DETERMINISTIC OPTIMAL CONTROL AND VISCOSITY SOLUTIONS

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1 Notations and problem description

Consider the controlled dynamical system, with state space $\mathbb{R}^n$ and finite time horizon $0 \leq t \leq T$, given by

$$\begin{aligned} \dot{x}(t) &= f(x(t), t, u(t)); \quad t_0 \leq t \leq T, & (1)\\ x(t_0) &= x_0. & (2) \end{aligned}$$

Here $t_0 \in [0, T]$, $f : \mathbb{R}^n \times [0, T] \times U \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$ is compact, $u(\cdot) \in \mathcal{U}[t_0, T] := L^\infty([t_0, T]; U)$.

We first discuss a set of sufficientt conditions on the initial value problem (IVP) given by (1), (2) that ensure existence of a unique solution (state trajectory) $x(\cdot)$ corresponding to each control input $u(\cdot)$.

1.1 Assumptions for existence and uniqueness

(A1) The function f is bounded, continuous on $\mathbb{R}^n \times [0, T] \times U$, and there exists a positive constant K such that

$$|f(x, t, u) - f(y, t, u)| \leq K|x - y|, \quad \forall x, y \in \mathbb{R}^n, t \in [0, T], u \in U. \tag{3}$$

Theorem 1 *Assume (A1). Then, for each $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$ and $u(\cdot) \in \mathcal{U}[t_0, T]$, there exists a unique solution (in the Caratheodory sense) for the IVP (), () which exists for all $t \in [t_0, T]$.*

PROOF: See [2]. ■

Notation for state trajectory: The unique solution of the IVP (1), (2) is denoted by $\phi(t) = \phi(t; x_0, t_0, u(\cdot))$. This notation is very indicative and is motivated by the paper [4] by Kalman.

1.2 Optimal control problem

Consider the objective)or cost) functional:

$$J[x_0, t_0, u(\cdot)] := \int_{t_0}^{T} \ell(\phi(t), t, u(t))\ dt + g(\phi(T)), \tag{4}$$

where $\ell : \mathbb{R}^n \times [0, T] \times U \to \mathbb{R}_+$ is the running cost, $g : \mathbb{R}^n \to \mathbb{R}_+$ is the terminal cost, and $\phi(t) = \phi(t; x_0, t_0, u(\cdot))$. We assume that
(A2) The function g is bounded and uniformly continuous. The function ℓ is bounded, continuous, and there exists a positive constant L such that

$$|\ell(x, t, u) - \ell(y, t, u)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}^n, t \in [0, T], u \in U. \tag{5}$$

We now can define the value function $V : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ as

$$V(x_0, t_0) := \inf_{u(\cdot) \in \mathcal{U}[t_0, T]} J[x_0, t_0, u(\cdot)]. \tag{6}$$

Remark 1 *By definition, $V(x_0, T) = g(x_0)$ for all $x_0 \in \mathbb{R}^n$.*

Statement of optimal control problem: For each $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$, find $u^*(\cdot) \in \mathcal{U}[t_0, T]$ that minimizes the cost functional J. That is, to find $u^*(\cdot)$ such that $V(x_0, t_0) = J[x_0, t_0, u^*(\cdot)]$.

Definition 1 *A control (input) $u^*(\cdot)$ that minimizes the cost functional is called an optimal control (input).*

Remark 2 *Unless otherwise stated, we always make the assumptions (A1) and (A2) for the remainder of these notes.*

2 Properties of the value function V

We first prove (a version of) Gronwall inequality that will be useful in the sequel.

Lemma 1 (Gronwall inequality) *Let $\alpha \in L^1[t_0, t_1]$. Let C and D be constants such that*

$$\alpha(t) \leq C + D \int_{t_0}^{t} \alpha(s)\, ds, \quad a.e.\ t \in [t_0, t_1].$$

Then $\alpha(t) \leq Ce^{D(t-t_0)}$ for a.e. $t \in [t_0, t_1]$.

PROOF: Let $\beta(t) = e^{-D(t-t_0)} \int_{t_0}^{t} \alpha(s)\, ds$. Then it follows that

$$\dot{\beta}(t) = -D\beta(t) + \alpha(t)e^{-D(t-t_0)} \leq Ce^{-D(t-t_0)}, \quad \text{a.e. } t \in [t_0, t_1].$$

Inegrating from t_0 to t, and then multiplying by $e^{D(t-t_0)}$, we get the required inequality. ■We use Gronwall inequality to compare trajectories emanating from different initial states x_0 and y_0.

