Random structures: Phase transitions, scaling limits, and universality

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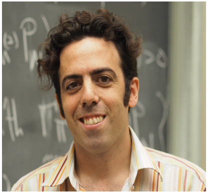
Based on joint work with



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INRIA

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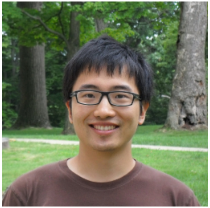
Stanford University

Remco van der Hofstad
TU Eindhoven



Xuan Wang
Georgia Tech/Databricks

Outline of talk

- Phase transition in random graphs and intrinsic geometry of components:
  - light-tailed degree distribution
  - heavy-tailed degree distribution
- Vacant set left by random walks on random graphs
- Minimal spanning tree on supercritical discrete structures (strong disorder regime)
- Spatial minimal spanning trees

Classical example: Erdős-Rényi graphs

- The study of random graphs began with the work of Paul Erdős and Alfred Rényi who introduced (in 1959) and studied several properties of a random graph process which we now call the Erdős-Rényi random graph (ERRG) process.
- Extensive study by Bollobás, Janson, Knuth, Pittel, Ruciński, Łuczak, Wierman until the 90's. A huge literature on ERRGs grew purely out of mathematical interest.

- n vertices and $0 < p = p(n) < 1$.
- Place edges independently with probability p between each pair of vertices.
- Call the resulting graph $G(n, p)$.

- $p = c/n$, c fixed.
- $|\mathcal{C}_1^{(n)}| =$ size of largest component of $G(n, p)$.
- $|\mathcal{C}_1^{(n)}| = \Theta_P(n)$ when $c > 1$.
- $|\mathcal{C}_1^{(n)}| = \Theta_P(\log n)$ when $c < 1$.
- Phase transition at $c = 1$. **Criticality.**

Late 90's onwards

In late 90's, physicists and computer scientists noticed that many real-life networks (e.g. social and biological networks, scientist collaboration), **viewed as a graph**, share fascinating features. For example,

Phase transition: Criticality and emergence of the giant

- One can associate a parameter t (often related to **edge density**) and a "critical time" t_c to these graphs.
- We can do edge percolation (i.e., remove edges independently) to change the edge density and hence the parameter t.
- When $t < t_c$ no giant component ($|\mathcal{C}_1^{(n)}(t)| = o_P(n)$).
- When $t > t_c$ then $|\mathcal{C}_1^{(n)}(t)| \sim f(t)n$. **Giant component.**
- Criticality at $t = t_c$.

Late 90's onwards

In late 90's, physicists and computer scientists noticed that many real-life networks (e.g. social and biological networks, scientist collaboration), **viewed as a graph**, share fascinating features. For example,

- Existence of **hubs** or **vertices having very high-degree**.

  This cannot be explained by the classical ERRG model.

- **Scale-free property:** Empirical degree distribution behaves like a power law asymptotically:

$$\mathbb{P}(D \geq k) \sim k^{-\tau+1},$$

  where D is the degree of a uniformly chosen vertex.

  The constant τ is called the degree exponent.

  Degree distribution for ERRG has exponentially decaying tail ($\tau = \infty$).

Random graphs as models for complex networks

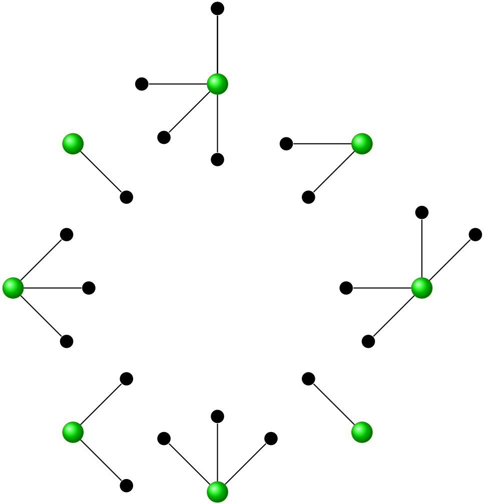
- Real-world networks can be modeled as random graphs and these properties can be studied rigorously.
- Earlier work by Barabasi, Albert, Newman, Watts, Stanley ...
- Many random graph models have been proposed by probabilists, physicists, combinatorialist, and computer scientists for the purpose of modeling and studying real-world networks.

Model I: Configuration model

Model I: Configuration model (Bollobás; Molloy and Reed)

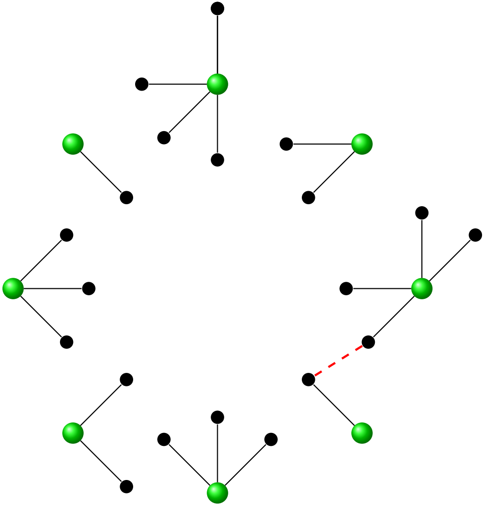
- Given a degree sequence $\boldsymbol{d} := (d_1, d_2, \ldots, d_n)$. Assume $\sum d_i$ is even.
- Uniformly pair half-edges to form full edges.

Given degree sequence $\boldsymbol{d}$



Configuration Model: the construction

Choose a half-edge $\longrightarrow$ Pair uniformly with another



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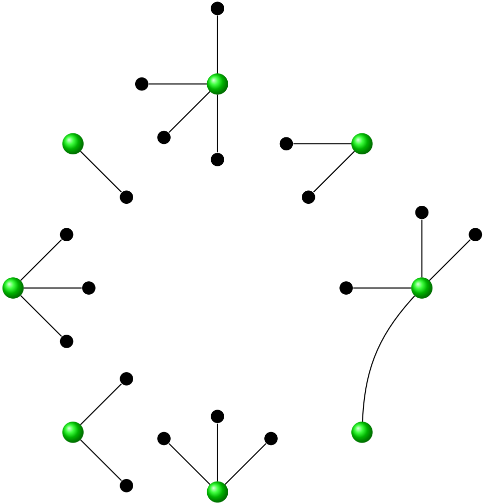

