Finite size effects and finite size scaling

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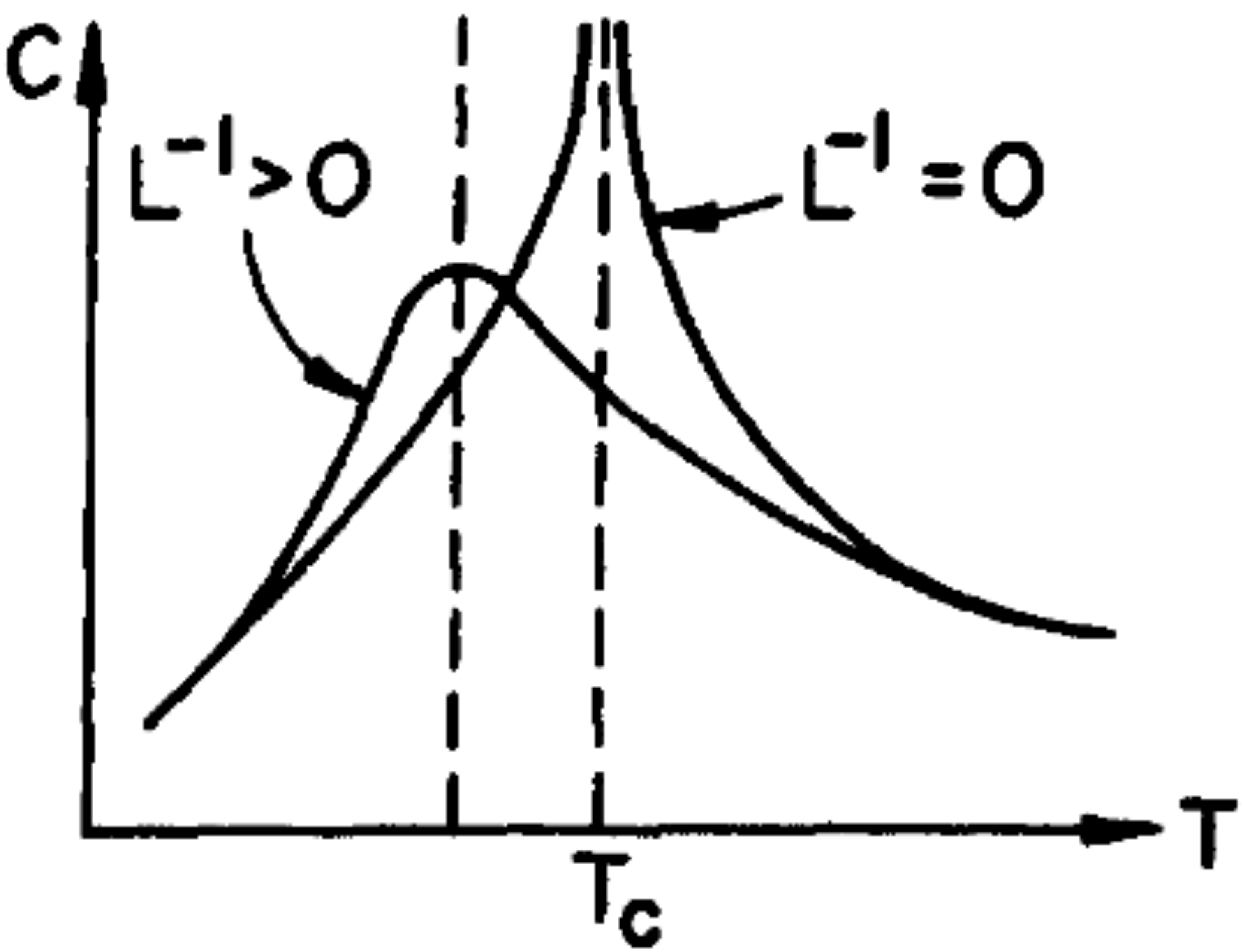
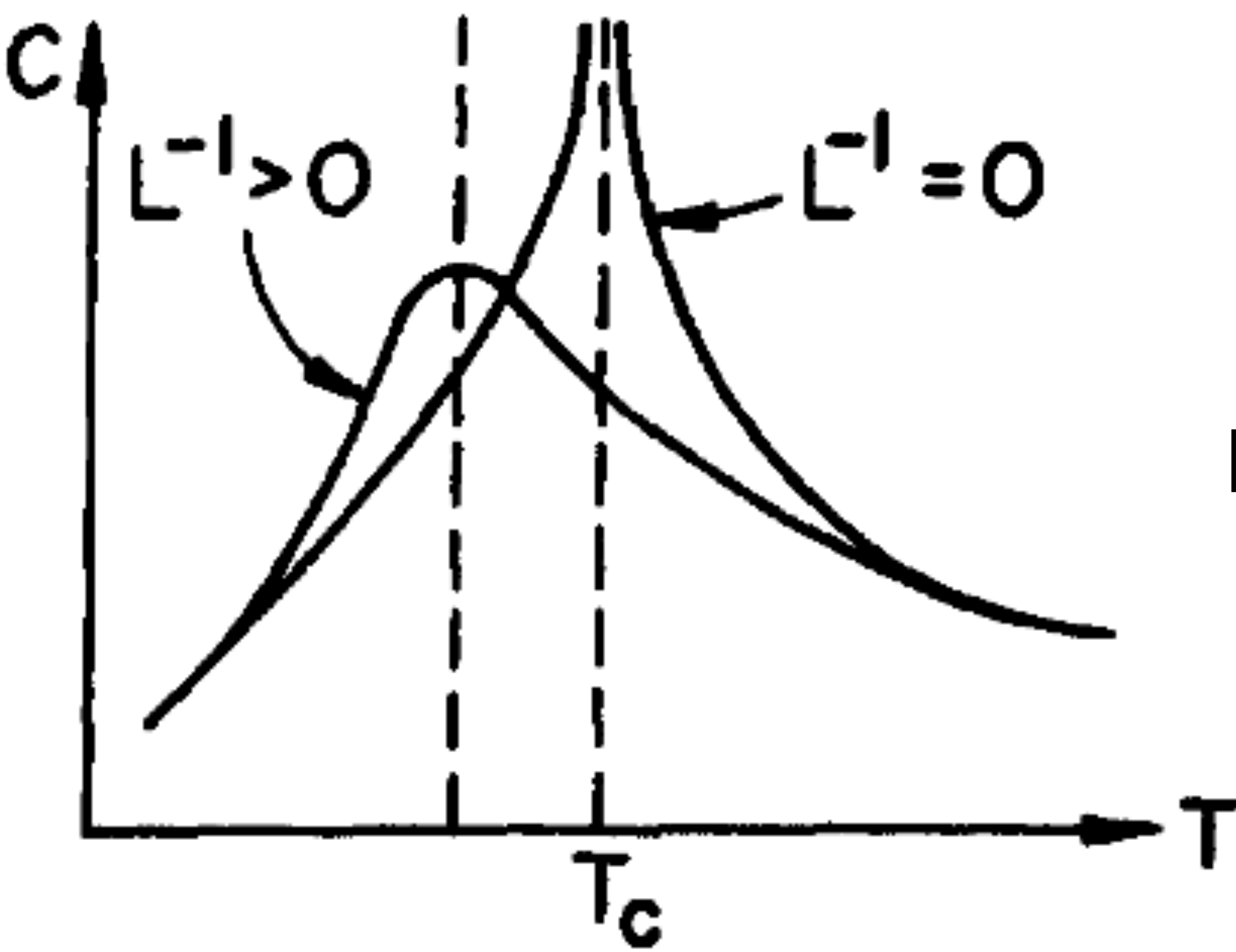


