 3, 4$ as the Wilson parameter $r$ is increased. We realize that
- The reflection positivity is not guaranteed when $\mid r \mid > 1$, but in the present context we adjust r only to make the doubler mass heavy.

# Decoupling of the doublers

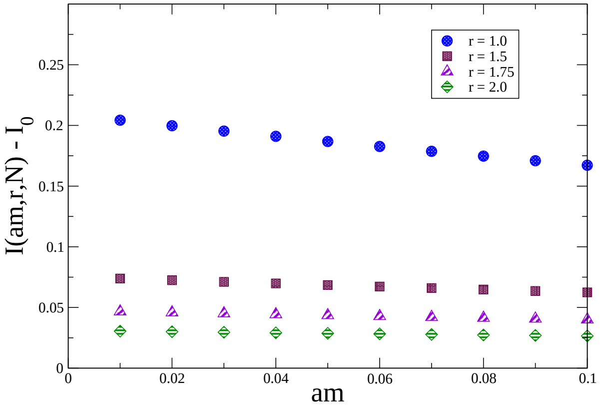



Figure: Total contribution of the doublers to the anomaly integral for $.01 < am < 0.1$ for naive Wilson fermions at $L = 40$ for $1 \leq r \leq 2$.

# Volume Dependence

Since finite size artifacts are different for the three fermion actions considered in this work, and finite volume and finite cutoff ($1/a$) effects may interfere with each other, the anomaly integral provides us an opportunity to explore this issue in a simple setting.

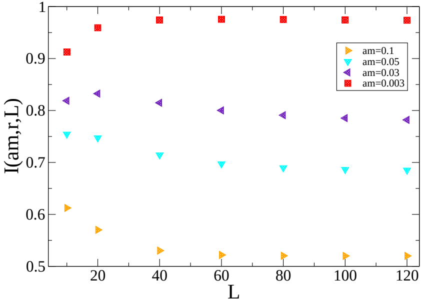



Figure: Volume dependence of the anomaly integral for $r = 1$ and $0.01 \leq am \leq 0.1$ for naive Wilson fermions with PBC.

The behavior qualitatively changes as quark mass varies from heavy to light. Moreover, irrespective of the quark mass, approximate $L$ independence is achieved only for $L \gtrsim 40$.

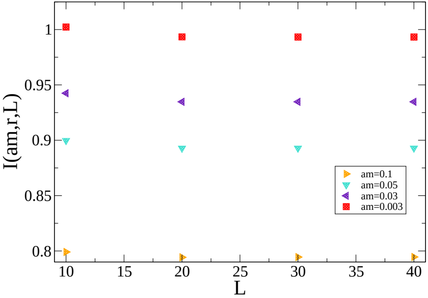



Figure: Volume dependence of the anomaly integral for $r = 1$ and $0.003 \leq am \leq 0.1$ for $\mathcal{O}(a)$ improved Wilson fermions with PBC.

The $L$ dependence is qualitatively the same across the mass range, and furthermore, $L$ independence is achieved for $L > 20$.

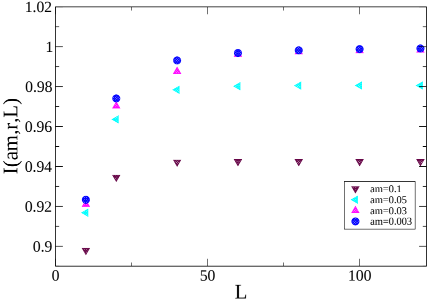



Figure: Volume dependence of the anomaly integral for $r = 1$ and $0.003 \leq am \leq 0.1$ for OStm Wilson fermions with PBC.

The $L$ dependence is qualitatively the same across the mass range but $L$ independence is achieved only for $L > 50$.

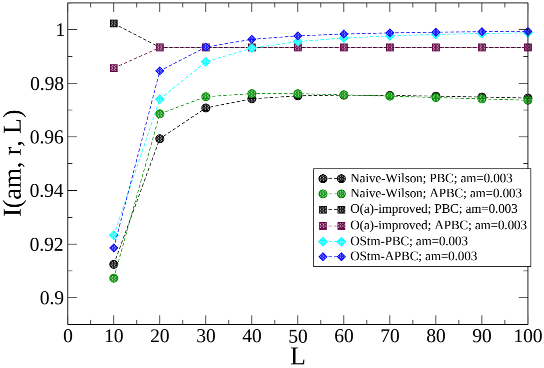



Figure: Volume dependence of the anomaly integral for $am = 0.003$ and $10 \leq L \leq 120$ for naive Wilson, $\mathcal{O}(a)$ improved Wilson and OStm Wilson fermions with PBC and APBC.

