Fermion-Boson Duality in 2+1 dim. large N Gauge Theories

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Puri Lectures — I 11/1/14

Topics of discussion:

1. Higher spin theories in 2+1 dim. (general discussion)

2. CS + matter theories involving both fermions and bosons

3. Level-rank duality of CS theory and the fermion-boson duality.

1.

i) We choose to study 1. in the context of CFT as it
~~ii)~~ enables many precise and model independent statements.
~~iii)~~ ii) We want to work with weakly coupled theories and hence it is natural to study various models in the large N limit. ($\frac{1}{N}$ is small).

The conformal group in 2+1 dim. is $SO(3,2)$:
$P_\mu, D, M_{\mu\nu}, K_\mu$ ($SO(4,1)$ in euclidean metric)

Unitary ~~#~~ representations are labeled by $SO(3) \times SO(2) \in SO(3,2)$
(The Euclidean theory in 3 dim. has conformal group $SO(4,1)$).

$$\begin{array}{ccc} SO(3) & \times & SO(2) \\ \downarrow & & \downarrow \\ S & & \Delta \end{array}$$

Rep. theory implies for primary fields.

$S \geq 1$	$\Delta \geq S+1$
$S = \frac{1}{2}$	$\Delta \geq 1$
$S = 0$	$\Delta \geq \frac{1}{2}$ (we will consider $\Delta = 1$ or 2)

for $S \geq 1$ the inequality $\Delta \geq S+1$ is saturated by the short representations:

$$\Delta = S+1$$

Higher spin currents and higher spin symmetry

e.g. Take a theory of massless Dirac fermions:

$$\mathcal{L} \sim \bar{\Psi}_i \not{\partial} \Psi^i \implies \not{\partial}\Psi = 0$$

$$J^{(1)}_{\mu} = \bar{\Psi}_i \gamma_\mu \Psi^i$$

$$J^{(2)}_{\mu_1\mu_2} = \bar{\Psi}_i \gamma_{\mu_1} (\overrightarrow{\partial}_{\mu_2} - \overleftarrow{\partial}_{\mu_2}) \Psi^i$$

$$J^{(3)}_{\mu_1\mu_2\mu_3} = \frac{1}{6} \bar{\Psi}_i \gamma_{\mu_1} \Big(3 \overleftarrow{\partial}_{\mu_2} \overleftarrow{\partial}_{\mu_3} - 10 \overleftarrow{\partial}_{\mu_2} \overrightarrow{\partial}_{\mu_3} + 3 \overrightarrow{\partial}_{\mu_2} \overrightarrow{\partial}_{\mu_3} + 2 (\overleftarrow{\partial}_\sigma \overrightarrow{\partial}_\sigma) \eta_{\mu_2\mu_3}\Big) \Psi^i$$

$\vdots$ (all μ indices above are to be symmetrized)

- $J^{(s)}_{\mu_1 \cdots \mu_s}$ are traceless symmetric tensors
- they are conserved.
- they have spin s and $\Delta = s+1$ (by construction)

Similarly for a free bosonic theory.

These are the primary fields of the CFT $|\Delta = s+1, s\rangle$

Now it is a general statement that

$$\Delta = s+1 \quad (\tau = \Delta - s = 1) \iff \partial \cdot J = 0.$$

$$\langle \partial_\mu \cdot J_{\mu\cdots} | \partial_\nu \cdot J_{\nu\cdots} \rangle = \langle J_{\mu\cdots} | \, [K_\mu, P_\nu] \, | J_{\nu\cdots} \rangle$$

$$= (\Delta - s - 1) \langle J_{\cdot\cdot} | J_{\cdot\cdot} \rangle$$

where we have used $[P_\mu, K_\nu] = -i\left[2\eta_{\mu\nu} D + 2 M_{\mu\nu}\right]$

and $K_\mu^+ = P_\mu$ (Euclidean conf. algebra).

Maldacena-Zhiboedov : $\partial \cdot J^{(s)} = 0 \Rightarrow$ CFT is 'free'
exact higher spin symmetry $\Rightarrow$ free fields.

Introduce interactions:

couple fermions / bosons to gauge fields

($SU(N)$, $U(N)$, $O(N)$ ...)

we will work with $U(N)$:

$$\partial_\mu \to D_\mu = \partial_\mu - i A_\mu \quad , \quad A_\mu^+ = A_\mu$$

$$\mathcal{L} = \text{matter} + CS \qquad \text{(conformally invariant).}$$

$$CS \Rightarrow \quad S = \frac{k}{4\pi}\int d^3x \left(AdA + i\frac{2}{3}A^3\right).$$

k = level of CS theory, $k \in \mathbb{Z}$

We are interested in weak coupling:

$$\frac{1}{N} \to 0 \quad \text{and} \quad \frac{N}{|k|} = \lambda \ (\text{fixed}).$$

In the large N limit 'gauge invariant' operators (suitably normalized) factorize. $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle + O(\frac{1}{N})$.

~~e.g.~~ $\cancel{\langle J^{(s_1)} J^{(s_2)} J^{(s_3)} J^{(s_4)} \rangle}$

$$\cancel{= \langle J^{(s_1)} J^{(s_2)} \rangle \langle J^{(s_3)} J^{(s_4)} \rangle + O(\tfrac{1}{N})}$$

~~This tells us that the 'complete' set of spin s currents in the interacting theory~~

The spin s currents of the interacting theory are obtained from the currents of the 'free theory' by the substitution:

$$\partial_\mu \to \partial_\mu - iA_\mu = D_\mu$$

However since $[D_\mu, D_\nu] = F_{\mu\nu} \neq 0$ (unlike $[\partial_\mu, \partial_\nu] = 0$)

we need to work a bit to produce the symmetric traceless spin 's' currents of the interacting theory.

e.g.

$$J^{(3)}_{\mu_1\mu_2\mu_3} = \hat{J}^{(3)}_{\mu_1\mu_2\mu_3} - \frac{\pi}{5k}\,\eta_{(\mu_1\mu_2}\,\bar{\psi}\psi\bar{\psi}\gamma_{\mu_3)}\psi$$

Note: Using eqns. of motion we have

$$\partial^\mu J_{\mu\nu_1\nu_2} = -\frac{16\pi}{5k}\Big[\eta_{\nu_1\nu_2}\,(\partial^\mu J^{(0)})\,J^{(1)}_\mu - 3\,\partial_{(\nu_1}J^{(0)}\,J^{(1)}_{\nu_2)} + 2\,J^{(0)}\,\partial_{(\nu_1}J^{(1)}_{\nu_2)}$$

(classically)

where $J^{(0)} = \bar{\psi}_i\psi^i$ and $J^{(1)} = \bar{\psi}_i\gamma_\mu\psi^i$

Comments:

A) The spin 's' current operators, which are likely to are bi-local in the fermions / bosons, form a complete basis of operators at the leading order in large N, because they are gauge invariant and their products factorize to leading order.

B) The currents are not conserved.
What is the r.h.s. ?

$\partial \cdot J^{(s)} \sim o\left(\frac{1}{k}\right) \sim o\left(\frac{1}{N}\right)\lambda.$

CFT $\Longrightarrow \Delta - s - 1 \sim o\left(\frac{1}{N}\right)$

c) The operator $\left(\partial^{\mu} J^{(s)}_{\mu\mu_1\cdots\mu_{s-1}}\right)$ has

$$\Delta = (s+1) + 1 + o\left(\tfrac{1}{N}\right) = s+2+o\left(\tfrac{1}{N}\right)$$
$$\text{spin} = s-1 + o\left(\tfrac{1}{N}\right).$$

twist: $\tau = \Delta - \text{spin} = 3 + o\left(\frac{1}{N}\right)$

Since $\tau\left(J^{(s)}\right) = 1$

clearly $\nexists$ a single trace operator that can appear on the r.h.s !