Critical point *(also hidden couplings)*

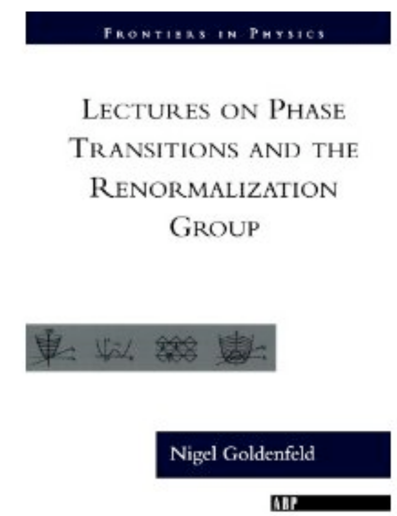
Finite size effects and finite size scaling



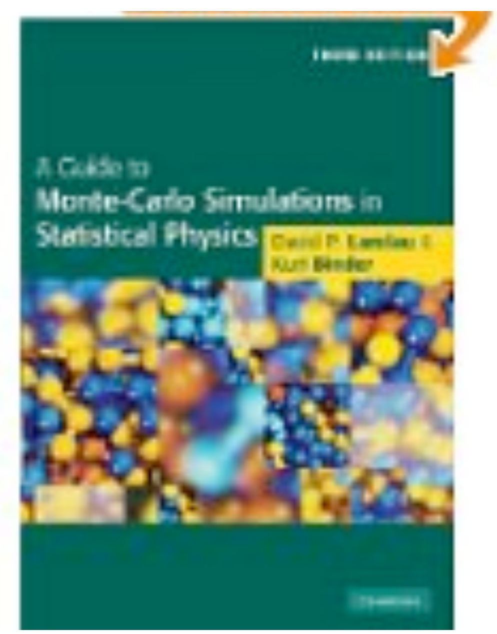
Critical point

(also hidden couplings)