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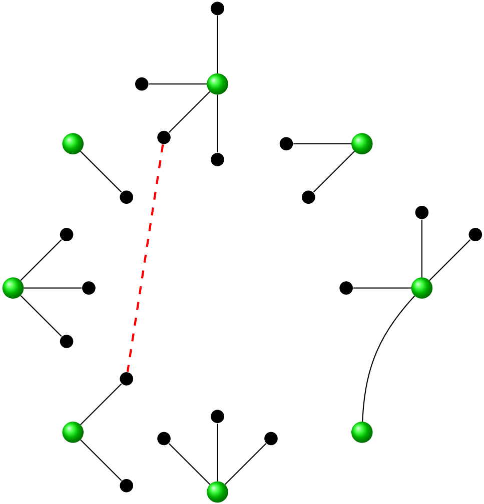

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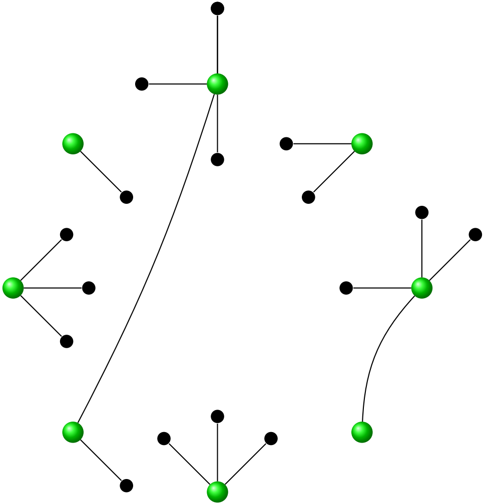

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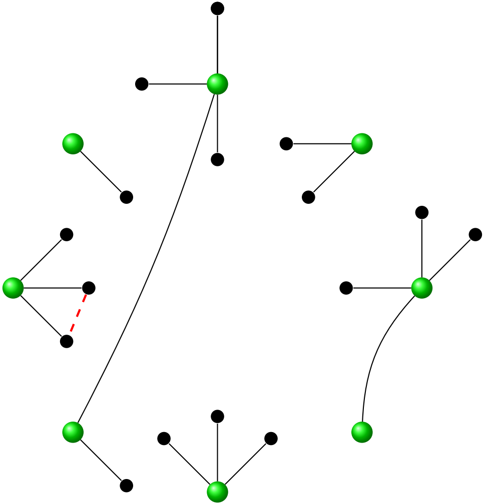

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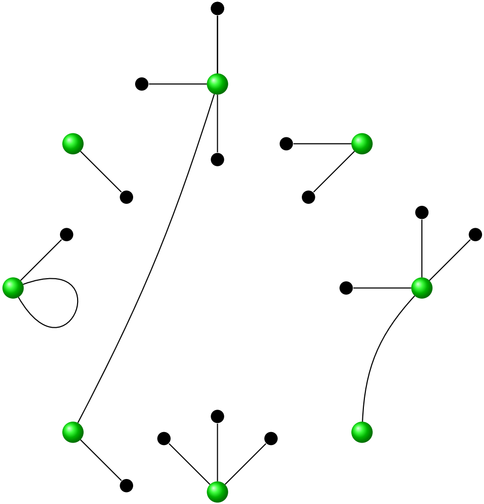

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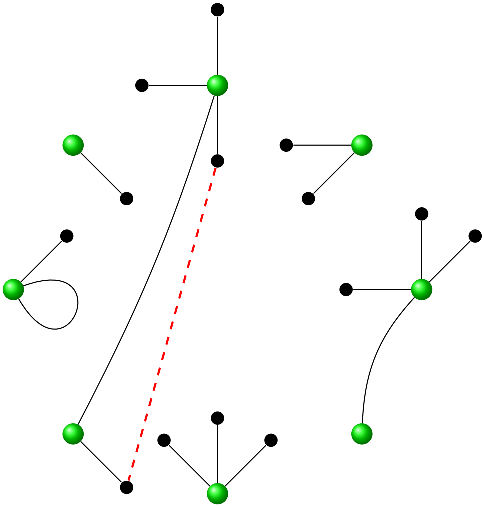

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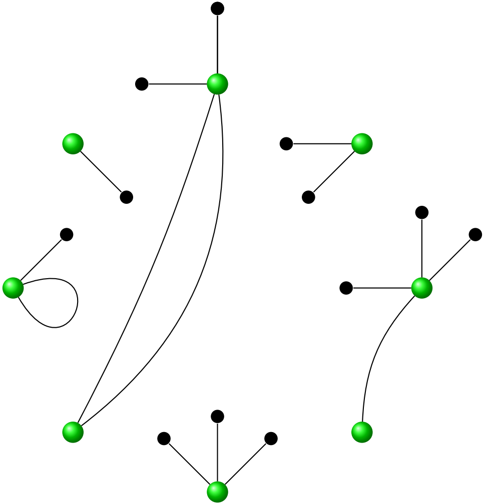
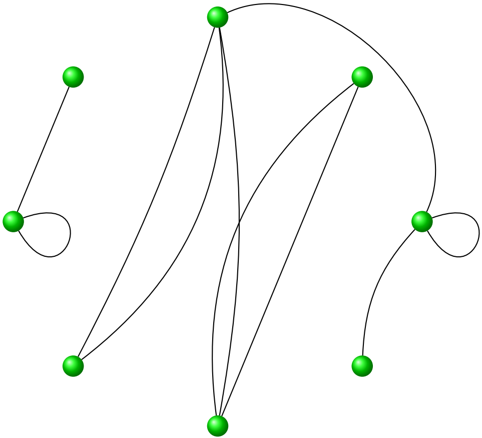


Configuration Model: the construction

Resulting multigraph...



Configuration model contd.

- The multigraph has scale-free nature if the empirical degree distribution $\frac{1}{n}\sum_{i=1}^{n}\delta_{d_i}$ has power-law tail.
- Exhibits phase transition under certain assumptions.

- Conditional on simplicity, this graph is uniformly distributed over the set of all simple graphs with degree sequence $\boldsymbol{d}$.
- A very popular model for generating scale-free networks.