Lemma 2 *For all $t_0 \in [0, T]$ and $u(\cdot) \in \mathcal{U}[t_0, T]$, we have the bound*

$$|\phi(t; x_0, t_0, u(\cdot)) - \phi(t; y_0, t_0, u(\cdot))| \leq |x_0 - y_0| e^{K(t-t_0)}, \quad t_0 \leq t \leq T. \tag{7}$$

PROOF: Using (A1), we get

$$|\phi(t; x_0, t_0, u(\cdot)) - \phi(t; y_0, t_0, u(\cdot))| \leq |x_0 - y_0| + K \int_{t_0}^{t} |\phi(t; x_0, t_0, u(\cdot)) - \phi(t; y_0, t_0, u(\cdot))|\, dt.$$

The required bound now follows from Gronwall inequality. ■The next lemma gives a bound for the difference between trajectories starting at different times, and proof is left as an exercise.

Lemma 3 *Let $0 \leq s_0 \leq t_0 \leq T$. Then for all $u(\cdot) \in \mathcal{U}[s_0, T]$, we have the yound:*

$$|\phi(t; x_0, t_0, u(\cdot)) - \phi(t; x_0, s_0, u(\cdot))| \leq \|f\|_\infty |s_0 - t_0| e^{K(t-t_0)}, \quad t_0 \leq t \leq T. \tag{8}$$

2.1 Continuity of V

Using the definition of the cost functional J, and lemmas 7,7, we get

$$\begin{aligned}|J[x_0,t_0,u(\cdot)]-J[y_0,s_0,u(\cdot)]| &\leq |J[x_0,t_0,u(\cdot)]-J[y_0,t_0,u(\cdot)]|+|J[y_0,t_0,u(\cdot)]-J[y_0,s_0,u(\cdot)]| \\ &\leq L\int_{t_0}^{T}|x_0-y_0|e^{K(t-t_0}\ dt+\omega_g(|x_0-y_0|e^{KT}) \\ &\quad +\|\ell\|_\infty|s_0-t_0|+L\int_{t_0}^{T}\|f\|_\infty|s_0-t_0|e^{K(t-t_0)}\ dt+\omega_g(\|f\|_\infty|s_0-t_0|e^{KT})\end{aligned}$$

This discussion proves the following regularity of the value function.

Theorem 2 *The value function V is bounded and uniformly continuous on $R^n\times[0,T]$. That is $V\in BUC(\mathbb{R}^n\times[0,T])$.*

The following example shows that, in general, V may fail to be differentiable ar some points.

Example 1

Consider the example where $n=1, m=1, U=[-1,1], f(x,t,u)=u, T=1, \ell\equiv 0, g(x)=\arctan(x^2)$.
Note that

$$\phi(1;x_0,t_0,u(\cdot))=x_0+\int_{t_0}^{1}u(t)\ dt.$$

Therefore

$$V(x_0,t_0)=\left\{\begin{array}{lll}0 & ; & |x_0|\leq 1-t_0, \\ g(|x_0|+t_0-1) & ; & \text{otherwise}\end{array}\right\}$$

This value function is not differentiable at all points, despote the fact that all the functions in the data of the optimal control problem are infinitely differentiable. ■

2.2 Dynamic programming principle

For $(x_0,t_0)\in\mathbb{R}^n\times[0,T]$, $\delta\in[0,T-t_0]$, and $\hat{u}(\cdot)\in\mathcal{U}[t_0,t_0+\delta]$, define

$$J_\delta[x_0,t_0,\hat{u}(\cdot)]:=\int_{t_0}^{t_0+\delta}\ell(\phi(t),t,\hat{u}(t))\ dt+V(\phi(t_0+\delta),t_0+\delta),$$

where $\phi(t)=\phi(t;x_0,t_0,\hat{u}(\cdot))$.
Also define

$$V_\delta(x_0,t_0):=\inf_{\hat{u}(\cdot)\in\mathcal{U}[t_0,t_0+\delta]}J_\delta[x_0,t_0,\hat{u}(\cdot)].$$

Clearly $V_0=V_{T-t_0}=V$. We want to show that $V(x_0,t_0)=V_\delta(x_0,t_0)$ for every $\delta\in[0,T-t_0]$.
To this end, we first prove:

Lemma 4 *We have the inequality $V(x_0,t_0)\leq V_\delta(x_0,t_0)$ for $\delta\in[0,T-t_0]$.*

PROOF: Fix $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$, $\delta \in [0, T - t_0]$, and $\hat{u}(\cdot) \in \mathcal{U}[t_0, t_0 + \delta]$.
Let $\phi(t) = \phi(t; x_0, t_0, \hat{u}(\cdot))$; $t_0 \le t \le t_0 + \delta$.
Let $\epsilon > 0$ be given. Choose $\bar{u}(\cdot) \in \mathcal{U}[t_0 + \delta, T]$ such that

$$J[\phi(t_0 + \delta), t_0 + \delta, \bar{u}(\cdot)] \le V(\phi(t_0 + \delta), t_0 + \delta) + \epsilon.$$