N. Goldenfeld



D. P. Landau and K. Binder



Scaling

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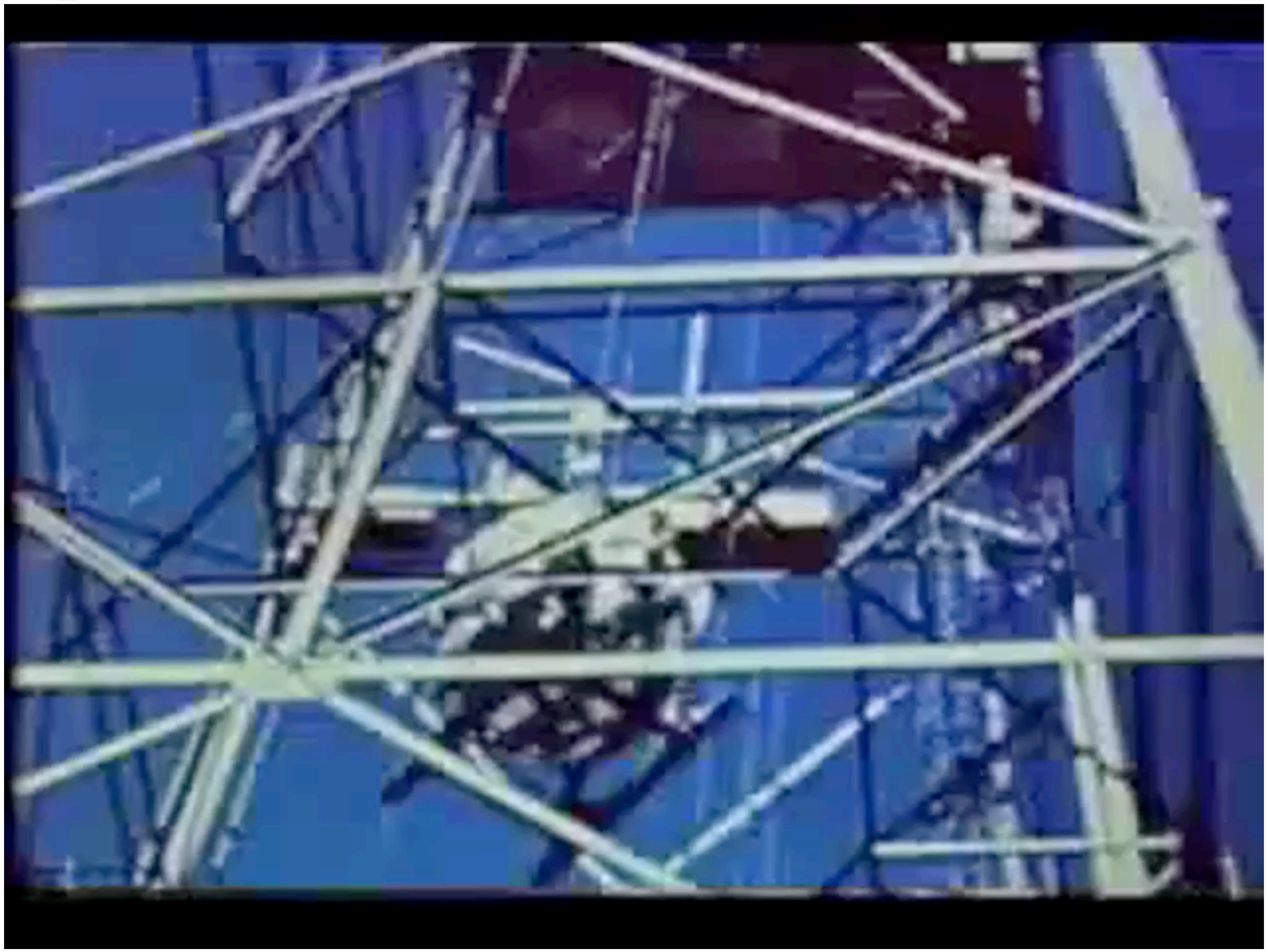

