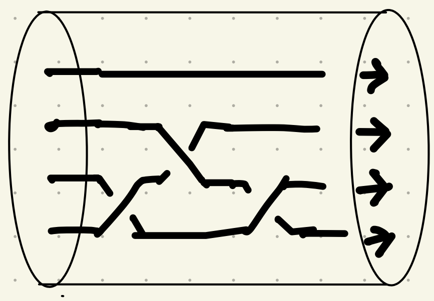
Braids II

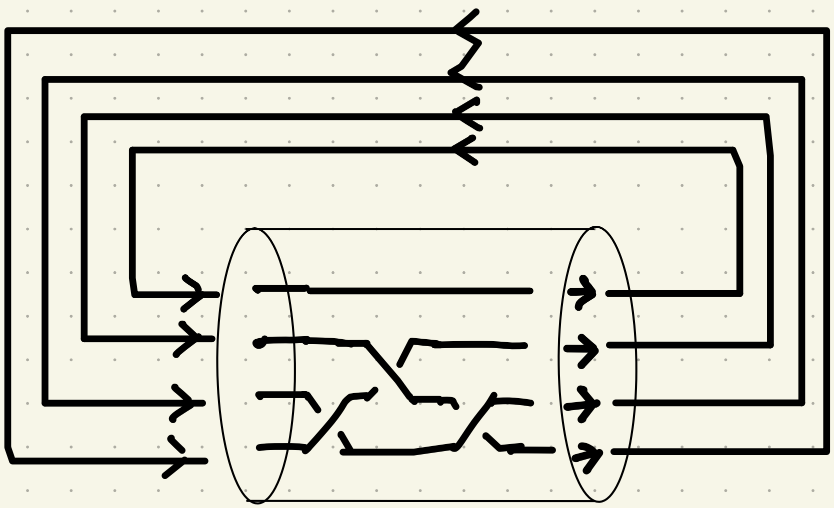
ICTS program "Knots through web"

Braids and links

$b \subset D^2 \times I$: a geometric n-braid



$\downarrow$



$\hat{b} \subset \mathbb{R}^3$: a closed braid

the closure of b

$\{\text{braids}\} \longrightarrow \{\text{links}\}$

Thm (Alexander)

Any link in $\mathbb{R}^3$ can be presented by the closure of a braid.

Thm (Markov)

Two braids b and b' present the same link if and only if they are related by a sequence of

- conjugations
- stabilizations
- destabilizations

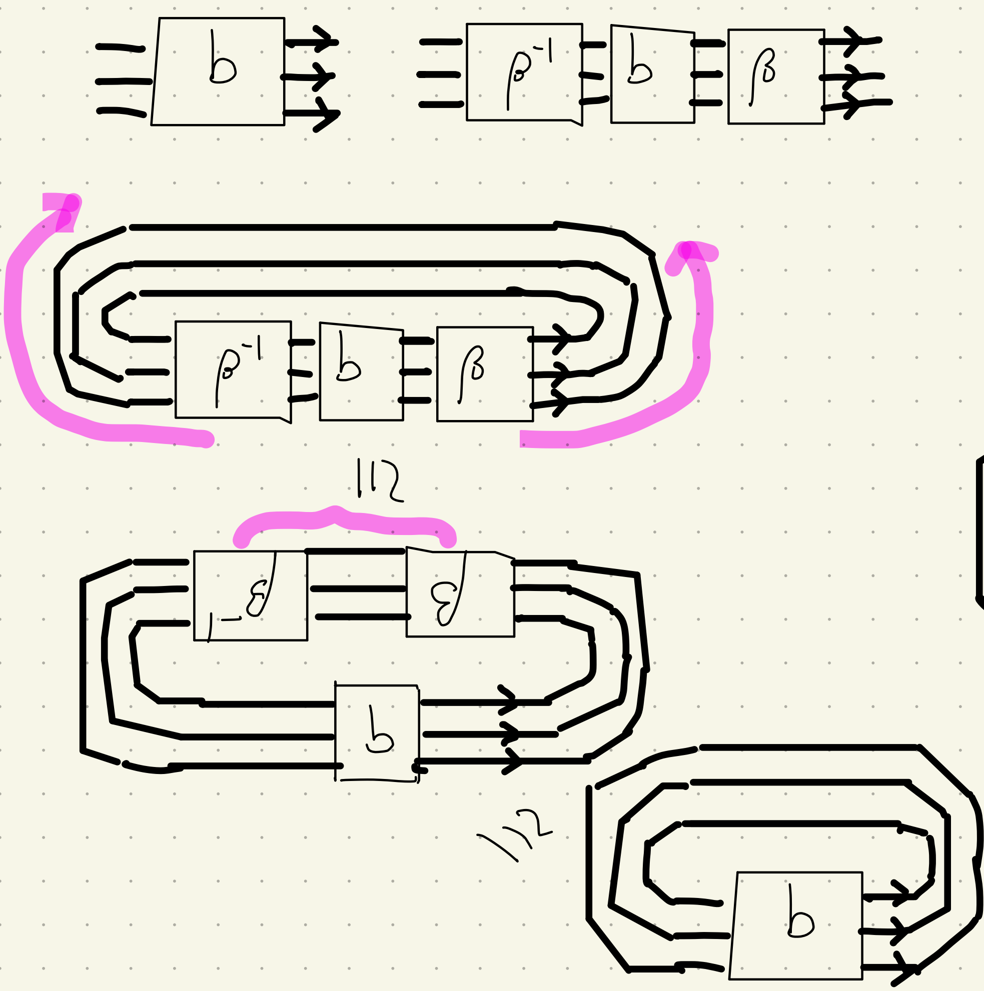
$$\{\text{braids}\}/\sim_{\text{Markov}} \xleftrightarrow{1:1} \{\text{links}\}$$

(Alexander–Markov)