Let $\tilde{u}(\cdot) \in \mathcal{U}[t_0, T]$ be the concatenation of $\hat{u}(\cdot)$ and $\bar{u}(\cdot)$. That is,

$$\tilde{u}(s) = \left\{ \begin{array}{lcl} \hat{u}(s) & ; & t_0 \le t \le t_0 + \delta, \\ \bar{u}(s) & ; & t_0 + \delta < t \le T. \end{array} \right\}$$

This implies that

$$\begin{aligned} V(x_0, t_0) &\le J[x_0, t_0, \tilde{u}(\cdot)] \\ &= \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, \hat{u}(t))\ dt + J[\phi(t_0 + \delta), t_0 + \delta, \bar{u}(\cdot)] \\ &\le \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, \hat{u}(t))\ dt + V(\phi(t_0 + \delta), t_0 + \delta) + \epsilon. \end{aligned}$$

This holds for every $\hat{u}(\cdot) \in \mathcal{U}[t_0, t_0 + \delta]$ and $\epsilon > 0$. Therefore $V(x_0, t_0) \le V_\delta(x_0, t_0)$. ■

Lemma 5 *(i). The map*

$$[0, T - t_0] \ni \delta \mapsto J_\delta[x_0, t_0, \hat{u}(\cdot)]$$

is monotonically increasing.
(ii). $J_\delta[x_0, t_0, \hat{u}(\cdot)] = V(x_0, t_0)$ *when* $\delta = 0$, *and* $J_\delta[x_0, t_0, \hat{u}(\cdot)] = J[x_0, t_0, \hat{u}(\cdot)]$ *when* $\delta = T - t_0$.
(iii). Let $u^*(\cdot) \in \mathcal{U}[t_0, T]$. *Then* $u^*(\cdot)$ *is an optimal control (for the initial profile* (x_0, t_0)*) iff the map*

$$[0, T - t_0] \ni \delta \mapsto J_\delta[x_0, t_0, u^*(\cdot)]$$

is a constant map.
(iv). For every $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$, *the map*

$$[0, T - t_0] \ni \delta \mapsto V_\delta(x_0, t_0)$$

is monotonically increasing.

PROOF: We prove (i). The proofs of (ii), (iii) and (iv) are left as exercise.
Let $0 \le \delta' < \delta \le T - t_0$, $\hat{u} \in \mathcal{U}[t_0, t_0 + \delta]$, and $\phi(t) = \phi(t; x_0, t_0, \hat{u}(\cdot))$.

$$\begin{aligned} J_\delta[x_0, t_0, \hat{u}(\cdot)] &= \int_{t_0}^{t_0+\delta'} \ell(\phi(t), t, \hat{u}(t))\ dt + J_{\delta-\delta'}[\phi(t_0 + \delta'), t_0 + \delta', \hat{u}(\cdot)] \\ &\ge \int_{t_0}^{t_0+\delta'} \ell(\phi(t), t, \hat{u}(t))\ dt + V_{\delta-\delta'}(\phi(t_0 + \delta'), t_0 + \delta') \\ &\ge \int_{t_0}^{t_0+\delta'} \ell(\phi(t), t, \hat{u}(t))\ dt + V(\phi(t_0 + \delta'), t_0 + \delta') \quad \text{(by Lemma 4)} \\ &= J_{\delta'}[x_0, t_0, \hat{u}(\cdot)]. \end{aligned}$$

This proves (i). ■

Theorem 3 *For every* $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$, *the map*

$$[0, T - t_0] \ni \delta \mapsto V_\delta(x_0, t_0)$$

is a constant map.

PROOF: Since $V_0 = V_{T-t_0} = V$, the theorem follows from Lemma 5(iv). ■

The above theorem may be restated in a more explicit way as follows.

Theorem 4 (Dynamic Programming Principle (DPP)) *For every* $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$ *and* $\delta \in [0, T - t_0]$,

$$V(x_0, t_0) = \inf_{\hat{u}(\cdot) \in \mathcal{U}[t_0, t_0+\delta]} \Big[\int_{t_0}^{t_0+\delta} \ell(\phi(t), t, \hat{u}(t)) \; dt + V(\phi(t_0 + \delta), t_0 + \delta) \Big],$$

where $\phi(t) = \phi(t; x_0, t_0, \hat{u}(t))$.

■

The following consequence of DPP is useful.