Model II: Inhomogeneous random graphs (IRG) with finite state space/Stochastic block model

Model II: IRG with finite state space (Bollobás, Janson and Riordan; Special case)

- Given n vertices $1, 2, \ldots, n$.
- Vertex i has type $t_i \in [K] = \{1, \ldots, K\}$.
- Kernel $\kappa_n : [K] \times [K] \to (0, \infty)$.
- Place edges between vertex i and j with probability

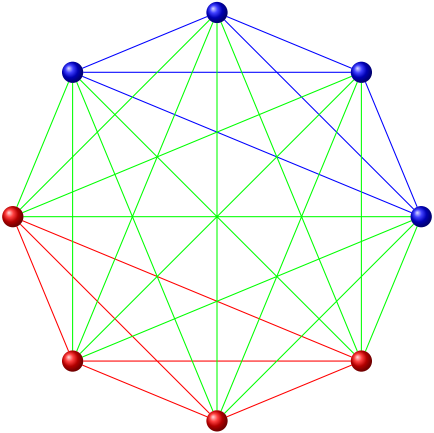
$$p_{ij} = 1 \wedge \frac{\kappa_n(t_i, t_j)}{n}$$

independently for each pair of vertices.

Also called **Stochastic block model**.

Inhomogeneous random graphs (with two types)

$\mathbb{P}(\text{red edge}) = 1 \wedge \kappa(\text{red}, \ \text{red})/8$
$\mathbb{P}(\text{blue edge}) = 1 \wedge \kappa(\text{blue}, \ \text{blue})/8$
$\mathbb{P}(\text{green edge}) = 1 \wedge \kappa(\text{red}, \ \text{blue})/8$



IRG with finite state space/Stochastic block model

- Standard model for **generating community structure**. Detecting community structure in networks is an important problem in statistics.

- Exhibits phase transition.

Model III: Rank-1 inhomogeneous random graphs/multiplicative coalescent

Model III: Rank-1 IRG (Aldous)

- Given n vertices $1, 2, \ldots, n$ and parameter $q > 0$.
- Vertex i has weight $x_i \in (0, \infty)$.
- Place edges between vertex i and j with probability

$$p_{ij} = 1 - \exp(-q x_i x_j)$$

independently for each pair of vertices.

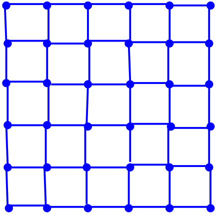
- Evolves as the **multiplicative coalescent** as a process in q.
- Scale-free under certain assumptions on the weight-sequence $x_i,\ i = 1, \ldots, n$.
- Phase transition under assumptions on $x_i,\ i = 1, \ldots, n$.
- Closely related to **Norros-Reittu model, Chung-Lu model**.

Physicists' prediction

- Distance in the largest components in the critical regime.
- Distance on the minimal spanning tree.

Consider a **connected graph** $G = (V, E)$ together with a **weight function** $w : E \to (0, \infty)$.

A subtree of G that has V as its vertex set is called a spanning tree of G.



Digression: Minimal spanning tree

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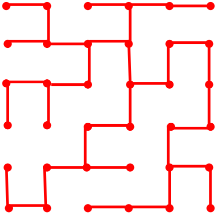


Digression: Minimal spanning tree

Consider a **connected graph** $G = (V, E)$ together with a **weight function** $w : E \to (0, \infty)$.

A subtree of G that has V as its vertex set is called a spanning tree of G.

A minimal spanning tree T of G is a spanning tree that minimizes the total weight:

$$\sum_{e \in T} w(e) = \min_{T' \text{ spanning tree}} \sum_{e \in T'} w(e).$$

An important functional

- Closely related to invasion percolation.
- Computer science: optimazition problem.
- Statistics: Single-linkage clustering, two-sample test.
- Statistical physics: Strong-disorder regime for weighted complex networks

Physicists' prediction

Based on non-rigorous argument, Braunstein, Buldyrev, Cohen, Havlin, Stanley (Phys. Rev. Letters, 2003) concluded that for scale-free random graphs on n vertices, the following hold:

When the degree exponent $\tau \geq 4$

- Typical distance between two vertices in the largest component in the critical regime is $\sim n^{1/3}$.
- Typical distance in the MST on the giant component in the supercritical regime is $\sim n^{1/3}$.

Erdős-Rényi type behavior in this regime.

When the degree exponent $\tau \in (3, 4)$

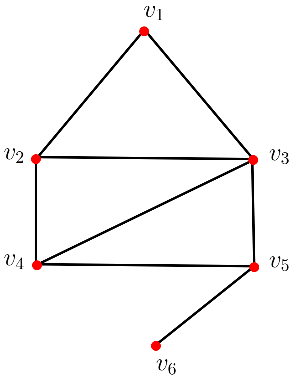
- Typical distance between two vertices in the largest component in the critical regime is $\sim n^{\frac{\tau-3}{\tau-1}}$.
- Typical distance in the MST on the giant component in the supercritical regime is $\sim n^{\frac{\tau-3}{\tau-1}}$.

Non-Erdős-Rényi behavior in this regime.

Critical window in Erdős-Rényi random graph

- The values $p(n, \lambda) = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$ (λ fixed) form the critical window.
- Huge literature on critical behavior of ERRG until late 90's: Janson, Knuth, Pittel, Rucinski, Łuczak, Wierman.
- Aldous (1997) showed that asymptotically the component sizes behave like the excursion lengths of a reflected Brownian motion with a parabolic drift.

Finite graphs as metric measure spaces



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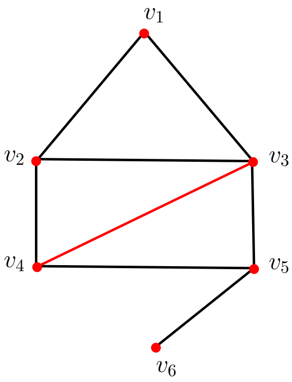

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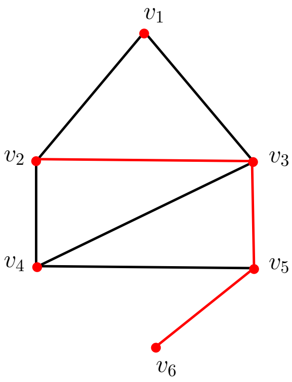

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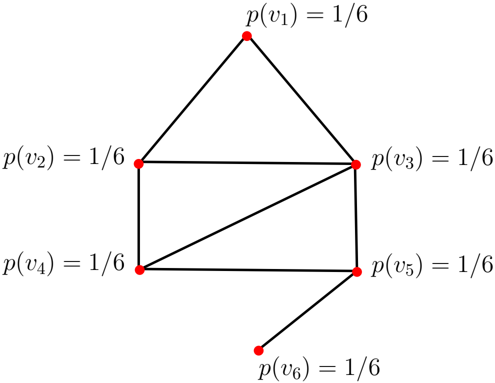


- Endow each component of the critical Erdős-Rényi graph with the graph distance scaled by $n^{1/3}$ (i.e. each edge has length $n^{-1/3}$).
- Give each component the uniform probability measure on the vertices of the component.
- Think of each component as a random compact metric measure space.