Conjugation

Markov move of type I

$b \in B_m \rightsquigarrow \beta^{-1} b \beta \in B_m$
for some $\beta \in B_m$

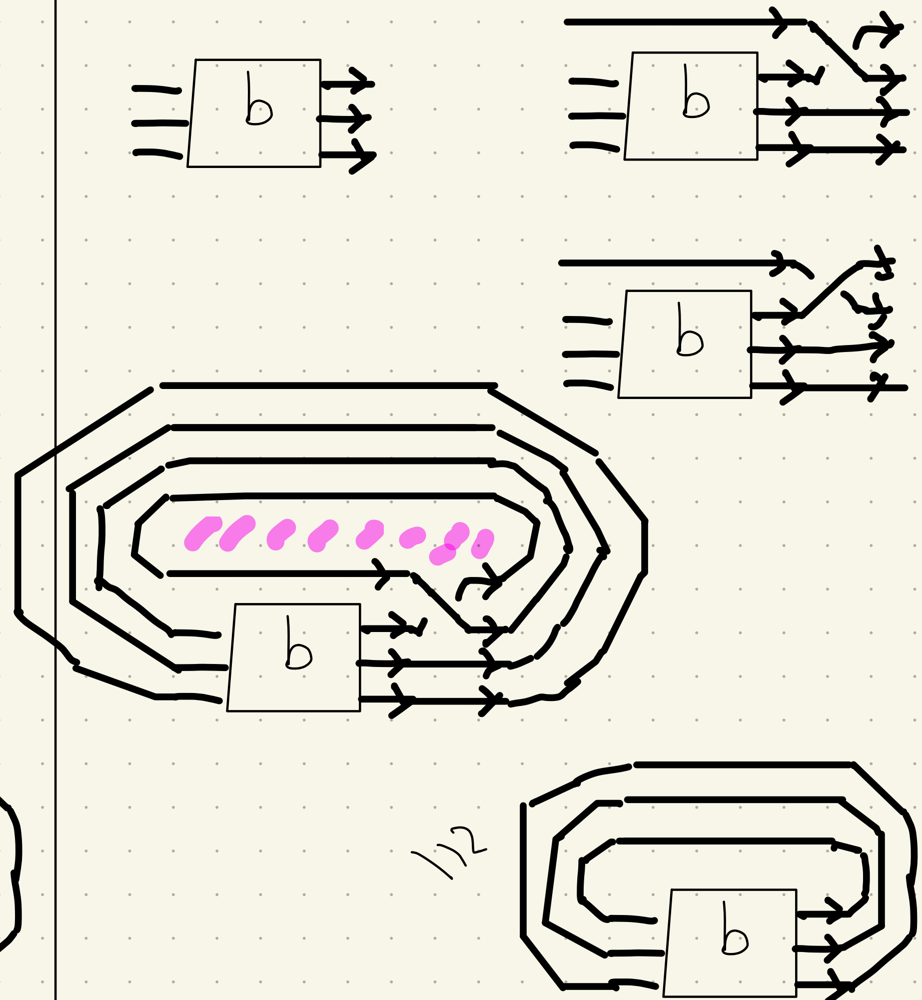


Stabilization

Markov move of type II

$b \in B_m \rightsquigarrow b\sigma_m \in B_{m+1}$

$\rightsquigarrow b\sigma_m^{-1} \in B_{m+1}$



Thm (Alexander and Markov)

$$\left. \coprod_{m=1}^{\infty} B_m \middle/ \sim_{\text{Markov}} \right. \overset{1:1}{\longleftrightarrow} \{\text{links in } \mathbb{R}^3\}$$

- Some invariants of knots and links are defined by using braids.

  e.g. Jones polynomial

- Some invariants of knots and links can be computed from braids.

  e.g. Knot group
  Knot quandle
  Quandle colorings
  Quandle cocycle invariants

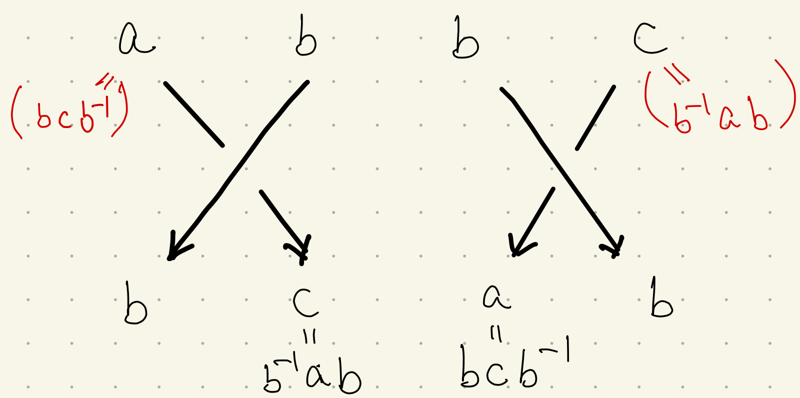
How to compute knot groups

K: a link in $\mathbb{R}^3$

$G(K)$: the knot group of K $= \pi_1(\mathbb{R}^3 - K)$

D: a diagram of K

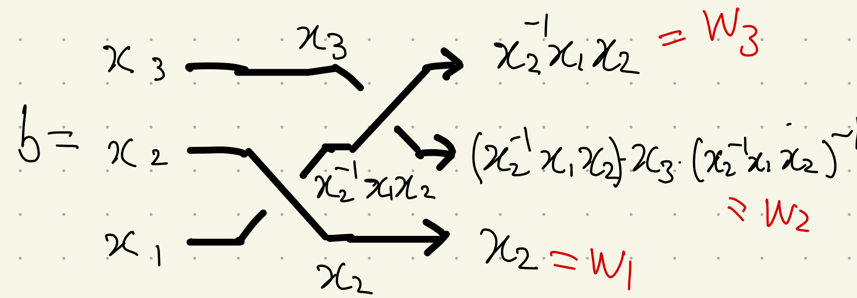
$G(D)$: the knot group of D $= \langle \text{arcs of } D \mid \text{relations at crossings} \rangle$

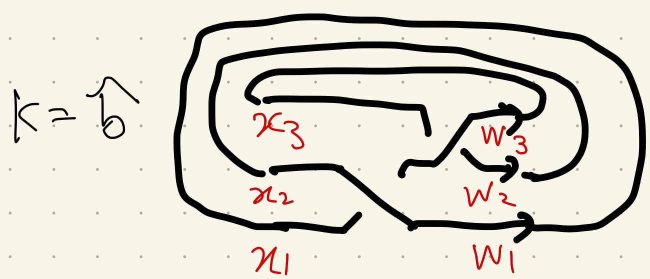


Thm (Wirtinger)

$G(D) \cong G(K)$

Example





$$\left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} x_1 = x_2 \\ x_2 = (x_2^{-1}x_1x_2)x_3(x_2^{-1}x_1x_2)^{-1} \\ x_3 = x_2^{-1}x_1x_2 \end{array} \right\rangle$$

Hurwitz action of B_m

The braid group B_m acts on G^m from the right for any group G.