Theorem 5 *For* $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$ *and* $\delta \in [0, T - t_0]$, *we have the following;*
(i). Let $u^*(\cdot) \in \mathcal{U}[t_0, T]$ *and* $\phi(t) = \phi(t; x_0, t_0, u^*(\cdot))$. *If* $J[x_0, t_0, u^*(\cdot)] = V(x_0, t_0)$, *then* $J_\delta[x_0, t_0, \hat{u}^*(\cdot)] = V_\delta(x_0, t_0)$ *and* $J[\phi(t_0 + \delta), t_0 + \delta, \bar{u}^*(\cdot)] = V(\phi(t_0 + \delta), t_0 + \delta)$, *where* $\hat{u}^*)\cdot)$ *and* $\bar{u}^*(\cdot)$ *are the restrictions of* $u^*(\cdot)$ *on the intervals* $[t_0, t_0 + \delta]$ *and* $[t_0 + \delta, T]$ *respectively.*
(ii). Let $\hat{u}^*(\cdot) \in \mathcal{U}[t_0, t_0+\delta]$, $\bar{u}^*(\cdot) \in \mathcal{U}[t_0+\delta, T]$ *and* $\phi(t) = \phi(t; x_0, t_0, \hat{u}^*(\cdot))$. *If* $J_\delta[x_0, t_0, \hat{u}^*(\cdot)] = V_\delta(x_0, t_0)$ *and* $J[\phi(t_0 + \delta), t_0 + \delta, \bar{u}^*(\cdot)] = V(\phi(t_0 + \delta), t_0 + \delta)$, *then* $J[x_0, t_0, u^*(\cdot)] = V(x_0, t_0)$, *where* $u^*(\cdot)$ *is the concatenation of* $\hat{u}^*(\cdot)$ *and* $\bar{u}^*(\cdot)$.

PROOF: Note that

$$\begin{aligned}
J[x_0, t_0, u^*(\cdot)] &= \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, \hat{u}^*(t)) \; dt + J[\phi(t_0 + \delta), t_0 + \delta, \bar{u}^*(\cdot)] \\
&\geq \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, \hat{u}^*(t)) \; dt + V(\phi(t_0 + \delta), t_0 + \delta) \\
&= J_\delta[x_0, t_0, \hat{u}^*(\cdot)] \\
&\geq V_\delta(x_0, t_0) \\
&= V(x_0, t_0).
\end{aligned}$$

The statements (i) and (ii) follow from these chain of inequalities. ■

Remark 3 *As mentioned earlier, the value function satisfies the terminal condition* $V(x, T) = g(x)$. *In the next section, we will derive a first order pde for* V *when it is smooth.*

3 Derivation of HJB pde when V is C^1

In this subsection, we assume that V is continuously differentiable on $\mathbb{R}^n \times [0, T]$.

Lemma 6 *Assume that $V \in C^1(\mathbb{R}^n \times [0, T))$. For every $(x_0, t_0) \in \mathbb{R}^n \times [0, T)$, we have the following:*

$$(i). \qquad -\frac{\partial V}{\partial t}(x_0, t_0) - DV(x_0, t_0).f(x_0, t_0, u) - \ell(x_0, t_0, u) \leq 0, \tag{9}$$

for $u \in U$.
(ii). If $u^*(\cdot) \in \mathcal{U}[t_0, T]$ is an optimal control and if it is a continuous function, then equality holds in (9) when $u = u^*(t_0)$.

PROOF: (i). By an abuse of notation, we may use u to denote the control input which assumes u at all times in the interval of interest.
Let $\phi(t) = \phi(t; x_0, t_0, u)$. Then by DPP, for every $\delta \in (0, T - t_0)$, we have

$$V(x_0, t_0) \leq \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, u)\ dt + V(\phi(t_0 + \delta), t_0 + \delta).$$

This implies that

$$-[\frac{V(\phi(t_0 + \delta), t_0 + \delta) - V(\phi(t_0), t_0)}{\delta}] - \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, u)\ dt \leq 0.$$

Letting $\delta \to 0+$, we get the required inequality.
(ii). If $u^*(\cdot)$ is an optimal control, then by Theorem 5, equality holds in the above two inequalities with u replaced by $u^*(t)$. We then take limit as $\delta \to 0+$ to get the desired result.
■