Hence the r.h.s. is of form

$$\partial \cdot J = \frac{1}{k}\, J\cdot J + \frac{1}{k^2}\, JJJ$$

sprinkling of derivatives

Why: $J\cdot J$ terms must have $\Delta - \text{spin} \geq 2$ $(\tau \geq 2)$

JJJ " " " $\tau \geq 3$

$JJJJ$ " " " $\tau \geq 4$ (not allowed since $\tau(\partial\cdot J) = 3 + o\left(\frac{1}{N}\right)$)

~~Hence weakly ~~

Hence the interacting theory has weakly broken higher spin symmetry encapsulated by the eqn: (independent of model)

$$\partial \cdot J = \frac{1}{N} JJ + \frac{1}{N^2} JJJ$$

(valid within correlation functions)

one should not set $N=\infty$ naively. e.g.

$$\langle \partial\cdot J\, J\, J \rangle = \frac{1}{N} \langle JJJJ \rangle + \cdots$$

$$= \frac{1}{N} \underbrace{\langle JJ \rangle}_{\propto N} \underbrace{\langle JJ \rangle}_{\propto N} = O(N) \neq 0.$$

Furthermore the spectrum of primary operators and their descendents is IDENTICLE for ~~whatever the~~ the type of matter (fermionic or bosonic) from which the currents are constructed.

SUMMARY OF SPECTRUM: Fermions / bosons at Wilson-Fischer point

$\Delta = s+1 + O(\frac{1}{N})$ independent of $\lambda = \frac{N}{k}$

Spectrum: $\boxed{(2,0)} + \sum_{s=1}^{\infty} (s+1, s) + O(\frac{1}{N})$. (reg. 4)

$\bar{\Psi}\Psi$ + $(\bar{\Phi}\Phi)_{WF\ point}$

It is one of the ~~this~~ facts that prompted the conjecture in Giombi et al* that there may be a duality between fermions & bosons.

* ref. 5

This is indeed true & now we come to the work of M+Z (Maldacena + Zhiboedov ref. 6)

Assumptions: They work with $O(N)$; ψ, ϕ are real and only even spins are allowed.

1. There is a parameter $\tilde{N}$ and the theory is weakly coupled for $\frac{1}{\tilde{N}} \to 0$.

· $\tilde{\lambda}$ parametrises the theories.

~~2. $\partial J \sim 1/\tilde{N}$ (No operator with $\tau = 3+\epsilon$ appears on the rhs)~~

2. All currents have $\tau = \Delta - s = 1 + o\left(\frac{1}{\tilde{N}}\right)$

3. $\partial \cdot J^{(s)} \sim o\left(\frac{1}{\tilde{N}}\right)$ rhs has NO operator which is single trace only double trace & higher

in particular $\partial \cdot J^{(4)}$

4. $\exists$ a unique spin 2 conserved current

5. $\exists$ a single $s=0$ scalar with $\Delta = 2$ or $\Delta = 1$

$\bar{\psi}\psi$ or $\bar{\phi}\phi$ (Wilson-Fisher point)

6. They work with real ψ & ϕ

MZ use the Ward-identities ~~for~~ for $J^{(4)}$

$$\partial \cdot J^{(4)} = a_2 J^{(2)} J^{(0)} + a_3 \left(J^{(0)} J^{(0)} J^{(0)} + J^{(2)} J^{(0)} J^{(0)} \right)$$

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = \tilde{N} \left[\frac{\tilde{\lambda}^2}{1+\tilde{\lambda}^2} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\substack{\text{boson} \\ \text{(single real)}}} + \frac{1}{1+\tilde{\lambda}^2} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\substack{\text{fermi} \\ \text{(single real)}}} + \frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}} \right]$$

the three conformal structures $\langle \ldots \rangle_{\text{boson}}$, $\langle \ldots \rangle_{\text{fermion}}$ + $\langle \ldots \rangle_{\text{odd}}$ were discussed in ref. 4, (earlier work by Giombi, Prakash + Yin.

parity violation terms, does not occur for free fields

(NB $\tilde{\lambda} = 0 \Rightarrow$ fermion).

s_1, s_2, s_3 are even since $O(N)$ was used.
arguement valid for s_i even for $U(N)$.

Note $\tilde{\lambda} \to \frac{1}{\tilde{\lambda}}$, $\tilde{N} \to \tilde{N}$ leaves $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ invariant and interchanges the first 2 terms.

$\tilde{\lambda} \to 0 \Rightarrow \langle J_{s_1} J_{s_2} J_{s_3} \rangle$ fermions
$\tilde{\lambda} \to \infty \Rightarrow \langle J_{s_1} J_{s_2} J_{s_3} \rangle$ bosons.

Aharony - Yacoby - Gurri (ref. 7) $\quad$ $N \le k$.

$$\tilde{N} = 2N \frac{\sin \pi\lambda}{\pi\lambda} \quad + \quad \tilde{\lambda} = \tan \frac{\pi\lambda}{2} \qquad \boxed{0 \le \lambda \le 1}$$

$$\tilde{N} \to \tilde{N} \quad + \quad \tilde{\lambda} \to \frac{1}{\tilde{\lambda}} \iff \begin{array}{l} N \to (k-N) \\ k \to k \end{array} \quad , \; k>0$$

more generally $\left(\begin{array}{l} N \to |k| - N \\ k \to -k \end{array} \right)$

$$\lambda \to (1-\lambda) \quad , \quad \lambda = \frac{N}{|k|}.$$

def. of level depends upon UV regulator:

$$\left[\; \underset{\text{dim. reg}}{k_{our}} = \underset{\text{YM regulator in UV}}{k_{YM}} + N \right.$$

$$\left. \Rightarrow \lambda_{our} = \frac{\lambda_{YM}}{1+\lambda_{YM}} \le 1 \qquad \begin{array}{l} \lambda_{YM} \to \infty \\ \lambda_{our} \to 1 \end{array} \right].$$

—— x —— x —— x ——

So fermi-bose duality is very likely related to level-rank duality.

∴ 4 ways to describe the same theory

1) N_s massless scalars + $O(N_s)_{k_s}$ CS

2) N_f " fermion + $O(N_f)_{k_f}$ CS

3) Algebraically (currents) by slightly broken higher spin sym. by double / triple trace operators

4) Vasiliev's higher spin theory $e^{i\theta/2}$

$0 \le \theta \le \pi$.

(quasi fermion theory) Free boson O(N) model $\xleftrightarrow{\text{CS}}$ UV Gross Neveu O(N) model

$\downarrow$ RG $\propto (\bar{\Phi}\phi)^2$ $\qquad\qquad$ $\downarrow$ RG

(quasi boson theory) IR critical O(N) model $\xleftrightarrow[\text{parity violating except at } \lambda = 0 + 1]{\text{CS}}$ Free fermion O(N) model

Vasiliev: ref. 2

$$A \xrightarrow{e^{i\theta/2}} B$$

0 $\qquad$ parity violating $\qquad$ π

except at $\theta = 0, \pi$.