$b \in B_m$, $\quad P(b) : G^m \to G^m$

① $P(\sigma_i)$: $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_m)$

$$\Downarrow$$

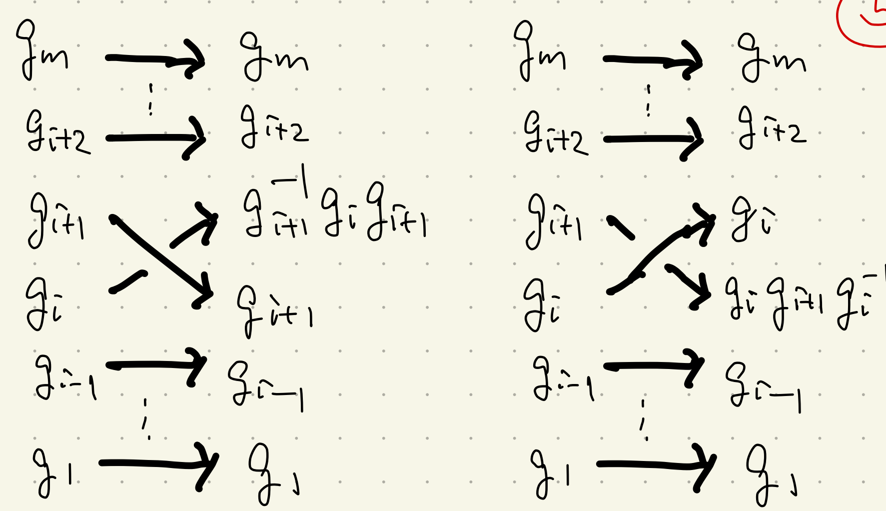
$$(g_1, \cdots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \cdots, g_m)$$

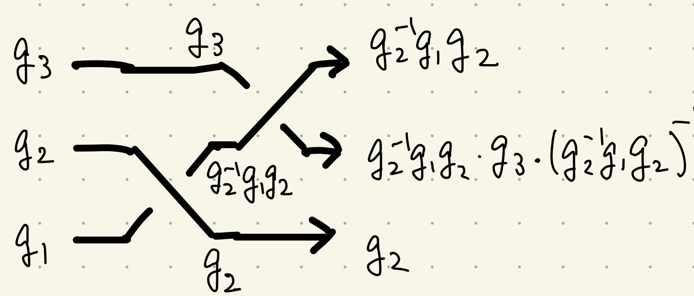
② $P(\sigma_i^{-1})$: $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_m)$

$$\Downarrow$$

$$(g_1, \cdots, g_i g_{i+1} g_i^{-1}, g_i, \cdots, g_m)$$

③ $\quad P(b_1 b_2) = P(b_2) \circ P(b_1)$





① $\mathrm{artin}(\sigma_i)$: $x_i \mapsto x_{i+1}$

$\quad x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}$

$\quad x_k \mapsto x_k \quad (k \neq i, i+1)$

② $\mathrm{artin}(\sigma_i^{-1})$: $x_i \mapsto x_i x_{i+1} x_i^{-1}$

$\quad x_{i+1} \mapsto x_i$

$\quad x_k \mapsto x_k \quad (k \neq i, i+1)$

③ $\mathrm{artin}(b_1 b_2) = \mathrm{artin}(b_1) \circ \mathrm{artin}(b_2)$

Thm (folklore, due to Artin)

$b \in B_m$. $G = F\langle x_1, \cdots, x_m \rangle$

$P(b) : G^m \to G^m$

$K = \hat{b}$: the closure of b

Let $P(b)(x_1, \cdots, x_m)$

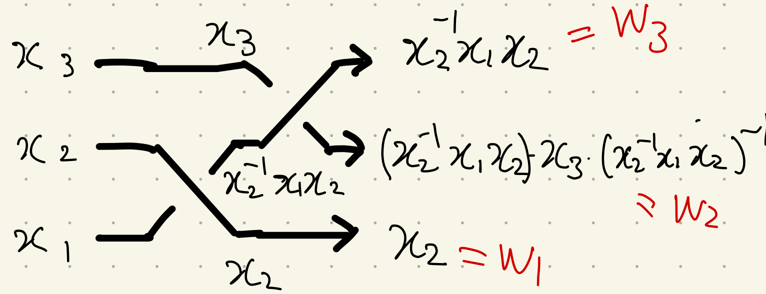
$= (w_1, \cdots, w_m)$.

Then $G(K)$ has a presentation

$$\left\langle x_1, \cdots, x_m \;\middle|\; \begin{array}{l} x_1 = w_1 \\ x_2 = w_2 \\ \vdots \\ x_m = w_m \end{array} \right\rangle$$

Example

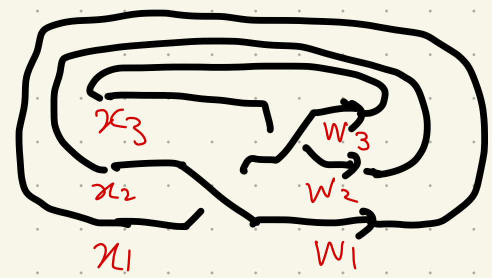
$b =$



$P(b)(x_1, x_2, x_3)$

$= (\underbrace{x_2}_{w_1}, \underbrace{(x_2^{-1}x_1x_2) \cdot x_3 (x_2^{-1}x_1x_2)^{-1}}_{w_2}, \underbrace{x_2^{-1}x_1x_2}_{w_3})$

$K = \hat{b}$



$G(K)$

$$\left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} x_1 = x_2 \\ x_2 = (x_2^{-1}x_1x_2)x_3(x_2^{-1}x_1x_2)^{-1} \\ x_3 = x_2^{-1}x_1x_2 \end{array} \right\rangle$$

Knot quandles and quandle colorings

Hurwitz action of B^m on quandles

X: a quandle $\quad X^m = \underbrace{X \times \cdots \times X}_{m}$

The braid group B_m acts on X^m from the right.