The above theorem shows that, at least under some simplifying assumptions, the value function satisfies the pde:

$$-\frac{\partial V}{\partial t} + H(x, t, DV) = 0 \quad \text{in } \mathbb{R}^n \times [0, T), \tag{10}$$

where

$$H(x, t, p) := \sup_{u \in U} [-f(x, t, u).p - \ell(x, t, u)]. \tag{11}$$

Remark 4 *The pde (10) is referred to as the Hamilton-Jacobi-Bellman (HJB) equation. The above theorem says that, if continuously differentiable, the value function satisfies the HJB equation (under some simplifying assumptions). As indicated earlier, the value function also satisfies the terminal time condition*

$$V(\cdot, T) = g(\cdot). \tag{12}$$

We further note that V satisfies (10) and (12) iff W ($W(x,t) := V(x,T-t)$) satisfies

$$\frac{\partial W}{\partial t} + H(x,t,DW) = 0 \quad \textit{in } \mathbb{R}^n \times (0,T] \tag{13}$$

and

$$W(\cdot,0) = g(\cdot). \tag{14}$$

In other words, via an affine transformation in the time variable, one can go from the terminal value problem (10),(12) to the initial value problem (13),(14), and vice versa. For this reason, from now onwards, we mainly consider the (more familiar) initial value problem.

Remark 5 *In general, the initial value problem (13),(14) may not have a continuously differentiable or classical solution. In the next section, we describe a way of defining a weak solution for such pdes. Our final aim is to show that the value function, even if not smooth, is the unique weak solution of the HJB pde.*

4 Viscosity solution of (13)

4.1 Motivation and definition of viscosity solution

Let us add a *viscous term* $\epsilon \Delta W$ on the R.H.S. of (13), and consider the resulting parabolic pde:

$$\frac{\partial W^\epsilon}{\partial t} + H(x,t,DW^\epsilon) = \epsilon \Delta W^\epsilon \quad \text{in } \mathbb{R}^n \times (0,T] \tag{15}$$

with the same initial condition

$$W^\epsilon(\cdot,0) = g(\cdot). \tag{16}$$

This parabolic pde, for every $\epsilon > 0$, has unique classical solution W^ϵ (see [3]). Furthermore, using Arzela-Ascoli theorem, one can show that, for some sequence $\epsilon_j \to 0$, the solutions $W^{\epsilon_j}(x,t)$ converges locally uniformly to a limit function $W \in C(\mathbb{R}^n \times [0,T])$.

Remark 6 *We now try to extract some nice properties of this limit function (which is a type of weak solution of (13)) that can be taken as the definition of the weak solution.*

We start by considering $\eta \in C^\infty(\mathbb{R}^n \times (0,T))$ and a strict loal maximum $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ of $W - \eta$.
This imples that, for all points $(x,t) \neq (x_0,t_0)$ in some closed ball $B_\delta[x_0,t_0]$ of positive radius δ around (x_0,t_0),

$$(W-\eta)(x,t) < (W-\eta)(x_0,t_0).$$

Take $\delta_k = \frac{\delta}{k}; k = 1,2,\cdots$. For each k, as $W^{\epsilon_j} \to W$ uniformly on $B_\delta[x_0,t_0]$, we should have

$$\max_{\partial B_{\delta_k}[x_0,t_0]} (W^{\epsilon_j} - \eta)(x,t) < (W^{\epsilon_j} - \eta)(x_0,t_0), \tag{17}$$

for j large enough. This implies that we can choose a subequence $\{\epsilon_{j_k}\}_{k=1}^\infty$ such that (17) holds for $j = j_k$. Clearly the maximum of $W^{\epsilon_{j_k}} - \eta$ on the closed ball $B_{\delta_k}[x_0,t_0]$ is attained at a

point, say (x_k, t_k) in its interior.
Thus $W^{\epsilon_{j_k}} - \eta$ has local maximum at (x_k, t_k), and $(x_k, t_k) \to (x_0, t_0)$ as $k \to \infty$. In particular, $\frac{\partial W^{\epsilon_{j_k}}}{\partial t}(x_k, t_k) = \frac{\partial \eta}{\partial t}(x_k, t_k)$, $DW^{\epsilon_{j_k}}(x_k, t_k) = D\eta(x_k, t_k)$, and $\Delta W^{\epsilon_{j_k}}(x_k, t_k) \le \Delta\eta(x_k, t_k)$.
Substituting these in (15), we get

$$\frac{\partial \eta}{\partial t}(x_k, t_k) + H(x_k, t_k, D\eta(x_k, t_k)) \le \epsilon_{j_k} \Delta\eta(x_k, t_k).$$

Now letting $k \to \infty$, we obtain

$$\frac{\partial \eta}{\partial t}(x_0, t_0) + H(x_0, t_0, D\eta(x_0, t_0)) \le 0.$$

Note that this inequality is true at all strict local maximum points of $W - \eta$; $\eta \in C^\infty \mathbb{R}^n \times (0, T)$. We leave it as as exercise to verify that this remains true even if we replace 'strict local maximum' with 'local maximum' and $\eta \in C^\infty \mathbb{R}^n \times (0, T)$ with $\eta \in C^1 \mathbb{R}^n \times (0, T)$.