The partition function at large N on $S^2 \times S^1_\beta$

(ref. 10)

$$Z = \int \mathcal{D}A_\mu \, e^{i S_{cs} + i S_{matter}}$$

$$S_{cs} = \frac{k}{4\pi} \int tr\left(A dA + \frac{2}{3} A^3 \right)$$

$$S^f_{matter} = \int \left(\bar{\Psi} \not{D} \Psi + A \bar{\Psi} \Psi \right) \quad \text{or}$$

$$S^b_{matter} = \int |D\phi|^2 + V(\phi) \ldots$$

In what we are going to discuss today, the matter action can have deformations away from being conformally invariant.

$$Z = \int \mathcal{D}A_\mu \, e^{i S_{cs}} Z'$$

$$Z' = \int \mathcal{D}(\text{matter}) \, e^{i S_{matter}(\text{matter}, A_\mu)}$$

The holonomy matrix $U(x) = e^{i \int_0^\beta dx^3 A_3(x^3, x)}$
$x \in S^2$.

In the gauge $\partial_3 A_3 = 0$, $U(x) = e^{i\beta A_3(x)}$

$$\beta = \frac{1}{T}$$

We will be working in the limit:

$$\cancel{\zeta N^2}\; V_2 T^2 N = N^2 \zeta$$

where V_2 is the vol. of S^2 and ζ is a fixed parameter.

$$Z_{matter} = e^{NT^2\left[\int \sqrt{g}\, v(U) + v_1(U)\, \mathrm{tr}(D_i U)(D_i U)^{+} + \cdots\right]}$$

The CS integral $\int \mathcal{D}A_\mu\, e^{iS_{cs}} \sim e^{N^2}$

N^2 should be comparable to (NTV_2)

$$Z_{matter} \propto e^{N(T^2 V_2)}$$

→ by conformal inv.

measure has N degrees of freedom

$$Z_{cs} \propto e^{N^2}$$

measure has N^2 degrees of freedom.

In order to have a well defined large N limit

$$NT^2V_2 = N^2\zeta \quad , \quad \boxed{T^2 V_2 = N\zeta}$$

for fixed V_2 + ζ $T^2 \propto N$, so large N is a high temperature expansion.

More precisely the form of Z' is.

$$Z' = e^{-N\cancel{T^2}\left(\int\sqrt{g}\left[v(U)T^2 + v_1(U)\,\mathrm{Tr}\,D_iU(D_iU)^\dagger + \dots\right]\right)}$$

(covariant derivation on S^2

where ~~$\sqrt{}$~~ g is the metric on the sphere, and $v(U)$ is a function of the holonomy metrix that will depend on the matter content and has to be calculated (Lecture 3).

- The factor of T is introduced by dimensional analysis.
- We expect the action to be ~~ttt~~ local because the matter fields have a large thermal mass $\propto T^2$.

$$(T^2 \propto N\zeta)$$

So the effective action at large N is a function of the holonomy metrix:

$$Z' = e^{-NT^2\int\sqrt{g}\,v(U)}$$

$$\therefore\ Z = \int \mathcal{D}A_\mu\, e^{iS_{CS} - NT^2\int\sqrt{g}\,v(U(x))}$$

$$= Z_{CS}\cdot\left\langle e^{-NT^2\int\sqrt{g}\,v(U(x))}\right\rangle_{CS}$$

Now CS is a topological theory and we expect no dependence on the metric of S^2 ~~instantaneous~~ and x

$$\therefore\ Z = Z_{CS}\left\langle e^{-NT^2 V_2\, v(U)}\right\rangle$$

where $V_2 = \int_{S^2}\sqrt{g}$.

zero mode of the holonomy metrix

Hence the calculation of the thermal partition function at large N is reduced to the calculation of the expectation value of a functional of the holonomy matrix in PURE CS theory.

Z' ~~V(U)~~ can be expanded in terms of the characters of $U(N)$.

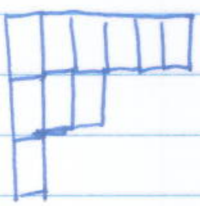
math begins

$$e^{-NT^2V_2 v(U)} = \sum_{Y} \chi_{Y}(U)\, a_{Y}(T^2V_2)$$

where $\chi_Y(U)$ is a character corresponding to a young tableaux Y of the group $U(N)$,

$$\therefore \left\langle e^{-NT^2V_2 v(U)} \right\rangle_{CS} = \sum_{Y} a_Y \left\langle \chi(U)_Y \right\rangle_{CS}$$

Let 'n' denote the number of boxes in ~~the~~ Y



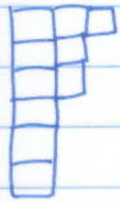
$$\chi(U)_Y = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{Y(\sigma)} \left(\prod_{m=1}^{n} (\mathrm{tr}\, U^m)^{k_m} \right)$$

$\chi_{Y(\sigma)}$ is the character of S_n corresponding to Y

$k_1, k_2, \ldots, k_n$ are cycles of length $1, 2, \ldots, n$

in the conjugacy class of the permutation ~~g.~~ σ

Now consider $Y \to \tilde{Y}$ (transposition of the tableaux, interchanges rows and columns).



$$\chi_{\tilde{Y}}(g) = \mathrm{sgn}(g)\, \chi_Y(g)$$

$$\mathrm{sgn}(g) = \prod_{m=1}^{n} \left[(-1)^{m+1}\right]^{k_m}$$

$$\Rightarrow \chi_{\tilde{Y}}(U) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{Y(\sigma)} \left(\prod_{m=1}^{n} \left[(-1)^{m+1} \mathrm{tr}\, U^m\right]^{k_m} \right)$$

$\Rightarrow$ $\boxed{\text{the simple rule for transposition } Y \to \tilde{Y} \text{ is: } \mathrm{tr}\, U^m \to (-1)^{m+1}\, \mathrm{tr}\, U^m}$

$$\sum_i e^{im\theta_i} \longrightarrow e^{i\pi} \sum_i e^{im(\theta_i + \pi)}.$$

end of math

Level-rank duality in CS theory:

Expectation value of Wilson loop in rep. Y theory (N, k) $=$

Expectation " " " " in rep $\tilde{Y}$ theory $(|k| - N, -k)$.

(it is an equality of 2-different Wilson loops corresponding to Y and $\tilde{Y}$)
in 2-different gauge theories: $U(N)$ and $U(|k|-N)$.

$$\therefore \quad \left\langle e^{V(\mathrm{tr}\,U^n)} \right\rangle_{N,k} = \left\langle e^{V((-1)^{n+1}\mathrm{tr}\,\tilde{U}^n)} \right\rangle_{k-N,-k}.$$

$$\Downarrow$$

———x———x———

The Density of eigenvalues of the zero mode of the holonomy matrix:

$$\rho(\alpha) = \sum_n e^{-in\alpha} \rho_n \,, \quad \rho_n = \frac{1}{N} \mathrm{tr}\, U^n$$

$$\left(= \frac{1}{N}\sum_i \sum_n e^{-in\alpha + in\theta_i} = \frac{1}{N} \sum_i \sum_n e^{in(\alpha+\theta_i)} = \frac{1}{N}\sum_i \delta(\alpha - \theta_i) \right)$$

same theory gauge $U(k-N)_k$

$$N\rho_n = \mathrm{tr}\,U^n \longrightarrow (-1)^{n+1}\mathrm{tr}\,\tilde{U}^n = (-1)^{n+1}(k-N)\tilde{\rho}_n$$

$$\boxed{N\rho_n = (k-N)(-1)^{n+1}\tilde{\rho}_n + k\,\delta_{n,0}}$$

$$\Rightarrow \quad \tilde{\rho}(\alpha) = \frac{\lambda}{1-\lambda}\left[\frac{1}{2\pi\lambda} - \rho(\alpha+\pi)\right] \geq 0.$$

$$\Leftrightarrow \quad \rho(\alpha) = {}^{1}\!/_{2\pi\lambda}\left(1 - 2\pi(1-\lambda)\tilde{\rho}(\alpha-\pi)\right).$$

———x———x———

$$\therefore \left\langle e^{V(\rho)} \right\rangle_{N,k} = \left\langle e^{V(\tilde{\rho})} \right\rangle_{k-N,k}$$

Now we do the CS integral:

$$Z = \int \mathcal{D}A_\mu \, e^{iS_{CS} - NT^2 v(U)}$$

Gauge fixing:

(1.) $\partial_3 A_3 = 0$.