$b \in B_m, \qquad P(b) : X^m \to X^m$

① $P(\sigma_i)$: $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_m)$
$$\downarrow$$
$(g_1, \cdots, g_{i+1}, g_i * g_{i+1}, \cdots, g_m)$

② $P(\sigma_i^{-1})$: $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_m)$
$$\downarrow$$
$(g_1, \cdots, g_{i+1} \bar{*} g_i, g_i, \cdots, g_m)$

③ $\quad P(b_1 b_2) = P(b_2) \circ P(b_1)$

Thm

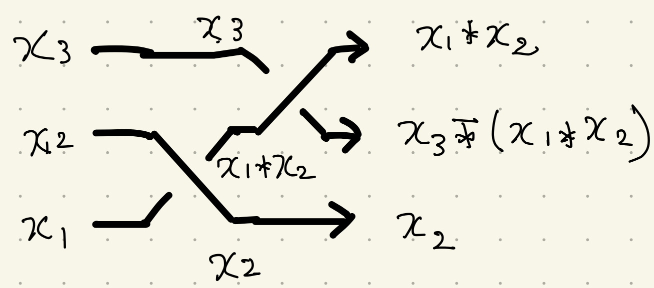
$b \in B_m, \quad X = FQ(\mathbb{X}) \quad \mathbb{X} = \{x_1, \cdots, x_m\}$

$K = \hat{b}$: the closure

Let $P(b)(x_1, \cdots, x_m) = (w_1, \cdots, w_m)$

Then $Q(K)$ has a quandle pres.

$$\left\langle x_1, \cdots, x_m \;\middle|\; \begin{array}{l} x_1 = w_1 \\ x_2 = w_2 \\ \vdots \\ x_m = w_m \end{array} \right\rangle$$



$Q(K)$

$$\left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} x_1 = x_2 \\ x_2 = x_3 \bar{*} (x_1 * x_2) \\ x_3 = x_1 * x_2 \end{array} \right\rangle$$

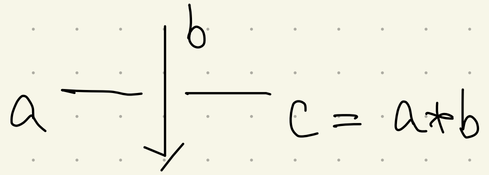
Quandle coloring

X: a finite quandle

D: a diagram of a knot/link K

$C: \mathrm{Arc}(D) \to X$ is a <u>coloring</u>

if for each crossing,



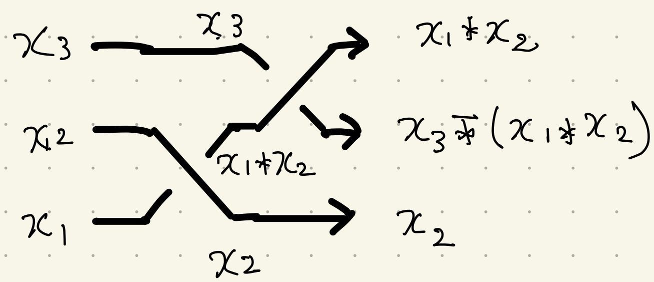
$$\mathrm{Col}_X(D) = \{\text{colorings of } D \text{ by } X\}$$

$$\updownarrow 1:1$$

$$\mathrm{Hom}(Q(D), X)$$

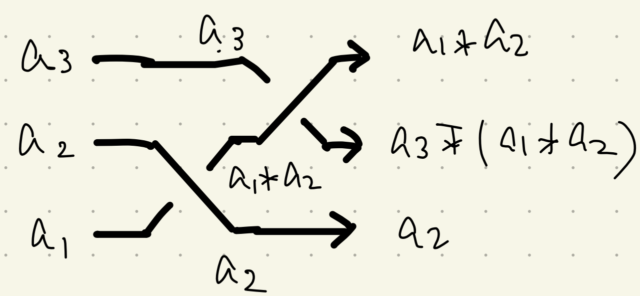
<u>Thm</u> $b \in B_m$, $K = \hat{b}$: the closure

$$\mathrm{Col}_X(K) \longleftrightarrow \left\{ (a_1, \cdots, a_m) \in X^m \;\middle|\; \begin{array}{l} (a_1, \cdots, a_m) \text{ is a solution of} \\ \quad (x_1, \cdots, x_m) = P(b)(x_1, \cdots, x_m) \end{array} \right\}$$



$$P(b)(x_1, x_2, x_3)$$
$$= (x_2, x_3 \,\bar{*}\, (x_1 * x_2), x_1 * x_2)$$

$$Q(K) \cong \left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} x_1 = x_2 \\ x_2 = x_3 \,\bar{*}\, (x_1 * x_2) \\ x_3 = x_1 * x_2 \end{array} \right\rangle$$



$$\mathrm{Col}_X(K)$$
$$\updownarrow$$
$$\left\{ (a_1, a_2, a_3) \in X^3 \;\middle|\; \begin{array}{l} a_1 = a_2 \\ a_2 = a_3 \,\bar{*}\, (a_1 * a_2) \\ a_3 = a_1 * a_2 \end{array} \right\}$$

Virtual braids and welded braids

B_m : the braid group

VB_m : the virtual braid group

WB_m : the welded braid group

B_m : generators: $\sigma_1, \cdots, \sigma_{m-1}$

relations:

braid relations $\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, & |i-j|>1 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, & |i-j|=1 \end{cases}$

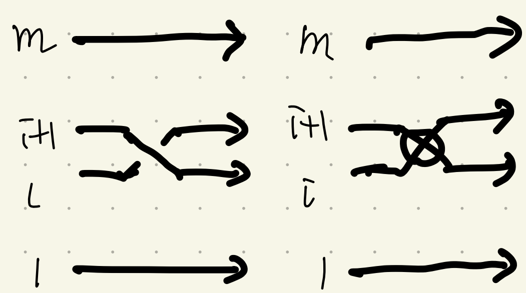
VB_m : generators $\begin{cases} \sigma_1, \cdots, \sigma_{m-1} \\ \tau_1, \cdots, \tau_{m-1} \end{cases}$

relations:

braid relations $\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, & |i-j|>1 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, & |i-j|=1 \end{cases}$