In a similar way, we can show that

$$\frac{\partial \eta}{\partial t}(x_0, t_0) + H(x_0, t_0, D\eta(x_0, t_0)) \ge 0,$$

for every local minimum (x_0, t_0) of $W - \eta$; $\eta \in C^1(\mathbb{R}^n \times (0, T)$.

These two properties of the weak solution will now be taken as its defining properties. Since the term $\epsilon \Delta W$ is a viscous term, this method of obtaining weak solution may be called *vanishing viscosity method* and the resulting weak solution *viscosity solution*.

Definition 2 *(i). We say that $W \in C(\mathbb{R}^n \times [0, T])$ is a viscosity subsolution of (13) if*

$$\frac{\partial \eta}{\partial t}(x_0, t_0) + H(x_0, t_0, D\eta(x_0, t_0)) \le 0,$$

for every $\eta \in C^1(\mathbb{R}^n \times (0, T))$ and local maximum $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ of $W - \eta$.

(ii). We say that $W \in C(\mathbb{R}^n \times [0, T])$ is a viscosity supersolution of (13) if

$$\frac{\partial \eta}{\partial t}(x_0, t_0) + H(x_0, t_0, D\eta(x_0, t_0)) \ge 0,$$

for every $\eta \in C^1(\mathbb{R}^n \times (0, T))$ and local minimum $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ of $W - \eta$.
(iii). We say that $W \in C(\mathbb{R}^n \times [0, T])$ is a viscosity solution of (13) if it is both viscosity sub- and super-solution.

4.2 Alternate definition of viscosity solution

For $W \in C(\mathbb{R}^n \times [0, T])$ and $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, the following sets arise naturally from Definition 2.

$$S^+W(x_0, t_0) \;:=\; \Big\{(p, r) \in \mathbb{R}^n \times \mathbb{R} \;:\; (p, r) = (D\eta(x_0, t_0), \frac{\partial \eta}{\partial t}(x_0, t_0)), W - \eta \text{ has local max.}$$

$$\text{at } (x_0,t_0), \eta \in C^1(\mathbb{R}^n \times (0,T))\Big\} .$$

$$S^- W(x_0,t_0) := \Big\{(p,r) \in \mathbb{R}^n \times \mathbb{R} \ : \ (p,r) = (D\eta(x_0,t_0), \frac{\partial \eta}{\partial t}(x_0,t_0)), W - \eta \text{ has local min.}$$
$$\text{at } (x_0,t_0), \eta \in C^1(\mathbb{R}^n \times (0,T))\Big\} .$$

We refer to [3] for a proof of the fact that

$$S^+W(x_0,t_0) = D^+W(x_0,t_0), \text{ subdifferential of} W \text{ at } (x_0,t_0),$$
$$S^-W(x_0,t_0) = D^-W(x_0,t_0), \text{ superdifferential of} W \text{ at } (x_0,t_0).$$

This implies the following equivalent definition of viscosity solution.

Definition 3 *[Alternate definition of viscosity solution]*
(i). We say that $W \in C(\mathbb{R}^n \times [0,T])$ is a viscosity subsolution of (13) if

$$r + H(x_0,t_0,p) \le 0,$$

for every $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and $(p,r) \in D^+W(x_0,t_0)$.
(ii). We say that $W \in C(\mathbb{R}^n \times [0,T])$ is a viscosity supersolution of (13) if

$$r + H(x_0,t_0,p) \ge 0,$$

for every $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and $(p,r) \in D^-W(x_0,t_0)$.
(iii). We say that $W \in C(\mathbb{R}^n \times [0,T])$ is a viscosity solution of (13) if it is both viscosity sub- and super-solution.

The proof of the next result may also be found in [3].

Lemma 7 *For $W \in C(\mathbb{R}^n \times [0,T])$ and $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$, the following are equivalent.*
(a). W is differentiable at (x_0,t_0).
(b). $D^+W(x_0,t_0)$ and $D^-W(x_0,t_0)$ are both nonempty sets.
If (a) or (b) holds, then $D^+W(x_0,t_0) = D^-W(x_0,t_0) = \{(DW(x_0,t_0), \frac{\partial W}{\partial t}(x_0,t_0))\}$.

This lemma clearly shows that if a viscosity solution is differential at a point, then the pde is satisfied at this point in the usual sense. We state this as a theorem.