(2.) $U(x) = e^{i\beta A_3(x)}$ is diagonal.

$$A_3(x)_{ij} = \frac{1}{\beta}\delta_{ij}\,\alpha_i(x). \quad (x \text{ dep. gauge trans.})$$

(3.) $\partial_1 A_1^{ii}(x) + \partial_2 A_2^{ii}(x) = 0$ when $A_{1,2} = \int_0^\beta dx^3 A_{1,2}(x, x_3)$.

(fixes the residual $U(1)\times U(1) \cdots U(1)_N$

Now we discuss the topology of the gauge fixing (2.)

example of $SU(2)$:

$$U(x) = g^{-1}(x)\begin{bmatrix} e^{i\theta(x)} & 0 \\ 0 & e^{-i\theta(x)} \end{bmatrix} g(x) \qquad , x \in S^2.$$

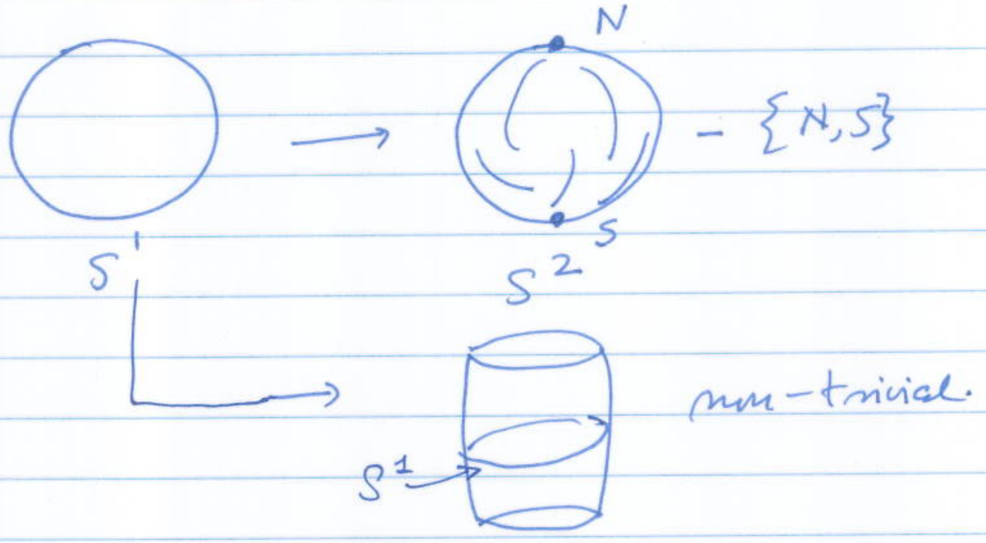
This choice fails if the diagonal matrix is $\mathbb{1}$ or $-\mathbb{1}$ when the 2 eigenvalues co-incide.

Hence the gauge can only be fix for $\{U(x) - \{\mathbb{1}, -\mathbb{1}\}\}$
In this case $\{U(x)\}$ fall into topological classes

because $\quad x \in S^2 \longrightarrow \left(SU(2) \simeq S^3 - \{1\!\!1, -1\!\!1\}\right)$

is ~~topologically non-t~~
maps are characterized by an integer:

e.g.



———— x ———— x ————

In general ~~when~~ when $e^{i\frac{\alpha_i}{\beta}} = \pm e^{i\frac{\alpha_{i+1}}{\beta}}$
of an ordered set (by Weyl)
2-eigenvalues, coincide upto a sign we have to excise such configuration from the $\{U(x)\}$

$$\begin{bmatrix} e^{i\frac{\alpha_1(x)}{\beta}} & & & \\ & e^{i\frac{\alpha_2(x)}{\beta}} & & 0 \\ & 0 & \ddots & \\ & & & e^{i\frac{\alpha_N(x)}{\beta}} \end{bmatrix}$$

~~However $U(x)$~~ We will see later that the maps are characterized for each $U(1)$ by $U(1)$ fluxes on S^2.

$$Z = \int \mathcal{D}A_3 \, \Delta_{FP}(A_3) \, e^{-T^2 N \, U(U)} \, Z'(A_3)$$

$$\Delta_{FP}(A_3) = \det{}'(\partial_3 + \mathrm{ad}\, A_3) \qquad \longleftrightarrow \qquad \partial_3 A_3 = 0$$

$$Z'(A_3) = \int \mathcal{D}A_1 \mathcal{D}A_2 \, e^{-(S_1+S_2)}$$

only x^3 ind. modes of $A_{1,2}$

$$S_1 = \frac{ik}{4\pi} \int d^3x \, \mathrm{tr}\, A_3 \left(\partial_1 A_2 - \partial_2 A_1\right)$$

$$S_2 = \frac{ik}{4\pi} \int d^3x \, \mathrm{tr}\left(D_3 A_1 A_2\right)$$

~~di~~ $A^{ii}_{1,2}(x)$ do not appear here

$$D_3 = \left(\partial_3 + i\, \mathrm{ad}\, A_3\right)$$

$$A_3^{ij}(x) = \underset{\beta}{\delta^{ij}} \, \alpha_j(x)$$

$$A_\alpha^{ij}(x) = a_\alpha^i(x)\,\delta_{ij} + \tilde{A}_\alpha^{ij}(x). \qquad \alpha = 1,2.$$

$$\int_0^\beta \tilde{A}^{ij}(x, x^3)\, dx^3 = 0.$$

$$S_1 = \frac{ik}{4\pi} \int_{S^2} \sum_i \alpha_i(x) \underbrace{\left(\partial_1 a_2^i - \partial_2 a_2^i\right)(x)}_{\mathcal{F}^i_{12}(x)}$$

$$S_2 = \frac{ik}{4\pi} \int_{S^2 \times S^1} \mathrm{tr}\left(\partial_3 + \mathrm{ad}\, A_3\right) \tilde{A}_1 \tilde{A}_2 \, dx_1 dx_2 dx_3$$

Now $\displaystyle\int \mathcal{D}\tilde{A}_1 \mathcal{D}\tilde{A}_2 \, e^{\frac{ik}{4\pi}\int \mathrm{tr}\,(\partial_3 + \mathrm{ad}\,A_3)\tilde{A}_1\tilde{A}_2}$

$$= \frac{1}{\sqrt{\det{}'_V\,(\partial_3 + A_3)}}$$

vector spherical harmonics.