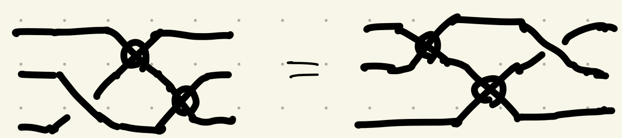
permutation group relations $\begin{cases} \tau_i^2 = 1 & \\ \tau_i \tau_j = \tau_j \tau_i & |i-j|>1 \\ \tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j & |i-j|=1 \end{cases}$

mixed relations $\begin{cases} \sigma_i \tau_j = \tau_j \sigma_i & |i-j|>1 \\ \sigma_i \tau_j \tau_i = \tau_j \tau_i \sigma_j & |i-j|=1 \end{cases}$



σ_i $\qquad\qquad$ τ_i

$\sigma_1 \tau_2 \tau_1 = \tau_2 \tau_1 \sigma_2$



WB_m : generators : $\begin{cases} \sigma_1, \cdots, \sigma_{m-1} \\ \tau_1, \cdots, \tau_{m-1} \end{cases}$

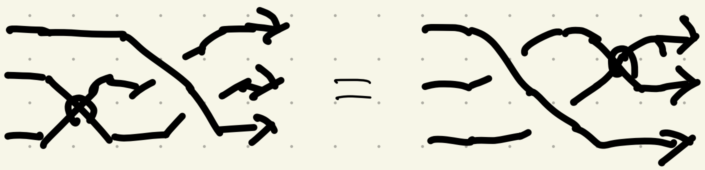
relations : the same with VB_m

$+$

$$\tau_i \sigma_j \sigma_i = \sigma_j \sigma_i \tau_j$$

(welded relation)

$$\tau_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \tau_2$$



Natural homomorphisms

$$B_m \xrightarrow{\mu} VB_m \xrightarrow{\lambda} WB_m$$

$$\begin{array}{ccccc} \sigma_i & \mapsto & \sigma_i & \longmapsto & \sigma_i \\ & & \tau_i & \longmapsto & \tau_i \end{array}$$

Thm

(1) $\mu : B_m \to VB_m$ is injective

(2) $\lambda \circ \mu : B_m \to WB_m$ is injective

(3) $\lambda : VB_m \to WB_m$ is surjective

(1) follows from (2).

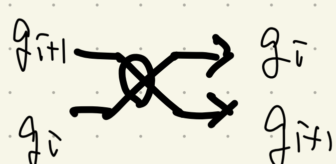
(2) is due to

[Fenn–Rimányi–Rourke, The braid-permutation group, 1997].

(3) is trivial

Artin automorphisms
$\quad \mathrm{Artin}(b) : F_m \to F_m$
$\quad \mathrm{artin}(b) : F_m \to F_m$
Hurwitz action
$\quad P(b) : G^m \to G^m$

$\Bigg\}$ are defined similarly.



τ_i

Thm (Artin)

Let b and $b' \in B_m$.

$b = b' \iff \mathrm{Artin}(b) = \mathrm{Artin}(b')$.

Let b and $b' \in VB_m$

$b = b'$ in $VB_m \Rightarrow \mathrm{Artin}(b) = \mathrm{Artin}(b')$

However $\Leftarrow$ is not true.

Let b and $b' \in B_m \subset VB_m$

$b = b'$ in $VB_m \iff \mathrm{Artin}(b) = \mathrm{Artin}(b')$

Thm (Fenn - Rimányi - Rourke)

Let b and $b' \in WB_m$.

$b = b'$ in $WB_m \iff \mathrm{Artin}(b) = \mathrm{Artin}(b')$

Alexander–Markov type theorem.

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$\{\text{braids}\}/\sim_{\text{Markov}} \xleftrightarrow{1:1} \{\text{links}\}$

$\{\text{virtual braids}\}/\sim_{\text{Markov}}$

$\updownarrow$ 1:1

$\{\text{virtual links}\}$

SK (2007)

S. Lambropoulou

$\{\text{welded braids}\}/\sim_{\text{Markov}}$

$\updownarrow$ 1:1

$\{\text{welded links}\}$

SK (2007)

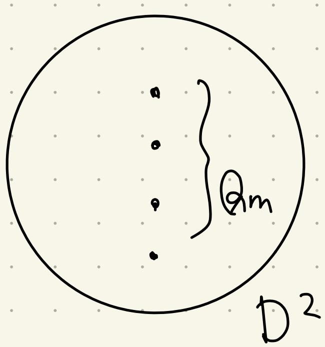
SK (2007) Braid presentation of virtual knots and welded knots, Osaka J. Math. 44 (2007), 441–458

(cf. arXiv: math.GT/0008092 (2000).

→ OJM (2007)

→ Kobe J. Math. 21 (2004).)

2-dimensional braids



Q_m is a fixed set of m interior points of D^2

D_1^2

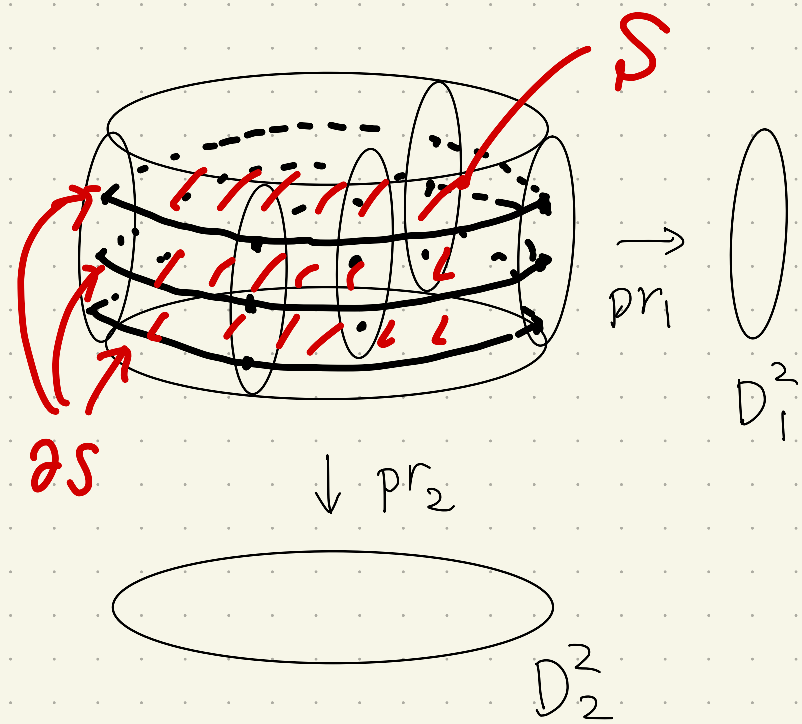
$pr_1 : D_1^2 \times D_2^2 \longrightarrow D_1^2$