Theorem 6 *Let $W \in C(\mathbb{R}^n \times [0,T])$ a viscosity solution of (13). Then the equation (13) is satisfied in the usual sense at all points $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ at which W is differentiable.*

4.3 Comparison and uniqueness of viscosity solutions

We consider the pde (**??**) where the Hamiltonian H is given by (11). Note that H satisfies

Theorem 7 *Let $W_1, W_2 \in UC(\mathbb{R}^n \times [0,T])$ be respectively sub- and supersolutions of (13). Then*

$$\sup_{\mathbb{R}^n \times [0,T]} (W_1 - W_2) = \sup_{\mathbb{R}^n \times \{0\}} (W_1 - W_2). \tag{18}$$

PROOF: Since W_1, W_2 are uniformly continuous, there exists a positive constant C such that

$$|W_i(x,t) - W_i(y,s)| \leq C(1 + |x-y|), \forall (x,t),(y,s) \in Q := \mathbb{R}^n \times [0,T]; i = 1,2. \tag{19}$$

Let A denote the R.H.S. of (18). Without loss of generality, we may assume that $A < \infty$.

The proof is by reductio-ad-absurdum. If possible, let (18) be not true. Then $\exists \sigma_0 > 0$ such that

$$\sup_Q (W_1 - W_2) > A + \sigma_0. \tag{20}$$

For $\epsilon, \alpha \in (0,1)$, consider the auxiliary function $\Psi : Q^2 \to \mathbb{R}$ defined by

$$\Phi(x,t,y,s) = W_1(x,t) - W_2(y,s) - \frac{1}{\epsilon}|x-y|^2 - \frac{1}{\alpha}|t-s|^2.$$

Using (19), we can conclude that $\sup \Phi < \infty$. This also implies that, for $\delta \in (0, \sigma_0/2)$, there exists $(x_0,t_0),(y_0,s_0) \in Q$ such that

$$\Phi(x_0,t_0,y_0,s_0) + \delta > \sup \Phi. \tag{21}$$

Now consider $\Psi : Q^2 \to \mathbb{R}$ defined by

$$\Psi(x,t,y,s) = \Phi(x,t,y,s) + \delta\zeta(x,y) - \sigma t,$$

where $\sigma \in (0, \frac{\sigma_0}{2T})$, and $\zeta \in C_c^\infty(\mathbb{R}^{2n})$ with $\zeta(x_0,y_0) = 1$, $0 \leq \zeta \leq 1$, $|D\zeta| \leq 1$.
It is left as an exercise to the reader to show that Ψ attains its maximum at a point, say $(\bar{x},\bar{t},\bar{y},\bar{s})$, in Q^2.

Clearly

$$\Psi(\bar{x},\bar{t},\bar{x},\bar{t}) + \Psi(\bar{y},\bar{s},\bar{y},\bar{s}) \leq 2\Psi(\bar{x},\bar{t},\bar{y},\bar{s}).$$

This implies that

$$\begin{aligned} \frac{1}{\epsilon}|\bar{x}-\bar{y}|^2 + \frac{1}{\alpha}|\bar{t}-\bar{s}|^2 &\leq C(1+|\bar{x}-\bar{y}|) + 4\delta + \sigma(\bar{s}-\bar{t}) \\ &\leq C(1+|\bar{x}-\bar{y}|) + 2\sigma_0, \quad \text{when } \delta \leq \frac{\sigma_0}{4}. \end{aligned}$$

This implies that there exists constants C_ϵ and C_α (with $\lim_{\epsilon\to 0+} C_\epsilon = 0 = \lim_{\alpha \to 0+} C_\alpha$) which are respectively upper bounds for $|\bar{x}-\bar{y}|$ and $|\bar{t}-\bar{s}|$.

We can exploit this fact to establish that $(\bar{x},\bar{t},\bar{y},\bar{s})$ belongs to the interior of Q^2 when ϵ and α are small enough (see p.58 in [3]).

Now, by using the definition of subsoilution and super solution, we get the two inequalities:

$$\begin{aligned} \sigma + \frac{2}{\alpha}(\bar{t}-\bar{s}) + H(\bar{x},\bar{t},\frac{2}{\epsilon}(\bar{x}-\bar{y}) - \delta D_x\zeta(\bar{x},\bar{y})) &\leq 0, \\ \frac{2}{\alpha}(\bar{t}-\bar{s}) + H(\bar{y},\bar{s},\frac{2}{\epsilon}(\bar{x}-\bar{y}) - \delta D_y\zeta(\bar{x},\bar{y})) &\geq 0. \end{aligned}$$

Subtracting, we get

$$\sigma \leq H(\bar{y}, \bar{s}, \frac{2}{\epsilon}(\bar{x} - \bar{y}) - \delta D_y \zeta(\bar{x}, \bar{y})) - H(\bar{x}, \bar{t}, \frac{2}{\epsilon}(\bar{x} - \bar{y}) - \delta D_x \zeta(\bar{x}, \bar{y})).$$

By the definition of H it is clear that the R.H.S tends to zero as ϵ and α tend to zero. This yields the required contractiction.