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a regular vector field on S^2

$$a^i_\alpha = \partial_\alpha \psi^i + \epsilon^{\alpha\beta}\partial_\beta \chi^i$$

$\partial_\alpha a^i_\alpha = 0 \Rightarrow \partial^2\psi^i = 0 \Rightarrow \psi^i$ is a const. & $\partial_\alpha \psi^i = 0$

$$\Rightarrow S_1 = \frac{ik\beta}{2\pi}\int d^2x \sum_i \frac{\alpha_i(x)\partial^2\chi_i}{\beta}$$

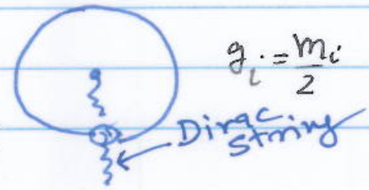
$$+ \frac{ik\beta}{2\pi}\int d^2x \sum_i \frac{\alpha_i(x)}{\beta}\mathcal{F}^i_{12}(x)$$

integration over $\chi^i \Rightarrow \delta(\partial^2\alpha_i) \Rightarrow \partial^2\alpha_i = 0$

$\Rightarrow \alpha_i(x) \equiv \alpha_i$ const. !

$\therefore S_1 = \frac{ik}{2\pi}\cancel{\beta}\sum_i \alpha_i \int d^2x\, \mathcal{F}^i_{12}$, $\int d^2x\,\mathcal{F}^i_{12} = \frac{m^i}{2}\cdot 4\pi$ (note the 2).

$e^{i\int d^2x\,\mathcal{F}^i_{12}} = e^{i\oint a^i_\alpha dx_\alpha} = e^{i4\pi g_i} = e^{i2\pi m_i}$



$g_i := \frac{m_i}{2}$

$$\Rightarrow S_1 = ik \sum_{i=1}^{N} \alpha_i m_i$$

Summing over <u>all</u> flux sectors for <u>each</u> $U(1)$.

$$\sum_{m_i=-\infty}^{+\infty} e^{ik\alpha_i m_i} = \sum_{m_i=-\infty}^{+\infty} \delta\left(\alpha_i - \frac{2\pi m_i}{k}\right)$$

for each m_i — electric-magnetic duality

$\frac{1}{k} \sim$ magnetic charge.

——— x ——— x ———.

<u>The measure:</u>

$$\int \mathcal{D}A_3 \frac{\det'(\partial_3 + \mathrm{ad}\,A_3)_S}{\sqrt{\det'(\partial_3 + \mathrm{ad}\,A_3)_V}}$$

$$= \int \prod_{i=1}^{N} d\alpha_i \prod_{i \neq j} 2 \sin\left(\frac{\alpha_i - \alpha_j}{2}\right)$$ (see ref. 8)

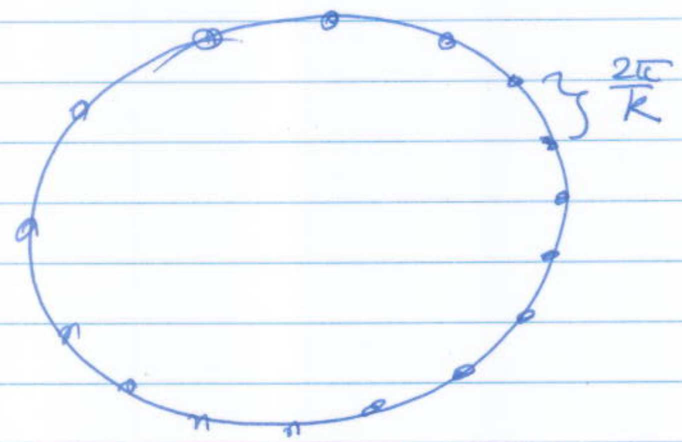
(only the zero mode survives)

Putting it all together: Blau-Thompson, g = genus.

$$Z = \prod_{i=1}^{N} \sum_{m_i=-\infty}^{+\infty} \left[\left\{ \prod_{i\neq j} 2 \sin \frac{2\pi}{k}\left(\frac{m_i - m_j}{2}\right) \right\}^{1-g} e^{-V(\{m\})} \right]$$

agrees with Blau + Thompson for $g = 0$ (sphere).
note that the measure is 1 for the $g=1$ (torus)....

Note that integral over eigenvalues is replaced by a sum over equally spaced discrete eigenvalues, which do not coincide.
$e^{i\frac{2\pi m_i}{k}}$ of the zeromode of the holonomy matrix on S^2.



$\frac{2\pi}{k} \times k = 2\pi$

$k \to \infty, \frac{2\pi}{k} \to 0$

Maximum no. of eigen values in an arc $\Delta\alpha$

$$\frac{\Delta\alpha}{\text{spacing}} = \frac{\Delta\alpha}{\frac{2\pi}{k}} \equiv \left[\Delta\alpha\, \tilde{\rho}(\alpha)\right]_{max} \Rightarrow \tilde{\rho}(\alpha)_{max} = \frac{k}{2\pi}$$

or for the normalized density $\boxed{\rho_{max} = \frac{1}{N}\tilde{\rho}_{max} = \frac{k}{2\pi N} = \frac{1}{2\pi\lambda}}$

In the previous lecture we indicated that CS + matter theories (any specific) ~~have the level~~ inherit the level-rank duality of CS theories via

$$\langle \chi_Y(U) \rangle_{N,k} = \langle \chi_{\tilde{Y}}(\tilde{U}) \rangle_{|k|-N,-k} \qquad \text{(A)}$$

χ_Y is the character of $U(N)$.

[We also showed that if

$$\chi_Y(U) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_Y(\sigma) \underbrace{\mathrm{tr}U^{k_1} (\mathrm{tr}U^2)^{k_2} \cdots (\mathrm{tr}U^n)^{k_n}}_{\prod_{m=1}^{n} (\mathrm{tr}U^m)^{k_m}}$$

then

$$\chi_{\tilde{Y}}(U) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_Y(\sigma) \prod_{m=1}^{n} \left[(-1)^{m+1} \mathrm{tr}U^m\right]^{k_m}$$

i.e. if you replace $\mathrm{tr}U^m$ in $\chi_Y(U)$ by $(-1)^{m+1} \mathrm{tr}U^m$ then you get $\chi_{\tilde{Y}}(U)$]

This fact applied to the duality eqn. (A) implies

$$\cancel{\langle \chi_Y(U) \rangle_{N,k}} = \langle \chi_Y(\{(-1)^{m+1} \mathrm{tr}\tilde{U}^m\}) \rangle_{k-N,k}$$

$$\langle \chi_Y(\{\mathrm{tr}U^n\}) \rangle_{N,k}$$

$$\langle \chi_Y(\{\mathrm{tr}U^n\}) \rangle_{k\lambda,k} = \langle \chi_Y(\mathrm{tr}\tilde{U}^m (-1)^{m+1}) \rangle_{k(1-\lambda),k}$$

Now ~~both sides~~ at the saddle points in the large k & N limit are evaluated by the value of the density of corresponding eigenvalues

$$\rho(\alpha) = \sum_{n} e^{in\alpha}\left(\frac{\mathrm{tr}U^n}{N}\right) = \sum_{n} e^{in\alpha}\rho_n$$

$$\tilde{\rho}(\alpha) = \sum_{n} e^{in\alpha}\,\frac{\mathrm{tr}\tilde{U}^n}{(k-N)} = \sum_{n} e^{in\alpha}\tilde{\rho}_n$$