$pr_2 : D_1^2 \times D_2^2 \longrightarrow D_2^2$

Def A <u>2-dimensional braid</u> of degree m is a compact oriented surface S embedded in $D_1^2 \times D_2^2$ such that

(i) $pr_2|_S : S \to D_2^2$ is a branched covering map of degree m,

(ii) $\partial S = Q_m \times \partial D_2^2$



A 2-dim braid is called <u>simple</u> if the branched covering $S \to D_2^2$ is simple.

Each branch value has a unique branch point of local deg $= 2$.

Motion picture

$S \subset D_1^2 \times D_2^2$: a 2-dim m-braid

$D_2^2 = I_3 \times I_4$

We assume I_4 is a time parameter.

For each $t \in I_4$, let

$$b_t \subset D^2 \times I = D_1^2 \times I_3$$

such that

$$b_t \times \{t\} = S \cap D_1^2 \times I_3 \times \{t\}.$$

The one-parameter family $\{b_t\}$ is called a motion picture of S.

Fact $\{b_t\}$ is a motion picture of S.

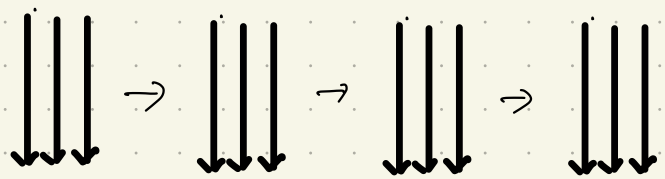
$\Rightarrow$

- $b_0 = b_1$ = trivial m-braid $Q_m \times I \subset D^2 \times I$
- b_t are m-braids except for some $t_1, t_2, \cdots, t_s \in I$, where b_{t_i} are singular braid

<u>Example</u> (trivial 2-dim braid)

$S = Q_m \times D_2^2$

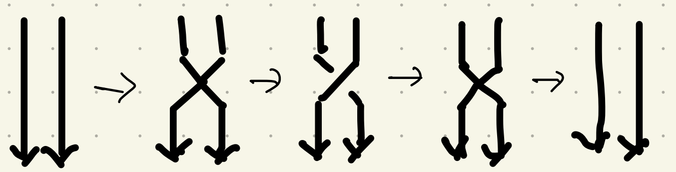
Then $b_t = Q_m \times I$, trivial braid



(m=3)

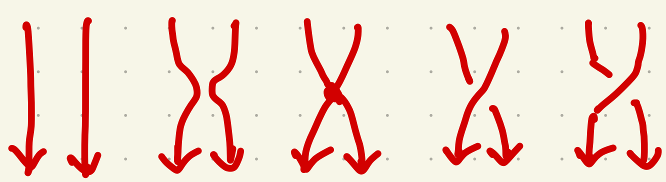
$t=0$ $t=1$

<u>Example</u> (m=2)

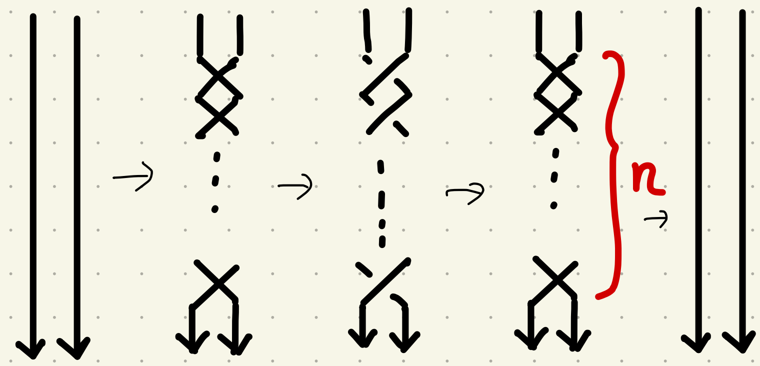


$t=0$ $t=\frac{1}{4}$ $t=\frac{3}{4}$ $t=1$

S is a 2-dim 2-braid with 2 branch points



<u>Example</u> (m=2)