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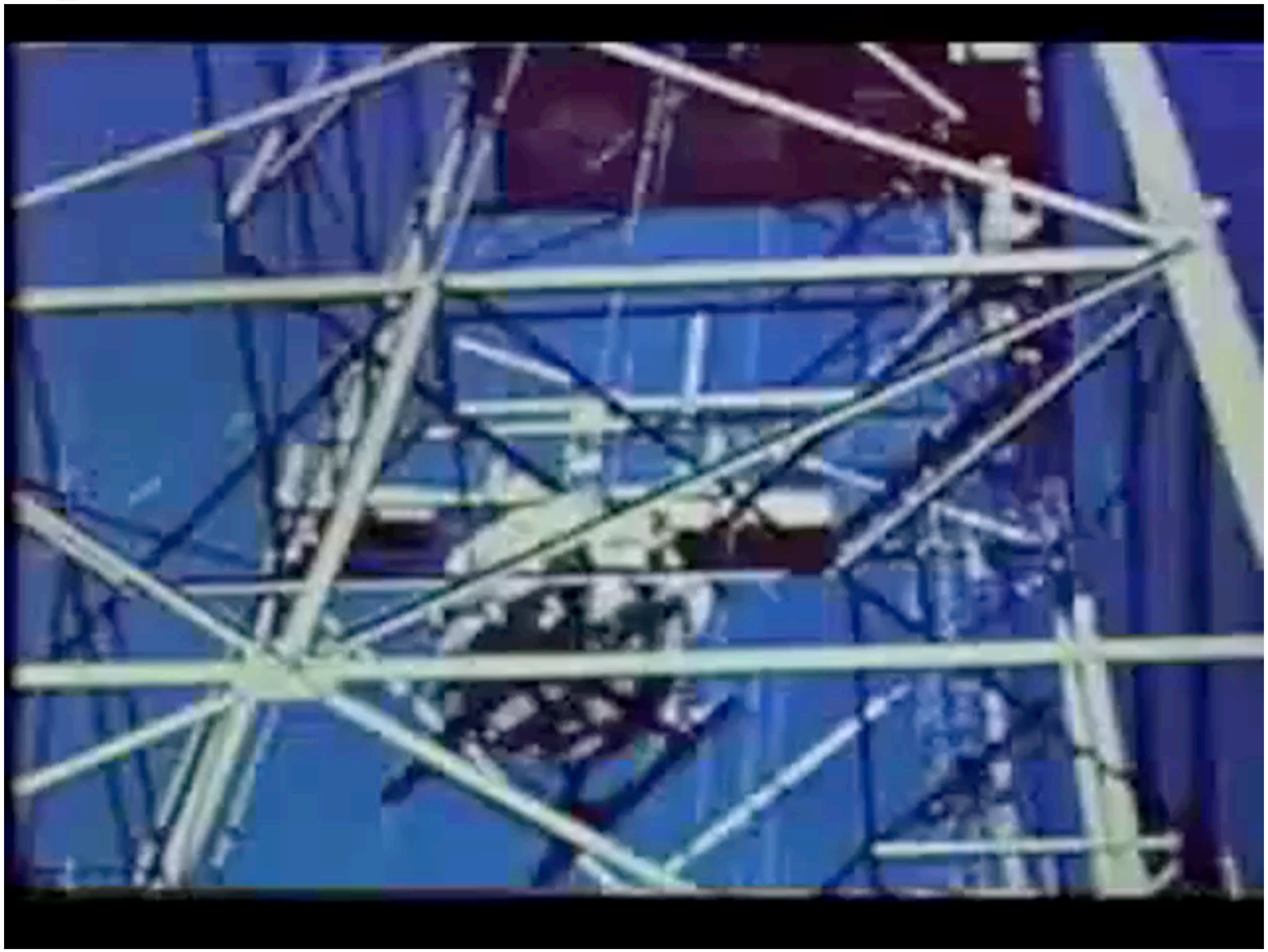
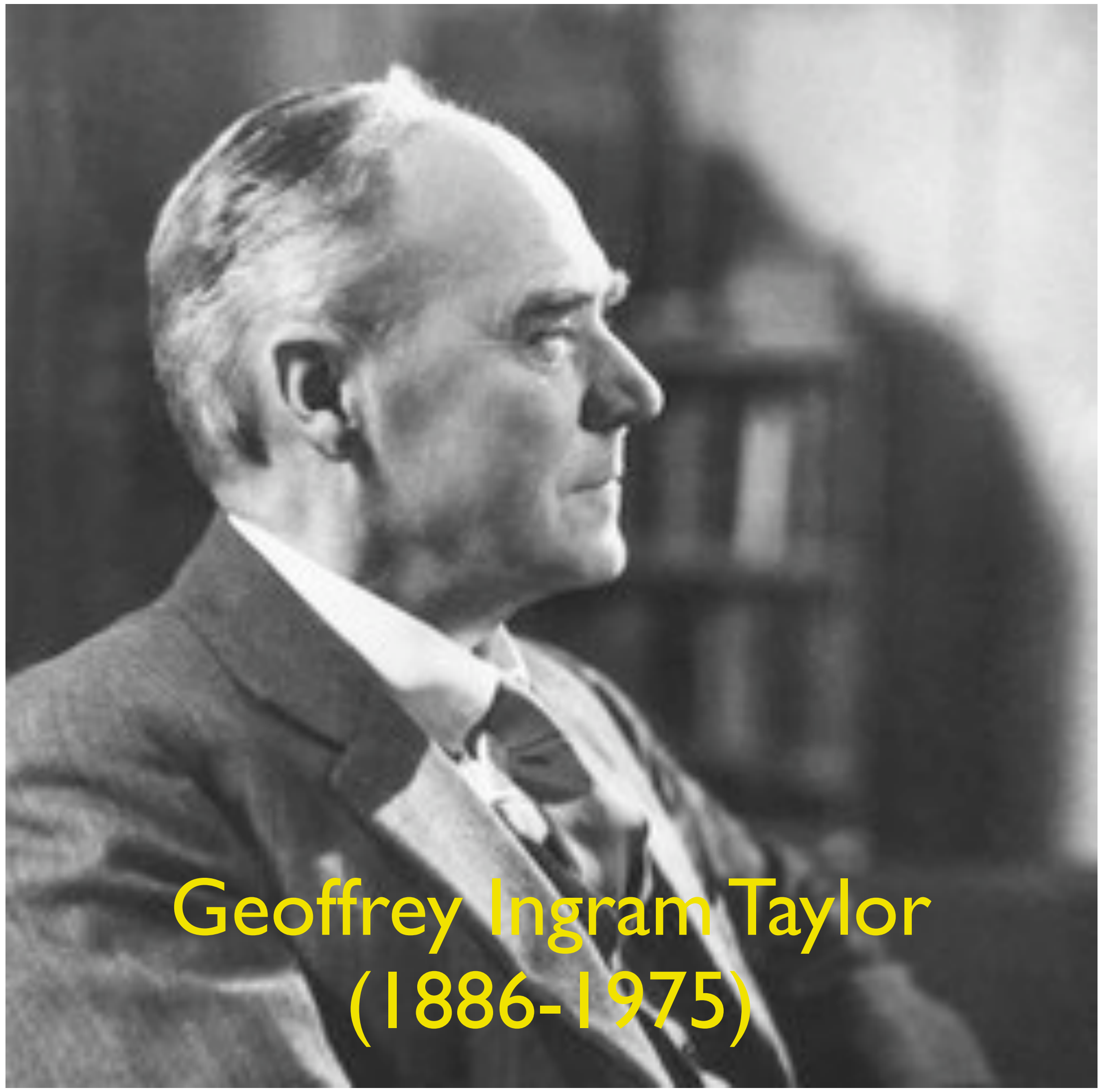


Trinity was the code name of the first nuclear weapons test of an atomic bomb. This test was conducted by the United States Army on July 16, 1945,[4][5][6][7] at a location about 35 miles (56 km) southeast of Socorro, New Mexico, at the White Sands Proving Ground, now the White Sands Missile Range.[

Scaling



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$$x_1 = f(x_2, x_3 ....., x_n)$$

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Table 1.1 RADIUS R OF BLAST WAVE AFTER TIME T

T/msec	R/m
0.10	11.1
0.24	19.9
0.38	25.4
0.52	28.8
0.66	31.9
0.80	34.2
0.94	36.3
1.08	38.9
1.22	41.0
1.36	42.8
1.50	44.4
1.65	46.0
1.79	46.9
1.93	48.7
3.26	59.0
3.53	61.1
3.80	62.9
4.07	64.3
4.34	65.6
4.61	67.3
15.0	106.5
25.0	130.0
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Actual yield : 18 kt Taylor estimate : 22 kt !!