~~Since~~ $\cancel{\langle \chi_r(\{\rho_n\})\rangle_{k\lambda,k} = \chi_r(\{\bar{\rho}_n\})_{k\lambda,k}}$

$$\cancel{\langle \chi_r(\{\tilde{\rho}_n\})\rangle_{k(1-\lambda),k} = \chi_r(\{\bar{\tilde{\rho}}_n\})_{k(1-\lambda),k}}$$

~~we have~~ $\cancel{\bar{\rho}_n = \bar{\tilde{\rho}}_n}$

Since $\langle \chi_Y(\{\mathrm{tr}U^n\})\rangle = \chi_Y(\{\langle \mathrm{tr}U^n\rangle\})$

and $\langle \chi_Y(\{(-1)^{n+1}\mathrm{tr}\tilde{U}^n\})\rangle = \chi_Y(\{\langle \mathrm{tr}\tilde{U}^n(-1)^{n+1}\rangle\})$

we get $\boxed{\mathrm{tr}U^n = \mathrm{tr}\tilde{U}^n(-1)^{n+1}}$

AT the saddle points of both sides.

$$N \rho_n = (k-N)(-1)^{n+1} \tilde{\rho}_n + k\delta_{n,0}$$

$\therefore$

$$\lambda \rho_n = (1-\lambda)(-1)^{n+1} \tilde{\rho}_n + \delta_{n,0}$$

$\therefore \quad \lambda \rho_n + (1-\lambda)(-1)^n \tilde{\rho}_n = \delta_{n,0} \qquad \times \sum_n \frac{e^{in\alpha}}{2\pi}$

$\therefore \quad \boxed{\lambda \rho(\alpha) + (1-\lambda)\tilde{\rho}(\alpha+\pi) = \frac{1}{2\pi}}$

This means that if $\mathcal{V}_a(\rho)_{k\lambda, k\lambda}$ is the answer for 'a' then

$$\mathcal{V}_{a,\lambda}(\rho) = \mathcal{V}_{a,1-\lambda}(\tilde{\rho}) = \tilde{\mathcal{V}}_{a,1-\lambda}(\rho)$$

where $\tilde{\mathcal{V}}(\rho) = \mathcal{V}(\tilde{\rho})$.

Now if $\tilde{\mathcal{V}}_{a,1-\lambda}(\rho) = \mathcal{V}_{b,\lambda}(\tilde{\rho})$

Then we can say that a + b are related by the level-rank duality

$$\lambda \to 1-\lambda$$
$$k \to -k$$

List of theories for which we know v:

1. CS + regular fermions

dual

2. CS + critical boson $\left(\delta S = A \int \bar{\Phi}\phi \right)$

3. CS + critical fermions $\left(\delta S = B \int \bar{\psi}\psi + \frac{N}{6} \lambda_6^f B^3 \right)$

dual

4. CS + regular bosons $\left(\delta S = \lambda_6 \int (\bar{\phi}\phi)^3 \right)$

5. CS + $N=2$ SUSY matter.

self dual

Now we have a list of various matter theories coupled to CS, for which we know $V_{a,\lambda}(\rho)$.

(1.) CS + massless fundamental bosons (with $\lambda_6 \phi^6$ term).
(need not mention).

$$V_{boson}[\rho] = -\frac{N}{6\pi}\sigma^3\left(1+\frac{2}{\hat{\lambda}}\right) + \frac{N}{2\pi}\int_{-\pi}^{\pi} d\alpha\,\rho(\alpha)\int_{\sigma}^{\infty} dy\, y\left[\ln\left(1-e^{-y+i\alpha}\right) + \ln\left(1-e^{-y-i\alpha}\right)\right]$$

$$\hat{\lambda} = \sqrt{\frac{\lambda_6}{8\pi^2} + \lambda^2}$$

σ is related to the thermal mass of the bosons:

$$\Sigma_B = \sigma^2 T^2$$

$$\sigma = -\frac{1}{2}\frac{\hat{\lambda}(\lambda,\lambda_6)}{\lambda}\int_{-\pi}^{\pi} d\alpha\,\lambda\,\rho(\alpha)\left[\ln 2\sinh\left(\frac{\sigma-i\alpha}{2}\right) + \ln 2\sinh\left(\frac{\sigma+i\alpha}{2}\right)\right]$$

$$\left(\Leftarrow \frac{\delta}{\delta\sigma} V[\rho,\sigma] = 0\right)$$

(2) CS minimally coupled to massless fermions:

$$\mathcal{V}_{\text{fermion}}[\rho] = -\frac{N}{6\pi}\left(\frac{\tilde{c}^3}{\lambda} - \tilde{c}^3 + 3\int_{-\pi}^{\pi} d\alpha\,\rho(\alpha)\int_{\tilde{c}}^{\infty} dy\, y\left[\ln(1+e^{-y+i\alpha}) + \ln(1+e^{-y-i\alpha})\right]\right)$$

$$\tilde{c} = \lambda\int_{-\pi}^{\pi} d\alpha\,\rho(\alpha)\left[\ln\left\{2\cosh\left(\frac{\tilde{c}+i\alpha}{2}\right)\right\} + \ln\left\{2\cosh\left(\frac{\tilde{c}-i\alpha}{2}\right)\right\}\right]$$

$$\frac{\delta \mathcal{V}_f[\rho,\tilde{c}]}{\delta\tilde{c}} = 0$$

is related to the thermal mass of the fermion.

$$\Sigma_T = \tilde{c}^2 T^2$$

(3) CS coupled to critical bosons. (gauged version of U(N) W-F theory)

W-F ≡ Wilson-Fischer

The lagrangian of the UV theory is simply that of massless minimally coupled fundamental boson deformed by

$$\delta S = \int d^3x\, A\,\bar{\Phi}\phi$$

$$\mathcal{V}[\rho] = -\frac{N}{6\pi}\left(\sigma^3 + \frac{2(\sigma^2 - A\beta^2)^{3/2}}{\lambda}\right) + \frac{N}{2\pi}\int_{\sigma}^{\infty} y\,dy\int_{-\pi}^{\pi} d\alpha\,\rho(\alpha)\left[\ln(1-e^{-y+i\alpha}) + \ln(1-e^{-y-i\alpha})\right]$$

$$\frac{\partial}{\partial\sigma}\mathcal{V} = 0 = \frac{\partial\mathcal{V}}{\partial A} = 0 \quad \text{determine } A \,\&\, \sigma$$

$A = \sigma^2 T^2$ and $\boxed{M^2 = \sigma^2 T^2}$

$$\int_{-\pi}^{\pi} d\alpha\, \rho(\alpha) \left(\ln 2 \sinh\left(\frac{\sigma - i\alpha}{2}\right) + \ln 2 \sinh\left(\frac{\sigma + i\alpha}{2}\right) \right) = 0$$

Simplification:

$$v[\rho] = -\frac{N}{6\pi}\sigma^3 + \frac{N}{2\pi}\int_\sigma^\infty y\, dy \int_{-\pi}^{\pi} d\alpha\, \rho(\alpha) \left[\ln\left(1 - e^{-y+i\alpha}\right) + \ln\left(1 + e^{-y-i\alpha}\right) \right]$$

$$\to \frac{\delta v}{\delta c} = 0.$$

———— × ———— × ————

~~CRITICAL~~

$$v^{\text{Crit. boson}}_{k-N,\, 1-\lambda}(\tilde{\rho}) = v^{\text{Reg. fermion}}_{N,\lambda}(\rho)$$

and $$\sigma^{\text{Crit. boson}}_{k-N,\, 1-\lambda}(\tilde{\rho}) = \tilde{c}^{\text{reg. fer}}_{N,\lambda}(\rho)$$

Prove.