Theorem (Addario-Berry, Broutin, Goldschmidt; 2012)

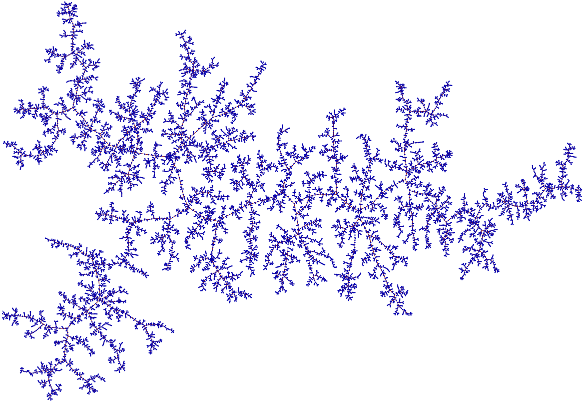
For $G(n, p(n,\lambda))$

$$\left(n^{-\frac{1}{3}} \cdot \mathcal{C}_1^{(n)}(\lambda), n^{-\frac{1}{3}} \cdot \mathcal{C}_2^{(n)}(\lambda), \dots\right) \xrightarrow{\text{w}} \boldsymbol{M}(\lambda) := \left(M_1(\lambda), M_2(\lambda), \dots\right),$$

w.r.t. Gromov-Hausdorff-Prokhorov topology.

In particular, the diameters of the largest components scale like $n^{1/3}$, and $n^{-1/3} \times$ diameter converges to some limiting random variable. Similar result for typical distance.

Limiting spaces: Aldous's Brownian continuum random tree



Taken from Igor Kortchemski's web page

Each of the spaces $M_i(\lambda)$ can be constructed by

- sampling a random real tree $\mathcal{T}_i$ whose law is absolutely continuous with respect to the CRT, and
- adding a random number of loops to $\mathcal{T}_i$.

Scaling limit of the minimal spanning tree (MST)

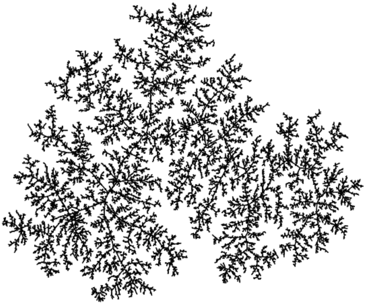
Consider the complete graph K_n on n vertices and let $\mathcal{M}_n$ be the MST on K_n.

Theorem (Addario-Berry, Broutin, Goldschmidt, Miermont; 2013)

There exists a binary, compact real tree $\mathcal{M}$ such that

$$n^{-1/3}\mathcal{M}_n \xrightarrow{\mathrm{w}} \mathcal{M},$$

w.r.t. Gromov-Hausdorff-Prokhorov topology. Further, the Minkowski dimension of $\mathcal{M}$ is 3. (Stark contrast to the Brownian CRT whose Minkowski dimension is 2.)



Taken from Nicolas Broutin's web page

Erdős-Rényi : Summary of results

- **Criticality:** Typical distance and diameter of $\mathcal{C}_i^{(n)}(\lambda)$ scale like $n^{1/3}$ and exact metric measure space scaling limit is known;
- **MST:** Typical distance and diameter of MST on giant component scale like $n^{1/3}$ and exact metric measure space scaling limit is known.

Goal of our project

Myriad of random graph models. Techniques are model specific.

Almost no results for typical distance/diameter at criticality or for the MST except for the simplest case of Erdős-Rényi graph.

Our aim is to provide rigorous justification for the predictions made by physicists. In particular ...

Give a streamlined method of proving that different critical random graph models have Erdős-Rényi scaling limit when the components are viewed as metric measure spaces. This, in particular, will yield scaling of typical distance.

We think of these critical random graphs as being in the Erdős-Rényi universality class.

Introduce non-Erdős-Rényi (metric space) universality classes that arise as the scaling limit of scale-free critical random graphs with degree exponent $\tau \in (3, 4)$.

Universality for scaling limit of the vacant set left by a random walk, components under critical percolation, and minimal spanning tree on giant components of different supercritical discrete structures.

WHEN DEGREE EXPONENT $\boldsymbol{\tau} \geq \mathbf{4}$

Starting point: rank-1 graphs

- Given positive numbers $x_1, \ldots, x_n$ and q, construct the random graph $\mathcal{G}(\mathbf{x}, q)$ by placing edges independently between i, j with probability
$$p_{ij} = 1 - \exp(-q x_i x_j).$$
- Aim: study $\mathcal{G}(\mathbf{x}, q)$ as a metric measure space.

Negligibility Assumptions

Let $\sigma_k = \sum_{i \in [m]} x_i^k$ for $k \geq 1$. Assume that
$$\frac{x_{\max}}{\sigma_2} \to 0, \quad \frac{\sigma_3}{(\sigma_2)^3} \to 1, \quad q - \frac{1}{\sigma_2} \to \lambda.$$

Rank-1 graphs: scaling limits

Theorem: Scaling limit for rank-1 (Bhamidi, S., Wang; 2016)

Treat $(\mathcal{C}_i : i \geq 1)$ as metric measure spaces using graph distance. Under above assumptions, for maximal components in $\mathcal{G}(\mathbf{x}, q)$,

$$(\sigma_2 \cdot \mathcal{C}_1,\ \sigma_2 \cdot \mathcal{C}_2, \ldots) \xrightarrow{\mathrm{w}} \boldsymbol{M}(\lambda) = (M_1(\lambda), M_2(\lambda), \ldots).$$

w.r.t. Gromov-Hausdorff-Prokhorov topology on each coordinate.