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$$\lim_{\lambda \to \infty} f = f(0) = \text{const.}$$

for large wavelengths the speed of the tsunami must be independent of wavelength

$$x_2 = \frac{h}{\lambda}$$

$$\therefore c \sim \sqrt{h}$$

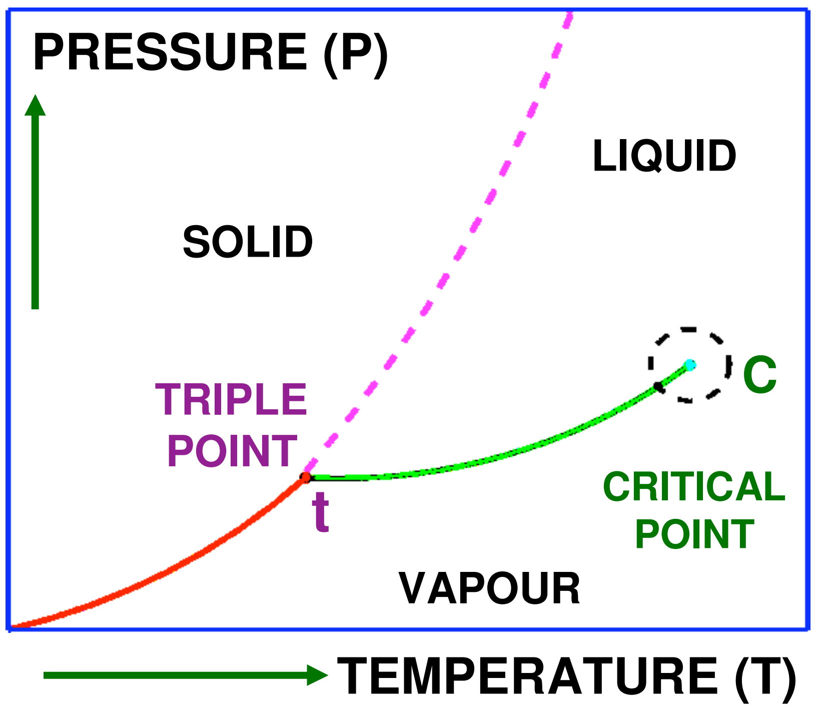
$$c^2 = \frac{g\lambda}{2\pi} \tanh\Big(\frac{2\pi h}{\lambda}\Big)$$

Thermodynamic Equilibrium:

N_1, V_1, E_1 T_1, P_1, μ_1	N_2, V_2, E_2 T_2, P_2, μ_2

$$T_1 = T_2$$
$$P_1 = P_2$$
$$\mu_1 = \mu_2$$

Thermodynamics of Phase Transition:

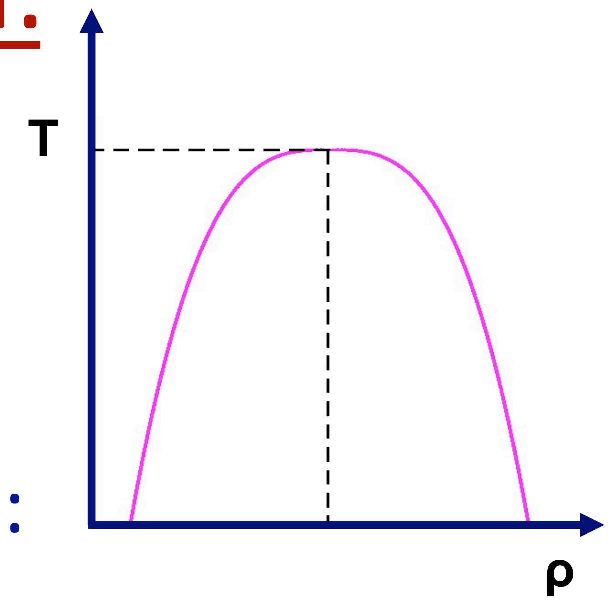


$(P_t, T_t) \rightarrow$ Triple point

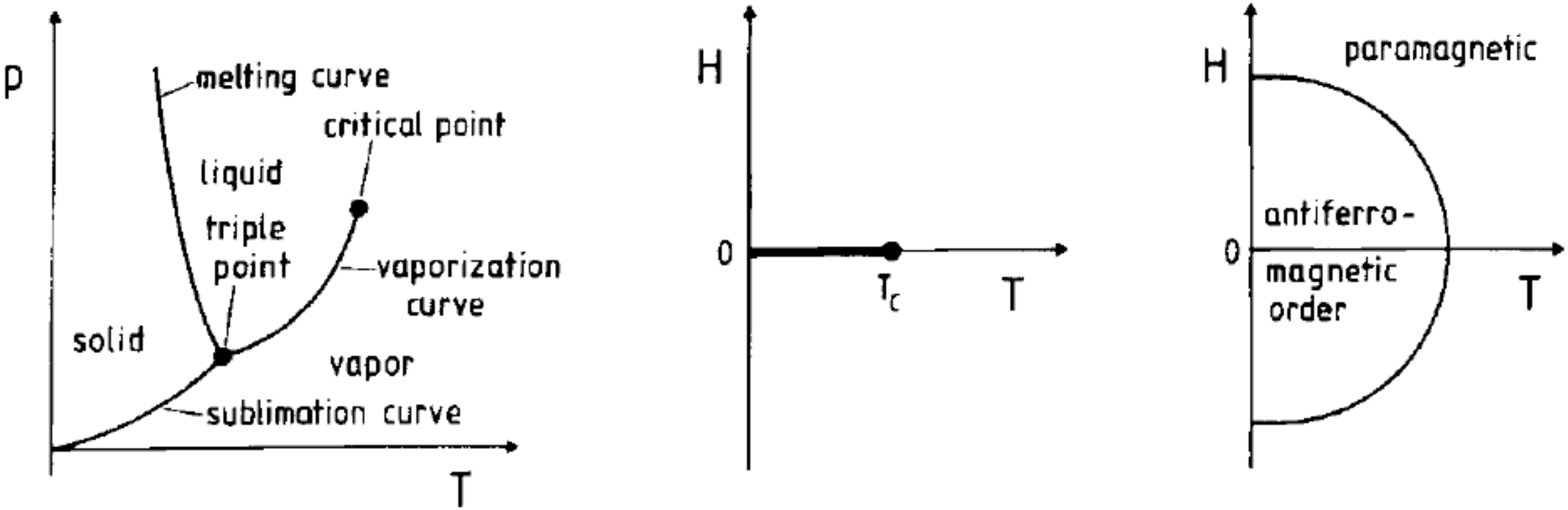
$(P_c, T_c) \rightarrow$ Critical point

Crossing Co-existence lines:
1st order phase transition

$C \rightarrow$ 2nd order phase transition



Magnetization	Specific Heat	Susceptibility	Correlation Length
$m \sim t^{\beta}$	$C \sim t^{-\alpha}$	$\chi \sim t^{-\gamma}$	$\xi \sim t^{-\nu}$



$$m = \frac{1}{N}\sum S_i \qquad \Gamma(r) = \langle S(r)S(0)\rangle - \langle S(r)\rangle\langle S(0)\rangle$$

$$\Gamma(r) \propto r^{-(d-1)/2}\exp(-r/\xi),$$

$$\varepsilon = |1 - T/T_c|$$

$$\alpha, \beta, \gamma, \delta, \nu, \eta$$

$$m = m_0\varepsilon^{\beta},$$

$$\chi = \chi_0\varepsilon^{-\gamma},$$

$$C = C_0\varepsilon^{-\alpha}, \quad \Gamma(r) = \Gamma_0 r^{-(d-2+\eta)}, \qquad r \to \infty,$$

$$\xi = \xi_0\varepsilon^{-\nu},$$

$$m = DH^{1/\delta},$$

critical exponents are not all independent

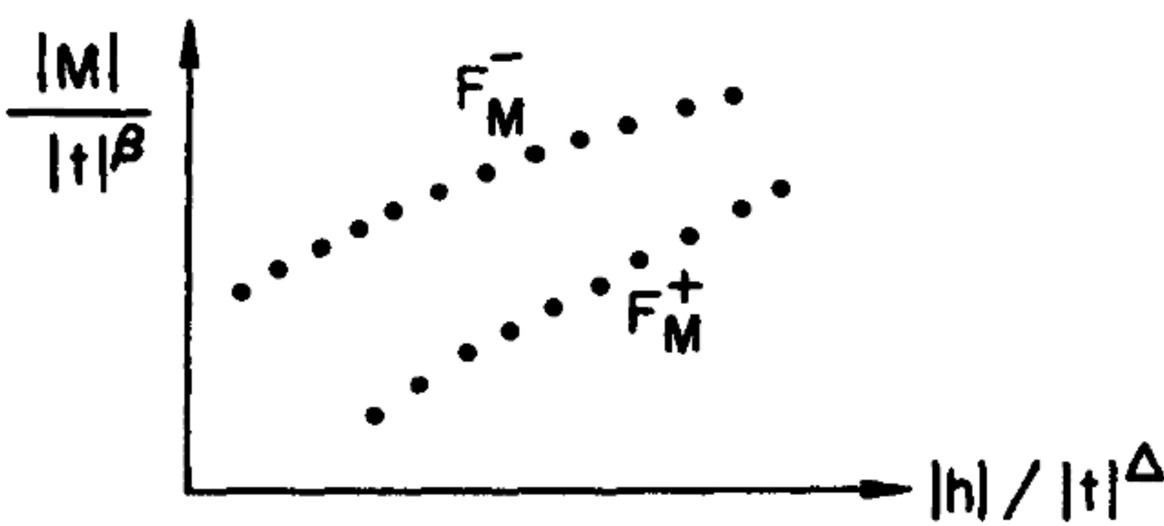
Scaling hypothesis

$$
\begin{aligned}
m &= m_0 \left| \frac{T_c - T}{T_c} \right|^{\beta} \\
m &= D h^{\frac{1}{\delta}}
\end{aligned}
$$

Can be combined into a single formula:

$$m(t,h) = (\pm t)^{\beta} F^{\pm}\left(\frac{h}{t^{\Delta}}\right)$$

$$t = (T/T_c - 1); h = H/k_B T$$



$$
\begin{aligned}
\chi_T(h=0) &= \frac{1}{k_B T}\frac{\partial m}{\partial h}|_{h=0} \\
&\sim |t|^{\beta - \Delta}(F'_M)^{\pm}(0) \\
\Rightarrow \Delta &= \beta + \gamma
\end{aligned}
$$

scaling relation

More scaling relations :

what happens on the critical isotherm? $t \to 0$

$$F_M(x) \sim x^\lambda$$

$$m(0,h) \sim |t|^\beta \left(\frac{h}{|t|^\Delta}\right)^\lambda \sim |t|^{\beta-\Delta\lambda} h^\lambda$$

since we know that the magnetization remains finite for non-zero field, $\beta = \Delta\lambda; \lambda = 1/\delta$

$$\boxed{\beta\delta = \beta + \gamma}$$

Scaling for the free energy

$$f_s(t,h) = t^{2-\alpha} F_f\left(\frac{h}{t^\Delta}\right)$$

can show similarly that (how?) $\boxed{\alpha + 2\beta + \gamma = 2}$

Scaling for the correlation function

$$\Gamma(k) = l^{-2-a}\xi^a\Big(g(k\xi)\Big[1 + A\Big(\frac{l}{\xi}\Big)^\sigma + \cdots\Big]\Big)$$

l is a microscopic length, at the c.p. $\xi \to \infty$.