———— × ———— × ————

Similarly one can prove statements about regular boson + critical fermion + self-duality of $N=2$ Susy theories:

$$v[\rho] = -\frac{N}{6\pi|\lambda|}\left(\tilde{c}^3 - 6|\lambda| \int_{-\pi}^{\pi} d\alpha\, \rho(\alpha)\, \mathrm{Re} \int_{\tilde{c}}^{\infty} dy\, y \left(\log \tanh \frac{y+i\alpha}{2}\right)\right)$$

$$\tilde{c} = 2\left|\mathrm{Re}\int_{-\pi}^{\pi} d\alpha\, \rho(\alpha) \log\coth\frac{\tilde{c}+i\alpha}{2}\right|, \quad \frac{\partial v}{\partial \tilde{c}} = 0$$

One can show that $\tilde{C}_{\lambda}^{SUSY}(\rho) = \tilde{C}_{1-\lambda}^{SUSY}(\tilde{\rho})$

$$v_{N,\lambda}^{SUSY}(\rho) = v_{1-\lambda}^{SUSY}(\tilde{\rho}).$$

Now to complete our program I have to tell you how to find:

1) $v_1(\rho)$ for a given theory and (these were computed in ref. 4)

2) how to find $\rho(\alpha)$.

Let us look at 2) first.

Recall:

$$Z = \prod_{i=1}^{N} \sum_{m_i=-\infty}^{+\infty} \left[\prod_{i \neq j} \sin\left(\frac{2\pi}{k} \frac{(m_i - m_j)}{2} \right) e^{-v(\{m_i\})} \right]$$

$$= \prod_{i=1}^{N} \sum_{m_i=-\infty}^{+\infty} e^{+\sum_{i\neq j} \ln \sin \frac{2\pi}{k}(m_i - m_j) - v(\{m_i\})}.$$

$$= \int \prod_{i=1}^{N} d\alpha_i \prod_{i=1}^{N} \delta\left(\alpha_i - \frac{2\pi}{k} m_i\right) e^{\sum_{i\neq j} \ln \sin \frac{\alpha_i - \alpha_j}{2} - v(\{\alpha_i\})}.$$

$$\simeq \int dU \, e^{-v(U)}$$

(1 unitary matrix model)

The saddle point eqn. for the density of eigenvalues

$$\rho(\alpha) = \frac{1}{N} \sum_{i=1}^{N} \delta(\alpha - \alpha_i) \quad \text{is found by}$$

extremizing

$$S(\{\alpha_i\}) = \sum_{m \neq l} \ln 2 \sin\left(\frac{\alpha_m - \alpha_l}{2}\right) - \sum_l V(\alpha_l)$$

potentials are single trace by calculation

$$\frac{\partial S}{\partial \alpha_i} = 0 \Rightarrow V'(\alpha_m) = \sum_{l \neq m} \cot\left(\frac{\alpha_m - \alpha_l}{2}\right)$$

$$\Rightarrow V'(\alpha_0) = N \; P\!\int d\alpha \, \rho(\alpha) \cot\left(\frac{\alpha_0 - \alpha}{2}\right)$$

$$\rho(\alpha) \geq 0 \quad \text{and} \quad \rho(\alpha) \leq \frac{1}{2\pi\lambda}$$

The upperbound comes from the discreteness of the eigenvalues as $N \to \infty$ and $|k| \to \infty$.

This is the same type of singular integral equation ~~enton~~ encountered by Gross-Witten-Wadia in 2-dim lattice gauge theory. However the upper bound leads to new solutions.

General solution is complex depending on the values of λ and ζ and exhibits various phase transitions including the GWW transition.

However in the $\zeta \to \infty$ limit, there is a universal solution independent of the potential because the measure factor dominates:

$$\rho(\alpha) = \frac{1}{2\pi\lambda} \quad , \quad -\pi\lambda \leq \alpha \leq \pi\lambda$$

This solution was first discovered in ref. 9, where the importance of the discreteness of the eigenvalues of the holonomy matrix was realized first.

Level-rank duality and the saddle point

It is easy to show that if $\rho(\alpha)$ is a solution of

$$V'(\alpha_0) = N\,P\int d\alpha\, \rho(\alpha) \operatorname{Cot}\left(\frac{\alpha_0 - \alpha}{2}\right)$$

then

$$\tilde{\rho}(\alpha) \quad \text{given by} \quad \lambda\rho(\alpha) + (1-\lambda)\tilde{\rho}(\alpha+\pi) = \frac{1}{2\pi}$$

solves

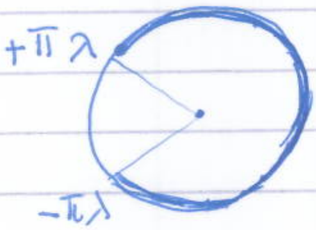
$$\tilde{V}'(\alpha_0) = (K-N)\,P\int d\alpha\, \tilde{\rho}(\alpha) \operatorname{Cot}\left(\frac{\alpha_0 - \alpha}{2}\right).$$

which shows that the saddle point solution is consistent with level rank duality.

The complete analysis of the solutions of the saddle pt. eqn is very difficult in general.

However in the limit $\zeta \to \infty$, only the CS measure term contributes and we have an exact solution independent of the potential V.

$$\rho(\alpha) = \frac{1}{2\pi\lambda}, \quad -\pi\lambda \leq \alpha \leq \pi\lambda$$



This solution first appeared in ref. 9.

For finite ζ in ref. 10 the toy model

$V(U) = -\frac{N\zeta}{2}\left(\mathrm{tr}U + \mathrm{tr}U^{+}\right)$ was worked out.

The solution is specified for λ, ζ by both

lower gaps where $\rho(\alpha) = 0$ on arcs and

upper gaps where $\rho(\alpha) = \frac{1}{2\pi\lambda}$ on arcs.

Result (quoted from ref. 10).

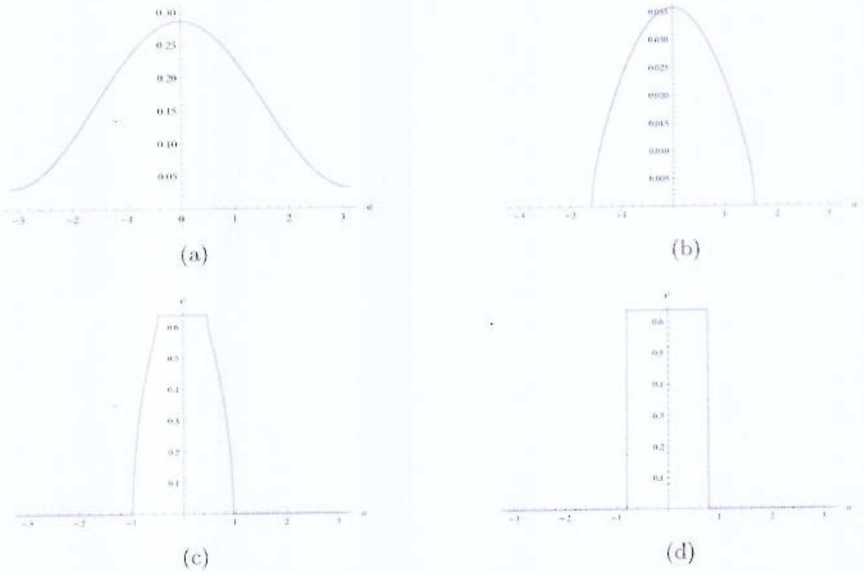


Figure 7: The eigenvalue distribution of the capped GWW model at $\lambda = 0.25$ for increasing values of ζ: Fig.7(a) is for $\zeta = \frac{1}{2}$, Fig.7(b) is for $\zeta = 2$, Fig.7(c)is for $\zeta = 4.6$ and Fig.7(d)is for $\zeta = 11.03$. In this case as we increase ζ, eigenvalue density distribution first develops a lower gap as Fig.7(b). Further increasing ζ, eigenvalue density distribution develops a upper gap as well Fig.7(c).