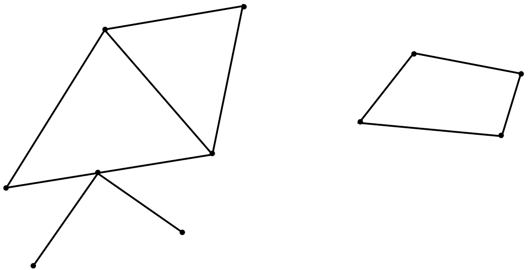
$\mathbf{p}$-trees and rank-1 graphs

- Each component of the rank-one graph has a spanning tree distributed as a tilted birthday tree or $\mathbf{p}$-tree.
- Under certain assumptions $\mathbf{p}$-trees converge to the Brownian CRT (Aldous, Miermont, Pitman).
- Can use this to prove convergence (as metric measure spaces) of the rank-1 graph.

Towards universality

Basic idea

- Start with $\mathcal{G}(\mathbf{x}, q)$ (**superstructure**). Scaling limit known for this.
- Replace each vertex of $\mathcal{G}(\mathbf{x}, q)$ by a “small” metric measure space; we like to call them blobs.
- If the blobs are nice enough, then the resulting space will have the same scaling limit.



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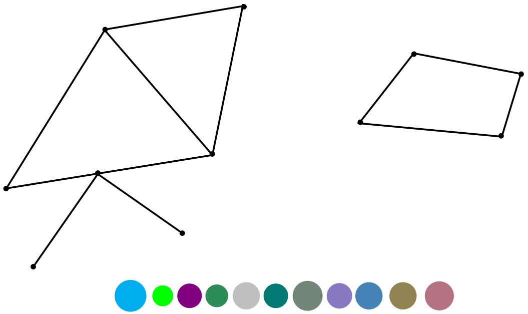

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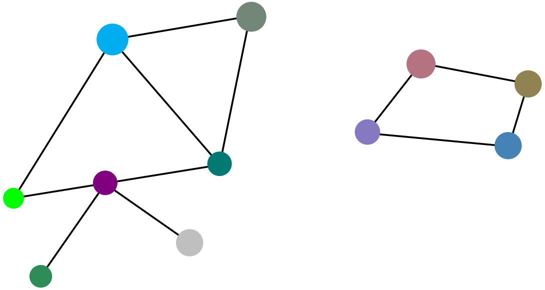


Universality: an abstract result

Setting and assumptions

- Choose two points $X_{i,1}$ and $X_{i,2}$ independently from Blob_i. Let $u_i = \mathbb{E}\big[d(X_{i,1}, X_{i,2})\big]$.
- $d_{\max} = \max_j(\mathrm{diam}(\mathrm{Blob}_j))$.
- **Assumptions:** In addition to previous assumptions, assume

$$\frac{x_{\max}}{\sigma_2^{3/2+\eta_0}} \to 0, \quad \frac{\sigma_2^{r_0}}{x_{\min}} \to 0, \quad \frac{d_{\max}\sigma_2^{3/2-\eta_0}}{\sum_{i=1}^{\infty} x_i^2 u_i + \sigma_2} \to 0, \quad \frac{\sigma_2 x_{\max} d_{\max}}{\sum_{i\in[n]} x_i^2 u_i} \to 0.$$

Theorem: Complete metric space scaling (Bhamidi, Broutin, S., Wang; 2017+)

Let $\mathcal{C}_i$ be the i-th largest component of $\mathcal{G}(\boldsymbol{x}, q)$. Let $\bar{\mathcal{C}}_i$ be the metric measure space obtained from $\mathcal{C}_i$ by replacing replacing vertex j by Blob_j. Then under above assumptions,

$$\left(\frac{(\sigma_2)^2}{\sigma_2 + \sum_{j\in[n]} x_j^2 u_j}\bar{\mathcal{C}}_i : i \geq 1\right) \xrightarrow{\mathrm{w}} \boldsymbol{M}(\lambda).$$

Applications to particular models

- Look at the barely subcritical regime of the model, and call the components in the barely subcritical regime blobs.
- Prove that blobs satisfy the properties from the previous slide. **Useful tool: Differential equations method.** (See earlier work by Tom Kurtz and comprehensive survey by Nick Wormald.)
- Approximate the original random graph model by the rank-1 graph that we get by replacing vertices by blobs.

- Will apply the abstract result to:
  1. Configuration model.
  2. General inhomogeneous random graph with finite state space/Stochastic block model.
- The general theorem should be applicable to most random graph models that are in the Erdős-Rényi class.

Model I: Percolation on supercritical Configuration model

- Fix pmf $\mathbf{p}_{\text{deg}} = \{p_k : k \geq 0\}$. Assume $p_2 < 1$. Also assume

$$\nu = \frac{\sum_k k(k-1)p_k}{\sum_k kp_k} > 1.$$

- Construct the configuration model with degree sequence $d_i \sim_{iid} \mathbf{p}_{\text{deg}}$.
- Now consider percolation with edge retention probability p.

Known results (Bollobás, Janson, Molloy and Reed, Riordan....)

- Edge retention probability $p > 1/\nu$: Giant component (supercritical).
- $p < 1/\nu$: $|\mathcal{C}_1| = o_P(n)$ (subcritical).

Critical percolation

- When

$$p(\lambda) = \frac{1}{\nu} + \frac{\lambda}{n^{1/3}},$$

all maximal component sizes $|\mathcal{C}_i(\lambda)| = \Theta_P(n^{2/3})$ [Nachmias-Peres (random regular graph); Joseph; Riordan (bounded degree)].

- Denote the corresponding graph by $\text{Perc}_n(\lambda)$.

Model I: Percolation on the configuration model

Theorem: Continuum scaling limits of metric structure for $\mathrm{Perc}_n(\lambda)$ (Bhamidi, Broutin, S., Wang; 2017+)

Let $\mathcal{C}_i^{(n)}(\lambda)$ be the ith largest component of $\mathrm{Perc}_n(\lambda)$. Then under some regularity assumptions,

$$\left(\frac{\alpha}{n^{1/3}}\mathcal{C}_i^{(n)}(\lambda) : i \geq 1\right) \xrightarrow{\mathrm{w}} \boldsymbol{M}\left(\alpha'\lambda\right) \quad \text{as } n \to \infty,$$

for model dependent parameters α and α'.

- Distances in maximal components scale like $n^{1/3}$.