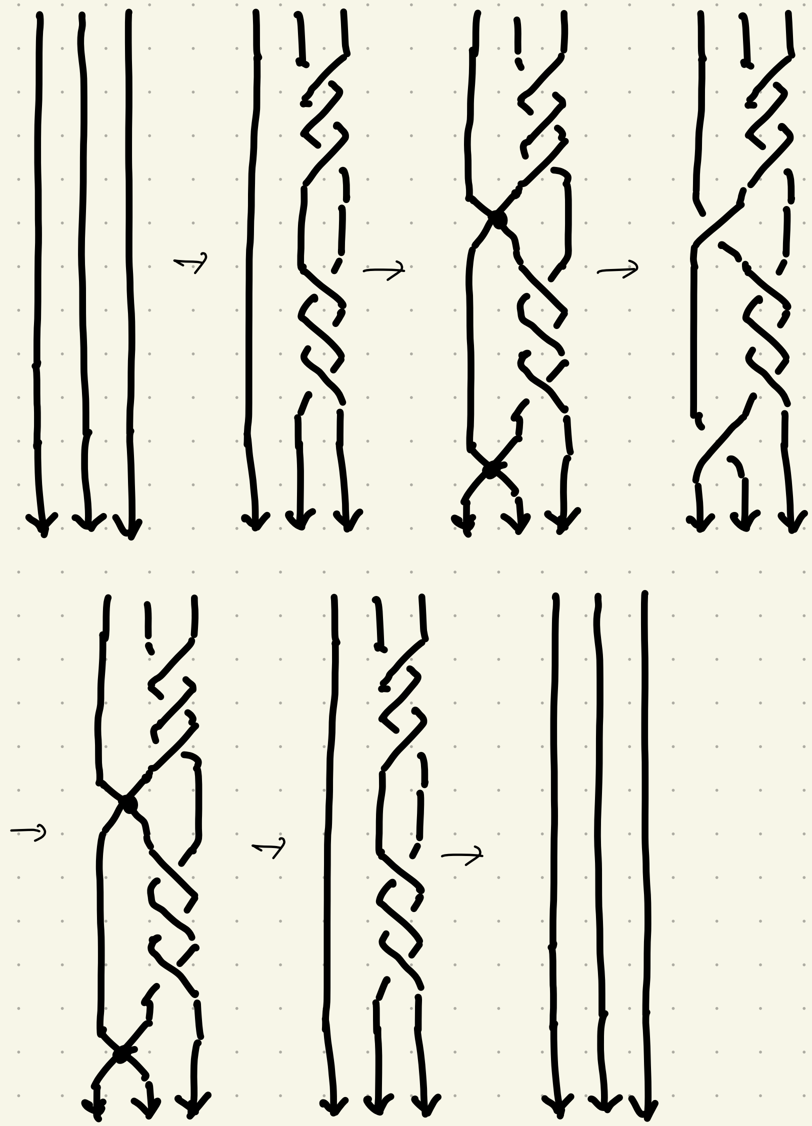


$t=0$ $t=1$

S is a 2-dim 2-braid with $2n$ branch points.

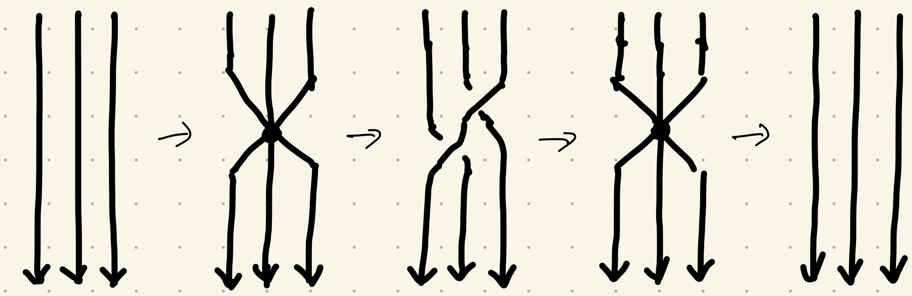
Any 2-dim 2-braid has an even number of branch points. It is equivalent to the above example.

Example ($m=3$)



S is a 2-dim 3-braid with 4 branch points.

Example ($m=3$)



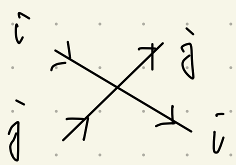
S is a <u>nonsimple</u> 2-dim 3-braid with 2 branch points.

Chart description of simple 2-dim braids

A chart of degree m or an m-chart is a graph in $I \times I$ whose edges are labeled by $\{1, 2, \dots, m-1\}$ and vertices are as follows:

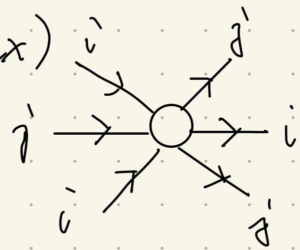
- (black vertex)

- (crossing)



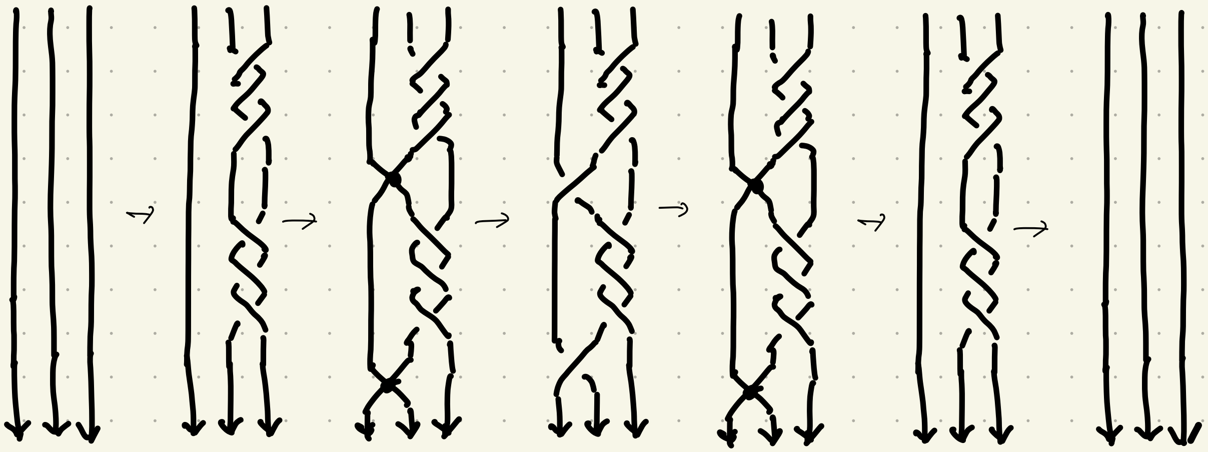
$|i-j|>1$

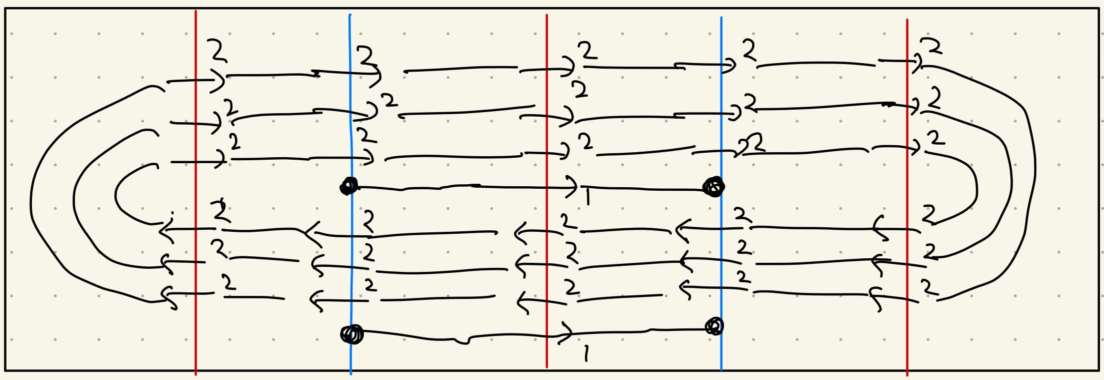
- (white vertex)