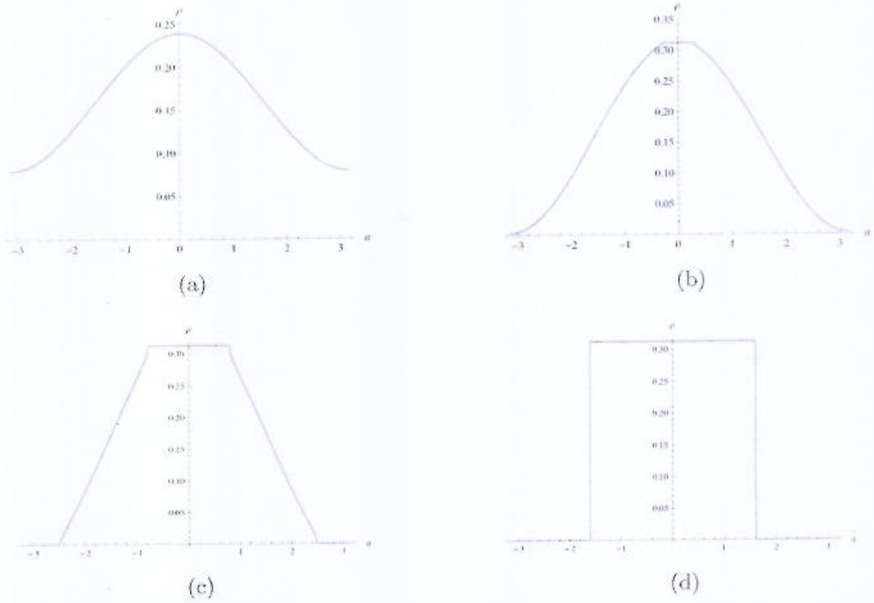


Figure 8: The eigenvalue distribution of the capped GWW model at $\lambda = 0.51$ for increasing values of ζ: Fig.8(a)is for $\zeta = \frac{1}{2}$, Fig.8(b)is for $\zeta = 0.98$, Fig.8(c)is for $\zeta = 1.13$ and Fig.8(d)is for $\zeta = 5.06$. In this case as we increase ζ, eigenvalue density distribution first develops a upper gap as Fig.8(b). Further increasing ζ, eigenvalue density distribution develops a lower gap as well Fig.8(c).

~~Now the complete analysis of the solution set is very difficult in general, however for~~

$$\zeta \gg 1$$

$$\rho(\alpha) = \frac{1}{2\pi\lambda} \qquad (-\pi\lambda \leq \alpha \leq \pi\alpha.)$$

$$\int_{-\pi\lambda}^{\pi\lambda} \frac{1}{2\pi\lambda}\, d\alpha = \frac{1}{2\pi\lambda}\, 2\pi\lambda = 1$$

~~(first appeared in ref 9.).~~

<u>Calculation of $v(U)$:</u> $\qquad N\zeta^2 = T^2 V_2 N.\,,\ T^2 \sim N\zeta$

V_2 large $\qquad S^2 \sim R^2.$

$$\cancel{R^4}\ S^2 \times S^1 \to R^2 \times S^1 \longrightarrow R^{(1,1)} \times S^1 \longrightarrow R^2 \times S^1$$

<u>Key</u> step: $A_- = 0$ gauge. $\qquad S_{CS} = \frac{k}{4\pi}\int A_+ \partial_- A_3$

Gauss law: $\cancel{F_{+-} = 0,\ F_{+3} = \partial_+ A_3 - \partial_3 A_+ + [A_+, A_3]}$

$$\partial_- A_3 = \frac{2\pi}{k} J_- \qquad R^{11} \to R^2$$

$$\partial_- \partial_+ A_3 = \frac{2\pi}{k}\, \partial_+ J_- \Rightarrow A_3 = \frac{2\pi}{k}(\partial_+\partial_-)^{-1}\partial_+ J_- + A_3^{(0)}(x^3)$$

choose $\partial_3 A^{(0)}(x^3) = 0$ gauge + Diagonalize $A_3^{(0)\,ij} = \delta^{ij}\, 2\pi \alpha_i / \beta$

$$\Rightarrow \delta^{ij}\partial_3 - i A_3^{ij} = \left(\partial_3 - i\frac{2\pi}{\beta}\alpha_i\right)\delta^{ij} - i\frac{2\pi}{R}(\partial_+\partial_-)^{-1}\partial_+ J_-$$

note that $R^{(1,1)} \to R^2$ means that x^+ and x^- are complex conjugates of each other and

$(\partial_+\partial_-)^{-1}$ is the inverse laplacian which is well defined.

All the calculations in ref. 4 can now be repeated to calculate $v(U)$ with the addition of including the eigenvalues of the holonomy matrix, bearing in mind that they are actually discrete as our calculation on S^2 ~~rele~~ revealed.

Summary & future directions:

1. A convincing case for Fermion-Boson duality has been presented in 2+1 dim. large N gauge theories both in terms of softly broken higher spin symmetry and an explicit calculation of the partition function.

2. It would be useful to understand these results entirely in terms of 'anyons'.

3. Calculate the fermion + boson S-matrix (of massive particles)
[in progress with S. Jain, S. Minwalla, Mangesh, S. Yokoyama].

References:

1. I. R. Klebanov and A.M. Polyakov, hep-th/0210114
'AdS Dual of the O(N) Vector model'

2. M. Vasiliev, hep-th/9910096
' Higher Spin Gauge theories'.

3. S. Giombi + X. Yin, arXiv: 1208.4036
' The Higher Spin / Vector model duality'.

4. S. Giombi, S. Minwalla, S. Prakash, S. Trivedi,
S. Wadia + Xi Yin. ArXiv: .1110.4386
'Chern - Simons Theory with vector fermion matter'.

5. O. Aharony, G. Gur-Ari, R. Yacoby, arXiv: 1110.4382
'd=3 bosonic vector models coupled to CS
gauge theoris'.

6. J. Maldacena and A. Zhiboedov, arXiv: 1204.4382
' constructing higher spin theories with a slightly
broken higher spin symmetry'.

7. O. Aharony, G. Gur-Ari and R. Yacoby, arXiv: 1207.4593
'Correlation functions of large N CS theories and
bosonization in 3-dim.'

8. S. Jain, S. Trivedi, S. Wadia and S. Yokoyama:
arXiv: 1207.4750
SUSY CS theories with Vector matter.

My.

9. O. Aharony, S. Giombi, G. Gur-Ari, J. Maldacena + R. Yacoby. ArXiv: 1211.4843
'The thermal free energy in large N CS-matter theories'.

10. S. Jain, S. Minwalla, T. Sharma, T. Takimi, S. Wadia + S. Yokoyama arXiv: 1301.6169
'Phases of large N vector CS theories'.

11. S. Jain, S. Minwalla, S. Yokoyama, arXiv: 1305.7235
'CS duality with a fundamental fermion + boson'