■

5 Characterization of V

Theorem 8 *The value function V is the unique viscosity solution (in $UC(\mathbb{R}^n \times [0, T])$) of (10) satisfying the terminal condition $V(\cdot, T) = g(\cdot)$.*

PROOF: We first show that V is a sub solution of (10). To this end, let $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ be a local maximum of $V - \eta$; $\eta \in C^1(\mathbb{R}^n \times [0, T])$.
Using DPP, as in Lemma 6, we can show that for every $u \in U$,

$$V(x_0, t_0) \leq \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, u)\ dt + V(\phi(t_0 + \delta), t_0 + \delta).$$

This implies that

$$-[\frac{V(\phi(t_0 + \delta), t_0 + \delta) - V(\phi(t_0), t_0)}{\delta}] - \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, u)\ dt \leq 0.$$

Since $V - \eta$ has local maximum at (x_0, t_0), for δ small enough, it follows that

$$-[\frac{\eta(\phi(t_0 + \delta), t_0 + \delta) - \eta(\phi(t_0), t_0)}{\delta}] - \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \ell(\phi(t), t, u)\ dt \leq 0.$$

Letting $\delta \to 0+$, we get

$$-\frac{\partial \eta}{\partial t}(x_0, t_0) - f(x_0, t_0, u).D\eta(x_0, t_0) - \ell(x_0, t_0, u) \leq 0.$$

Taking sup over $u \in U$, we obtain

$$-\frac{\partial \eta}{\partial t}(x_0, t_0) + H(x_0, t_0, D\eta(x_0, t_0)) \leq 0.$$

This proves that V is a viscosity sub solution of (10).
To show that V is a super solution, let $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ be a local minimum of $V - \eta$; $\eta \in C^1(\mathbb{R}^n \times [0, T])$.
Let $\epsilon > 0$ be given. Using DPP, for each δ small, there exists $u^\delta(\cdot) \in \mathcal{U}[t_0, t_0 + \delta]$ such that

$$V(x_0, t_0) \geq \int_{t_0}^{t_0+\delta} \ell(\phi^\delta(t), t, u^\delta(t))\ dt + V(\phi^\delta(t_0 + \delta), t_0 + \delta) - \epsilon\delta,$$

where $\phi^\delta(t) = \phi(t; x_0, t_0, u^\delta(\cdot))$. As in the first part of the proof, since $V - \eta$ has local minimum at (x_0, t_0), for δ small enough, it follows that

$$-[\frac{\eta(\phi^\delta(t_0+\delta), t_0+\delta) - \eta(\phi^\delta(t_0), t_0)}{\delta}] - \frac{1}{\delta}\int_{t_0}^{t_0+\delta} \ell(\phi^\delta(t), t, u^\delta(t))\ dt \geq -\epsilon.$$

It may be noted now that the difference between the L.H.S. of this inequality and the expression

$$-\frac{\partial\eta}{\partial t}(x_0, t_0) - \frac{1}{\delta}\int_{t_0}^{t_0+\delta} [f(x_0, t_0, u^\delta(t)).D\eta(x_0, t_0) + \ell(x_0, t_0, u^\delta(t))]\ dt$$

is $o(1)$ as $\delta \to 0+$. Hence

$$-\frac{\partial\eta}{\partial t}(x_0, t_0) - \frac{1}{\delta}\int_{t_0}^{t_0+\delta} [f(x_0, t_0, u^\delta(t)).D\eta(x_0, t_0) + \ell(x_0, t_0, u^\delta(t))]\ dt \geq -\epsilon + o(1).$$

From this, we get

$$-\frac{\partial\eta}{\partial t}(x_0, t_0) + H(x_0, t_0, D\eta(x_0, t_0)) \geq -\epsilon + o(1).$$

This shows that V is a super solution (let $\delta \to 0+$ and note that ϵ is an arbitrary positive number).

It remains to show the uniqueness. Let $\hat{V} \in UC(\mathbb{R}^n \times [0, T])$ be another viscosity solution of (10) satisfying $\hat{V}(\cdot, T) = g(\cdot)$. Then, by a comparison result analogous to Theorem 7, we get

$$\sup_{\mathbb{R}^n \times [0,T]} (V - \hat{V}) = 0 = \sup_{\mathbb{R}^n \times [0,T]} (\hat{V} - V).$$

Therefore $V = \hat{V}$. ∎

References

[1] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhauser, 2008.

[2] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, 1984.

[3] L. C. Evans, Partial Differential Equations, 2nd edn, AMS, 2010.

[4] R. E. Kalman, Contributions to the theory of optimal control, Bol. Soc. Matem. Max., pp.102-119, 1960.