Corollary: Random r-regular graph with $r \geq 3$

$$p(\lambda) = \frac{1}{r-1} + \frac{\lambda}{n^{1/3}}.$$

Then the maximal components viewed as metric spaces satisfy

$$\left(\frac{(r(r-1)(r-2))^{2/3}}{r(r-1)}\frac{1}{n^{1/3}}\mathcal{C}_i^{(n)}(\lambda) : i \geq 1\right) \xrightarrow{\mathrm{w}} \boldsymbol{M}\left(\frac{(r-1)^2}{(r(r-1)(r-2))^{2/3}}\lambda\right),$$

Model II: Inhomogeneous random graphs/Block model at criticality

- **Vertex type space:** $[K] = \{1, 2, \ldots, K\}$. Each vertex $i \in [n]$ has type $t_i \in \mathscr{X}$.
- **n-dependent kernel:** $\kappa_n : [K] \times [K] \to \mathbb{R}_+$.
- **Empirical distribution of types:** $\mu_n(x) = \#\{i \in [n] : x_i = x\}/n$.
- Connect vertex i, j with probability

$$p_{ij} := 1 \wedge \frac{\kappa_n(t_i, t_j)}{n}.$$

Criterion for phase transition **(Bollobas, Janson, Riordan, 2007) :**

Assume $\kappa_n \approx \kappa$, $\mu_n \approx \mu$.

Let ρ be the **spectral radius** (or the **Perron root**) of $M = \big((\kappa(i,j)\mu(j))\big)$.

- Supercritical regime: If $\rho > 1$, then $|\mathcal{C}_1| \sim f(\kappa, \mu)n$.
- Subcritical regime: If $\rho < 1$, then $|\mathcal{C}_1| = o_P(n)$.
- **Critical regime:** $\rho = 1$: content of this talk.

Model II: Inhomogeneous random graphs/Block model at criticality

Theorem: Continuum scaling limits of metric structure of critical IRG (Bhamidi, Broutin, S., Wang; 2017+)

Consider the critical IRG with some additional technical assumptions. View it as a metric measure space with usual graph metric. Then

$$\left(\frac{\alpha}{n^{1/3}}\mathcal{C}_i(\mathcal{G}_{\mathrm{IRG}}^{(n)}) : i \geq 1\right) \xrightarrow{\mathrm{w}} \boldsymbol{M}\left(\alpha'\right),$$

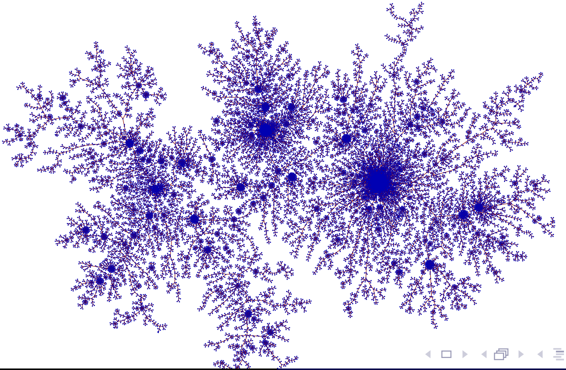
for model dependent parameters α and α'.

- In particular, for the maximal components, typical distance and diameter scale like $n^{1/3}$.

$\mathcal{WHEN\ \ DEGREE\ \ EXPONENT}$ $\boldsymbol{\tau \in (3,4)}$

The case $\tau \in (3, 4)$: A new universality class

- When $\tau \in (3, 4)$, the critical rank-1 random graph has a (metric measure space) scaling limit that can be constructed using the so-called inhomogeneous continuum random trees.
- This is the first model with $\tau \in (3, 4)$ for which the metric space scaling limit has been established.
- Existence of exploration process and connection to Lévy trees is still a conjecture. A new approach is required in this case.



Theorem (Bhamidi, van der Hofstad, S. (2017))

Consider the **multiplicative coalescent** and let $\mathcal{C}_{(1)}, \mathcal{C}_{(2)}, \ldots$ be its components in decreasing order of mass.

There exist random metric measure spaces S_i, $i \geq 1$ that can be constructed from suitable ICRTs such that under the appropriate entrance boundary conditions,

$$\sigma_2(\mathcal{C}_{(1)}, \mathcal{C}_{(2)}, \ldots) \xrightarrow{\mathrm{w}} (S_1, S_2, \ldots)$$

with respect to Gromov-weak topology.

The case $\tau \in (3,4)$: metric space scaling limit of the Chung-Lu/Norros-Reittu model

- n vertices labeled $1, 2, \ldots, n$.
- $w_i = c\left(\frac{n}{i}\right)^{1/(\tau-1)}$. (Also possible to take more general sequences.)
- Place an edge between i and j with probability

$$p_{ij} = \left(1 + \lambda n^{-\frac{\tau-3}{\tau-1}}\right) w_i w_j / \sum w_k.$$

Theorem (Bhamidi, van der Hofstad, S. (2017))

When $\tau \in (3,4)$,

$$n^{-\frac{\tau-3}{\tau-1}}\left(\mathcal{C}_i^{(n)},\ i \geq 1\right) \xrightarrow{\mathrm{w}} \left(S_i,\ i \geq 1\right)$$

with respect to Gromov-Hausdorff-Prokhorov topology.

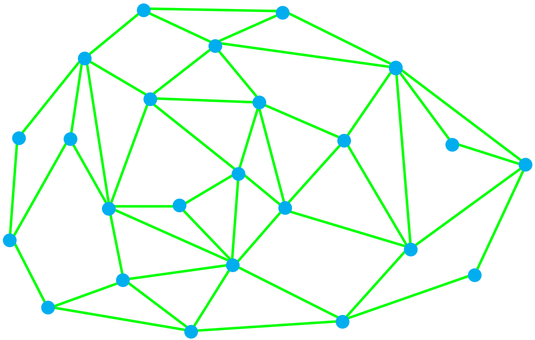
Further,
(a) the spaces S_i are compact and
(b) for all $i \geq 1$, the Minkowski dimension of S_i is $(\tau-2)/(\tau-3)$ almost surely.

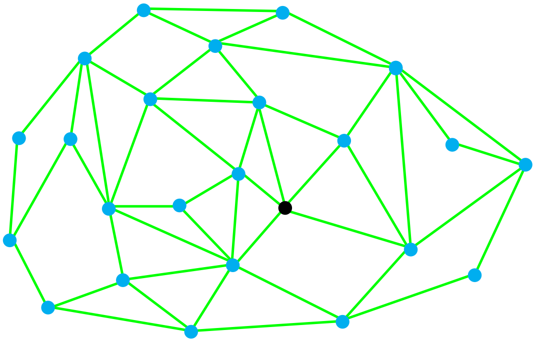
In particular, novel universality classes of metric measure spaces.

Diameter and typical distance scale like $n^{\frac{\tau-3}{\tau-1}}$, as predicted by physicists.