If $g(x) \sim x^{-a}$ then ξ dependence will cancel out. $\therefore a = 2 - \eta$

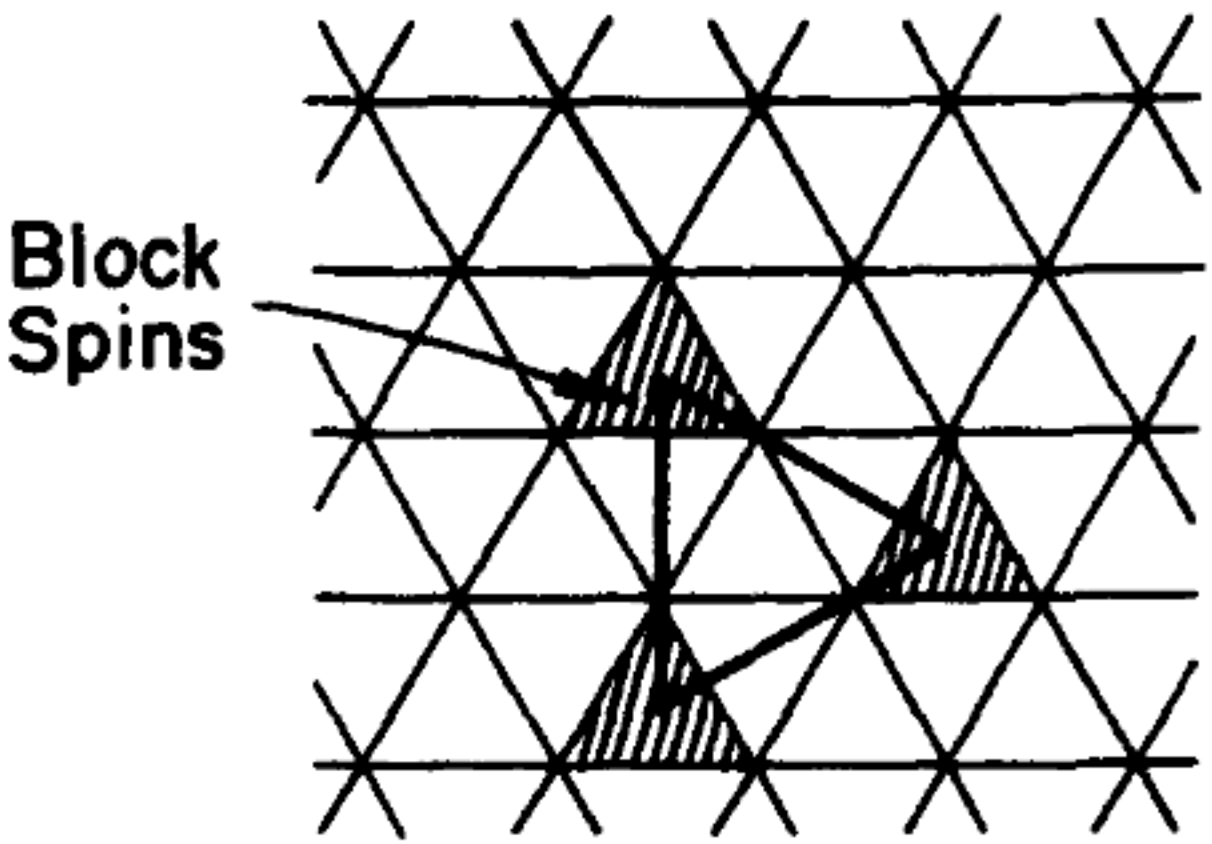
but.. $\Gamma(k = 0) \propto \chi_T$

$$\therefore \frac{\gamma}{\nu} = 2 - \alpha$$

But why do quantities scale?

Block transformation:

$$\beta H = -K \sum_{\langle ij \rangle} S_i S_j - h \sum_{i=1}^{N} S_i$$



$l = 3$

$$S_I = \frac{1}{|\bar{m}_l|} \frac{1}{l^d} \sum_{i \in I} S_i$$

$$\bar{m}_l = \frac{1}{l^d} \sum_{i \in I} \langle S_i \rangle$$

$$\langle S_I \rangle = \pm 1$$

$$-\beta H_l = K_l \sum_{\langle IJ \rangle}^{N l^{-d}} S_I S_J + h_l \sum_{I=1}^{N l^{-d}}$$

effective correlation length: $\xi_l = \frac{\xi_1}{l}$

implies *effective* temperature: $t_l > t$

effective field: $h_l = h\bar{m}_l l^d$

free energy density : $f_s(t_l, h_l) = l^d f_s(t, h)$

Now IF
$$\begin{aligned} t_l &= t l^{y_t} \\ h_l &= h l^{y_h} \end{aligned}$$

THEN can prove

$$f_s(t, h) = t^{2-\alpha} F_f\left(\frac{h}{t^\Delta}\right)$$

$$f_s(t, h) = l^{-d} f_s(t l^{y_t}, h l^{y_h})$$

l is arbitrary so free to choose $l = t^{-1/y_t}$

$$\begin{aligned} \Delta &= \frac{y_h}{y_t} \\ 2 - \alpha &= \frac{d}{y_t} \end{aligned}$$

All scaling ultimately related to the behavior of the correlation length vs. microscopic scale

Finite size scaling

Think of L^{-1} as another field

$$f_s([K], L^{-1}) = l^{-d} f_s([K'], lL^{-1})$$

$$L_l^{-1} = L^{-1} l^{y_L}; \Rightarrow y_L = 1$$

at $h = 0$ we get, $f_s(t, L^{-1}) = |t|^{2-\alpha} f_s(L^{-1}|t|^{-y_L/y_t})$

$$f_s(t, L^{-1}) = |t|^{2-\alpha} f_s\left(\frac{\xi_\infty}{L}\right)$$

True behavior recovered only in the limit $L \to \infty$ i.e. $f_s(0)$

almost there ...