For the $O(a)$ improved Wilson fermion, there is sensitivity to boundary condition only for very small $L$. On the other hand both for the naive and OStm Wilson fermion, the APBC improves the convergence with respect to $L$.

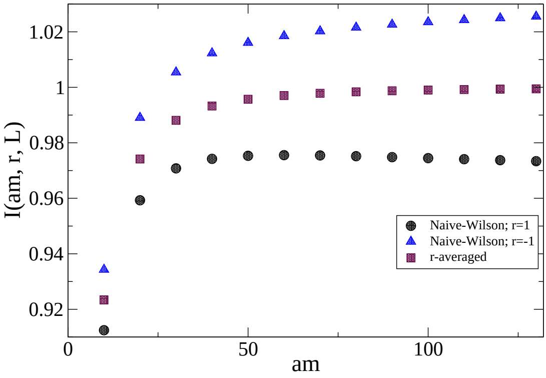



Figure: Volume dependence of the anomaly integral for $am = 0.003$ and $10 \leq L \leq 130$ for $r = \pm 1$ and r averaged for naive Wilson with PBC.

Previously we have seen some interplay between the finite volume and cut off effects for naive Wilson fermion. Finally the convergence of the anomaly integral with respect to $L$ is vastly improved by $r$ averaging, since it removes the cut off effects to a great extent.

# Pseudoscalar Density term

- Naively, the pseudoscalar density (PSD) term in the flavor singlet axial Ward Identity appears to vanish in the chiral limit. However, this conclusion is premature as was first shown by Schwinger in the context of sigma model J. Schwinger (*Particles, Sources and Fields*, Vol. III, (Addison-Wesley Publishing Company, Inc. (1989)).
- To $\mathcal{O}(g^2)$ it can be seen that the PSD term is independent of the fermion mass for a constant background gauge field and produces the negative of the anomaly term. In the context of $\mathcal{O}(a)$ improved Wilson fermions, in the chiral limit, this has been demonstrated by Hamber and Wu (Phys. Lett. B **133**, 351 (1983); Phys. Lett. B **136**, 255 (1984)).
- With naive Wilson fermions, we explicitly show the cancellation for finite lattice mass of the fermion. However, in this case both terms individually show strong lattice mass dependence.
- We show that $r$ averaging leads to remarkable mass independence for individual terms.

To $\mathscr{O}(g^2)$, the contribution from the PSD term is

$$
\begin{aligned}
2m\,\langle \bar{\psi}_x \gamma_5 \psi_x \rangle &= 2\,g^2\,\varepsilon_{\mu\nu\rho\lambda}\ \mathrm{trace}\ F_{\mu\nu}(x) F_{\rho\lambda}(x) \\
& \quad \frac{1}{(2\pi)^4} \sum_p \cos(p_\mu a)\cos(p_\nu a)\cos(p_\rho a) \\
&\times \quad m\Big[\cos(p_\lambda a)[m + W_0(p)] - \ 4\sin^2(p_\lambda a)\Big](\mathscr{G}_0(p))^3, \\
&= \frac{g^2}{16\pi^2}\,\varepsilon_{\mu\nu\rho\lambda}\ \mathrm{trace}\ F_{\mu\nu}(x) F_{\rho\lambda}(x) I_m(am, r, L)
\end{aligned}
$$

where

$$
\begin{aligned}
I_m(am, r, L) &= \frac{1}{32\pi^2} \sum_p \cos(p_\mu a)\cos(p_\nu a)\cos(p_\rho a) \\
&\times \quad m\Big[\cos(p_\lambda a)[m + W_0(p)] - \ 4\sin^2(p_\lambda a)\Big](\mathscr{G}_0(p))^3\ .
\end{aligned}
$$

$$
\mathscr{G}^{-1} = D^2 - (W + m)^2
$$

The PSD term is much more infrared sensitive than the anomaly term since one factor of $W_0(p)$ in the numarator is replaced by $m$.

The convergence with respect to $L$ deteriorates very fast as $am$ is around 0.01 or below.