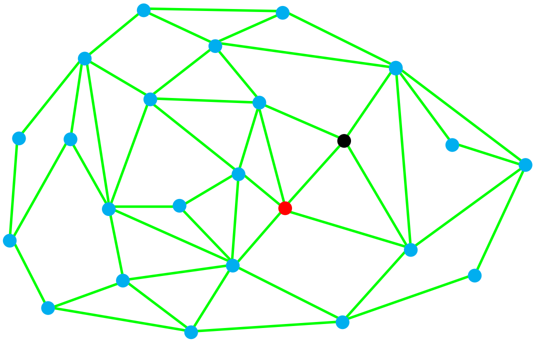
VACANT SET LEFT BY RANDOM WALKS

Vacant set left by random walks





Vacant set left by random walks



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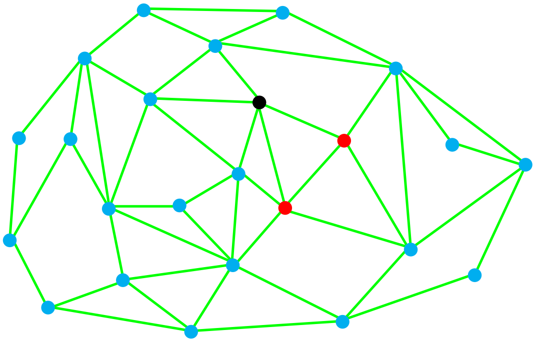

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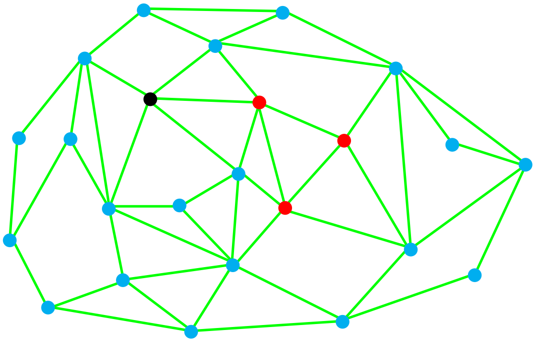

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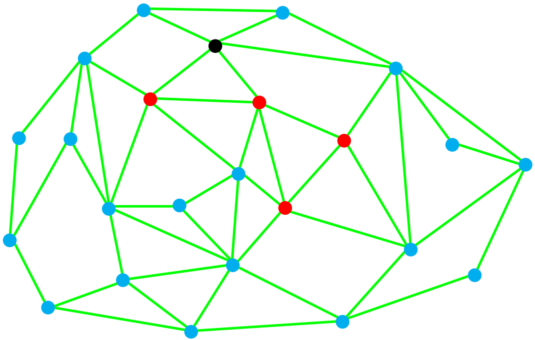

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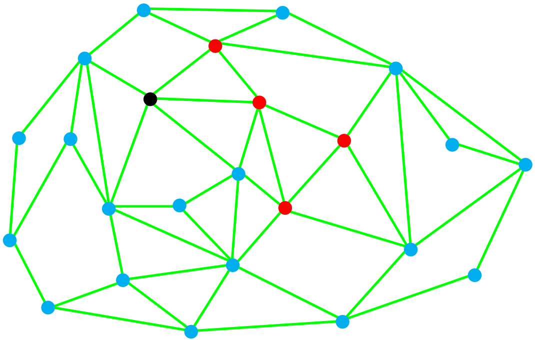

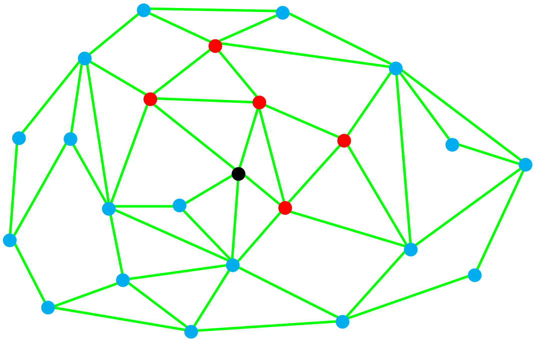

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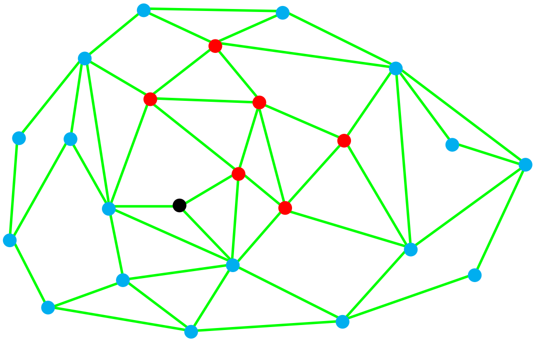
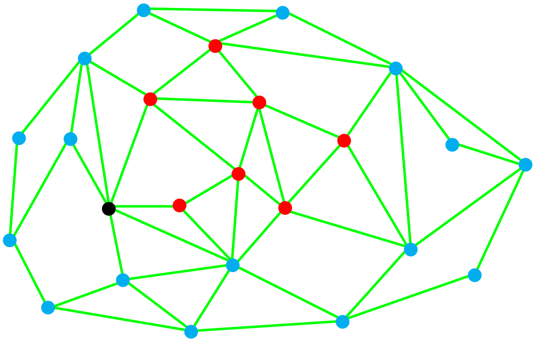
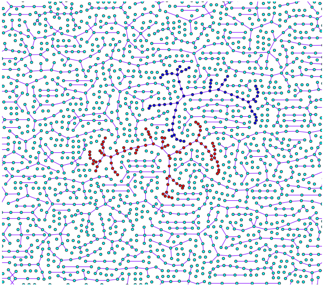


Vacant set left by random walks

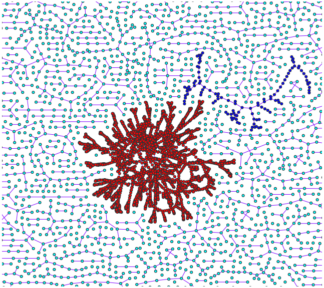


- Earlier work: Cooper and Frieze (2006)
- Consider the random r-regular graph ($r \geq 3$) and run a random walk on it upto time un. Consider the vacant set: the set of vertices not visited by the walk.
- The vacant set goes through a phase transition at $u = u_\star$ [Černy-Teixeira], where

$$u_\star := \frac{r(r-1)\ln(r-1)}{(r-2)^2}.$$



Connectivity structure at $u = u_\star + 0.5$



Connectivity structure at $u = u_\star - 0.5$

Černy and Teixeira (2011) raised the question: what can be said about the geometry of the vacant set around the point of phase transition?