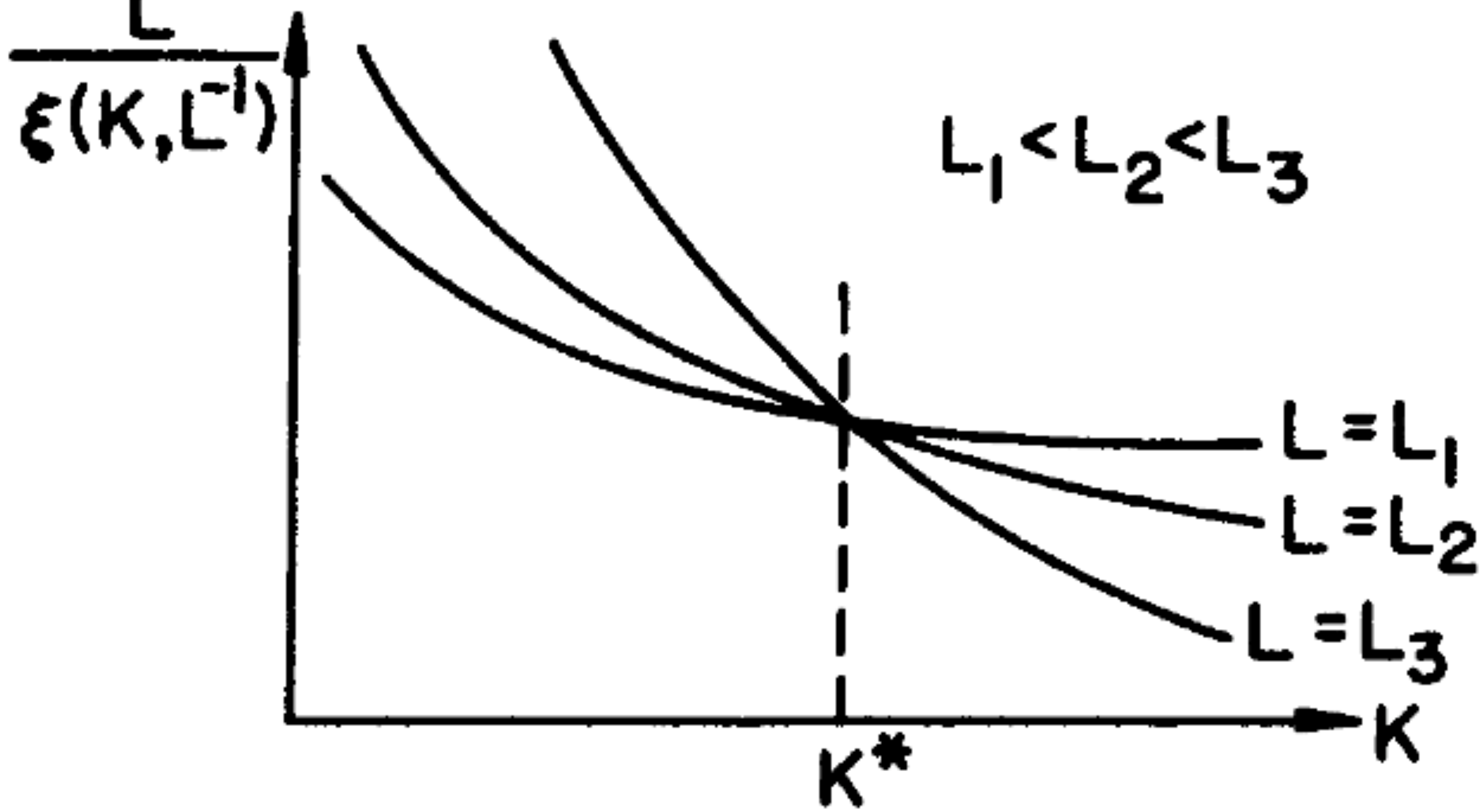
$$
\begin{aligned}
c(t, L^{-1}) &\propto \frac{\partial^2 f_s(t, L^{-1})}{\partial t^2} \\
&= |t|^{-\alpha} f_s(L^{-1} t^{-\nu}) \\
&= L^{\alpha/\nu} D(t L^{1/\nu})
\end{aligned}
$$

Remember if $f(x)$ is a scaling function, so is $x^{-\alpha/\nu} f(x)$!!

Finally,

$$
\begin{aligned}
\therefore \text{ peak position shift} &\propto L^{-1/\nu} \\
\text{peak height shift} &\propto L^{\alpha/\nu}
\end{aligned}
$$

We can do even better !!



Can show that : $\xi = LF(Lt^{\nu})$

$$\lim_{x\to\infty} F(x) \sim 1/x$$

$$\text{because at fixed } t \text{ as } L \to \infty, \xi \sim t^{-\nu}$$

On the other hand, if system size is finite there is no singularity even at $t = 0$

$$\frac{\xi}{L} = A + BtL^{1/\nu} + ...$$

Remarks

- It is often difficult to measure correlation functions directly, in that case one uses the Binder cumulant which has a similar behavior.

$$U_4 = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle}^{2}$$

- For first order transitions correlation lengths do not diverge so the finite size effects are weaker but increase with proximity of critical points.