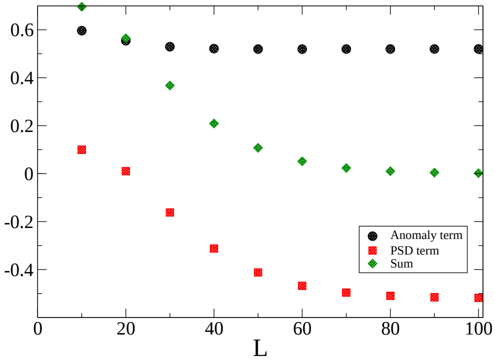
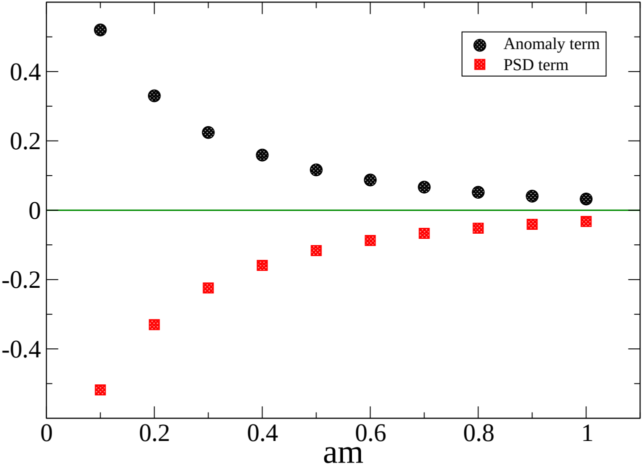
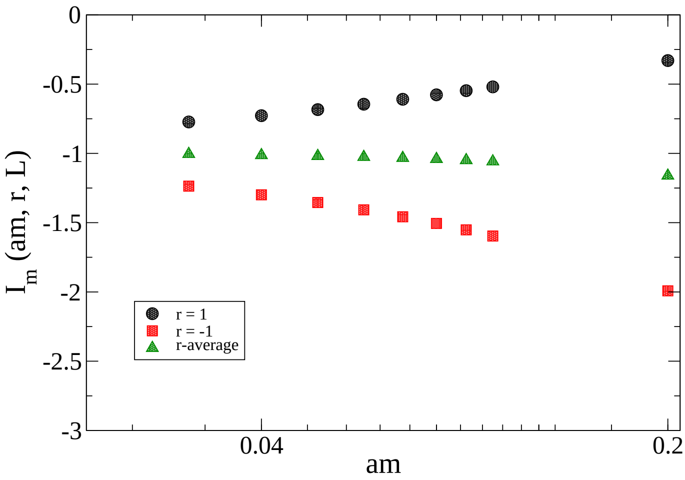



Figure: The $L$ dependence of the anomaly term, the PSD term and the sum for $am = 0.1$.

Anomaly term and the PSD term cancel each other independent of the value of *am*.



PSD term is almost independent of *am* as in the continuum when *r*-averaging is performed.



# Summary

- Calculation of chiral anomaly from chirally twisted Wilson action is shown. It produces better result even than the $\mathcal{O}(a^2)$ improved Wilson fermion.
- We have used one loop lattice calculation of chiral anomaly to study quantitatively the effects of $r$ averaging which has been proposed before to achieve better scaling behavior of observables with Wilson fermions. We find that $r$ averaging of naive Wilson fermion has a much better approach to continuum chiral limit compared to even OStm Wilson fermions.
- We have studied the doubler contributions as a function of $r$ and lattice fermion mass $am$. We have verified that the doubler contribution decreases as $r$ increases and it is not very sensitive to the lattice fermion mass.

- Next, we have studied the possible interference between finite size and cut off effects by investigating in detail naive, $\mathcal{O}(a)$ improved and OStm Wilson fermion cases for a range of volumes and fermion masses.
- Lastly we have studied the relative roles played by the mass term and the anomaly term in the flavor singlet axial Ward identity on the lattice at one loop level. We have shown that the anomaly contribution is completely cancelled by the PSD term even for finite lattice fermion mass. In this case, however, individual contributions show substantial lattice mass dependence for naive Wilson fermion, which appears to be different from the continuum theory. However, in this again, $r$ averaging substantially reduced the dependence on the fermion mass.

*THANK YOU*