Theorem [Bhamidi, S. (2016)]

Let $\mathcal{C}_i^{(n)}$ denote the ith largest component of the vacant set at $u = u_\star$. Then

$$n^{-\frac{1}{3}}\left(\mathcal{C}_1^{(n)}, \mathcal{C}_2^{(n)}, \ldots\right)$$

converges to an "Erdős-Rényi type" scaling limit.

In particular, the vacant set, in the case of regular graphs, belongs to the Erdős-Rényi universality class.

- This answers a question raised by Černy and Teixeira (2011).
- This is the only model for which the scaling limit of the vacant set has been established.

MST ON SUPERCRITICAL DISCRETE STRUCTURES

Corollary

Let $\mathcal{M}_n$ be the MST on the giant component in the supercritical regime for any of the models discussed. A corollary to our result is a lower bound of the form:

- $\mathbb{P}(\mathrm{diam}(\mathcal{M}_n) \geq c_\epsilon n^{1/3}) > 1 - \epsilon$, when degree exponent $\tau > 4$.
- $\mathbb{P}(\mathrm{diam}(\mathcal{M}_n) \geq c_\epsilon n^{\frac{\tau-3}{\tau-1}}) > 1 - \epsilon$, when degree exponent $\tau \in (3, 4)$.

- This lower bound matches the prediction in the conjecture.
- In general an upper bound of the same order is missing.

Theorem [Addario-Berry, S. (2017)]

Let D_n be the diameter of the MST on the random 3-regular simple graph/configuration model. Then

$$D_n = \Theta_P(n^{1/3}).$$

THE SPATIAL MST

Stochastic geometry: The spatial MST

- Consider a Poisson process $\mathcal{P}$ of intensity one in $\mathbb{R}^d$, $d \geq 2$.
- Consider the complete graph on $\mathcal{P} \cap [-n, n]^d$ and weight each edge by its Euclidean length.
- Let $\mathcal{M}_n$ be the MST on this graph.

- Study of spatial MST has a long history, but nothing known about the scaling limit of $\mathcal{M}_n$!
- A lot of work on the total weight $W_n := \sum_{e \in \mathcal{M}_n} |e|$.
- Law of large number type results were obtained in the case of n i.i.d. points by Steele; Aldous and Steele.

- Kesten and Lee (1996) showed that $(W_n - \mathbb{E}W_n)/\sqrt{\mathrm{Var}(W_n)}$ satisfies a CLT.
- They also raised the question: how to get an error bound in this CLT?
- The question of error bound has been answered for many other functionals studied in stochastic geometry. But this problem had remained open for the MST.

Stochastic geometry: The spatial MST

Theorem (Chatterjee, S. (2016))

There exists a positive constant η such that for every $p > 1$ and $n \geq 2$,

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{W_n - \mathbb{E}[W_n]}{\sqrt{\mathrm{Var}[W_n]}} \leq x \right) - \Phi(x) \right| \leq \begin{cases} c_1 n^{-\eta}, & \text{if } d = 2, \\ c_2 (\log n)^{-\frac{d}{4p}}, & \text{if } d \geq 3, \end{cases}$$

where c_1 is a positive constant which depends only on p and c_2 is a positive constant which depends only on p and d.

- A variation of Stein's method obtained by Chatterjee (2008) is used to reduce the problem to obtaining a bound on the probability of the so-called "two-arm event" in the setup of continuum percolation in $\mathbb{R}^d$.
- We obtained (Chatterjee and S) this error bound by quantifying the classical Burton-Keane argument for uniqueness of the infinite cluster in percolation on $\mathbb{Z}^d$.
- This is a robust technique and was later generalized by Duminil-Copin, Ioffe, and Velenik (2016).

Summary, related results, and open problems

- In a series of two papers, **Bhamidi, S., Wang (2016)** and **Bhamidi, Broutin, S., Wang (2017+)**, a general abstract universality principle was developed, which was used to prove $n^{1/3}$ scaling and Erdős-Rényi type scaling limit for three classes of standard random graph models at criticality.

  This also justified a conjecture of **Braunstein et al. (2003)**.

- **Kang, Perkins, and Spencer's (2012)** conjecture about a critical exponent of the so-called Bohman-Frieze process was proved in **S. (2016)**. Then the metric space scaling limit for this model was established in **Bhamidi, Broutin, S., Wang (2017+)**.

- Dembo, Levit, and Vadlamani's (2014) work suggests that the quantum random graph model should be in the same universality class. The general universality principle should also be applicable here.

- A major open problem in this field is showing that the high-dimensional discrete torus and the hypercube belong to the Erdős-Rényi universality class.

Summary, related results, and open problems

- **Bhamidi, van der Hofstad, and S. (2017)** proved the first result about $n^{\frac{\tau-3}{\tau-1}}$ scaling for the rank-1 inhomogeneous random graph models and introduced novel classes of metric space scaling limits.

  This justifies another conjecture of **Braunstein et al. (2003)**.

- Building on this work, **Bhamidi, Dhara, van der Hofstad, and S. (2017)** established $n^{\frac{\tau-3}{\tau-1}}$ scaling for the configuration model.

- Goldschmidt, Haas, and Sénizergues (work in progress), Conchon-Kerjan and Goldschmidt (work in progress), and Broutin and M. Wang (work in progress) are working on similar problems using different approaches.

- A question of **Černy and Teixeira (2011)** was answered by **Bhamidi and S. (2016)** where the intrinsic geometry of the vacant set left by a random walk on random regular graph was characterized.

- The metric space structure of the vacant set left by a random walk should be universal for a large class of random discrete structures. The general problem is open.

- An open question of **Kesten and Lee (1996)** about total weights of spatial minimal spanning trees was answered by **Chatterjee and S. (2016)**.
- What is the intrinsic diameter of the spatial MST?

- **Addario-Berry, and S. (2017)** shows that the diameter of the MST on the random 3-regular graph is $\Theta_P(n^{1/3})$.
- **Addario-Berry, Bhamidi, and S. (2017)** proves the leader problem for the multiplicative coalescent which is a step towards universality in the strong disorder regime.
- More work is needed to settle the universality conjecture in the strong disorder regime.

THANK YOU.