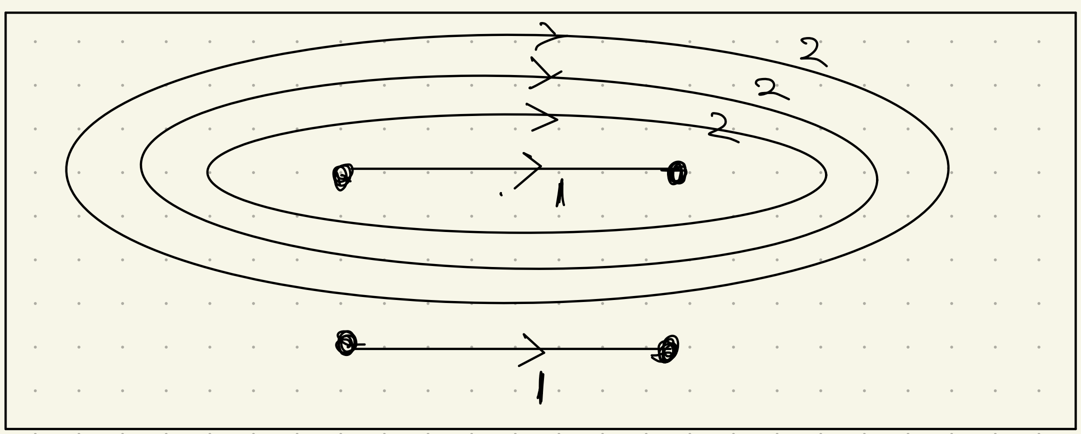


$|i-j|=1$

A motion picture $\{b_t\}$ of a simple 2-dim braid can be described by a chart.





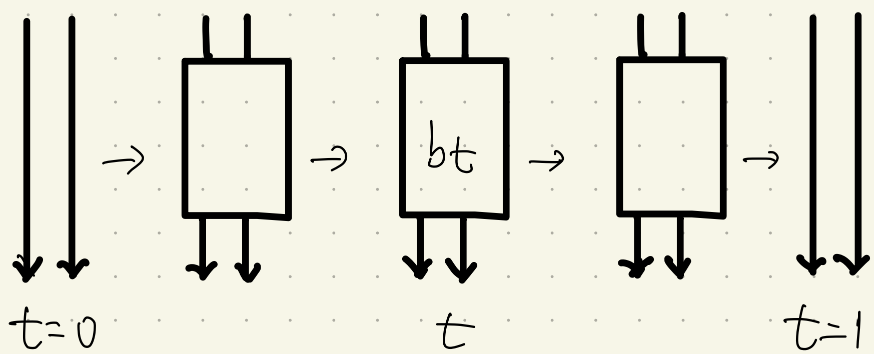


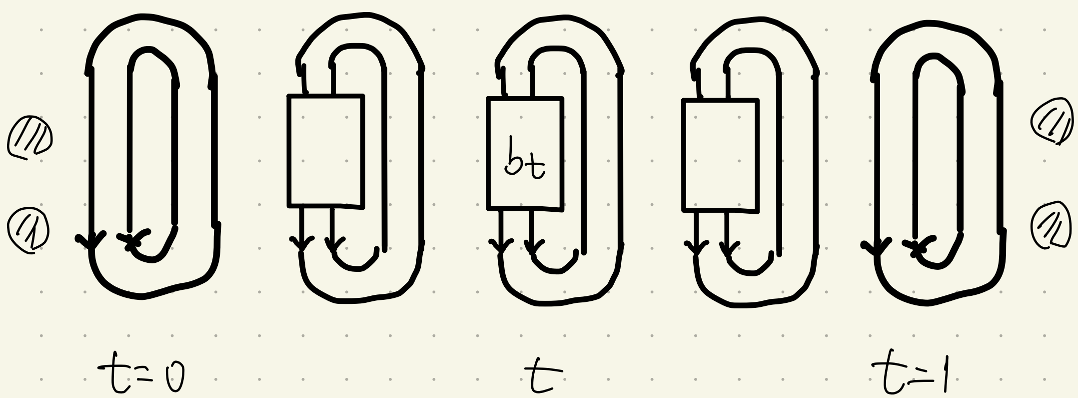
(m=3)

Alexander and Markov theorem in dimension four.

S : a 2-dim m-braid

$\{b_t\}_{t\in[0,1]}$: motion picture of S





$\hat{S}$: the closure of S, which is an oriented surface embedded in $\mathbb{R}^4$.

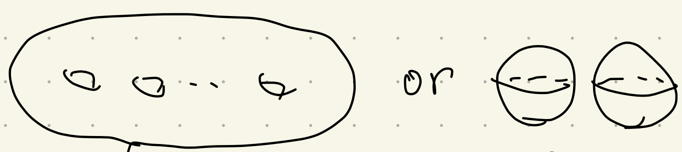
Thm (Viro, K.) Any oriented surface link is ambient isotopic to the closure of a (simple) 2-dimensional braid.

Def The braid index $\mathrm{Braid}(F)$ of an oriented surface link F is

$$\min\{\, m \mid F \cong \hat{S} \text{ for some simple 2-dim } m\text{-braid} \,\}$$

- $\mathrm{Braid}(F) = 1 \iff F$ is a trivial 2-knot.

- $\mathrm{Braid}(F) = 2 \iff F$ is a trivial surface link, which is

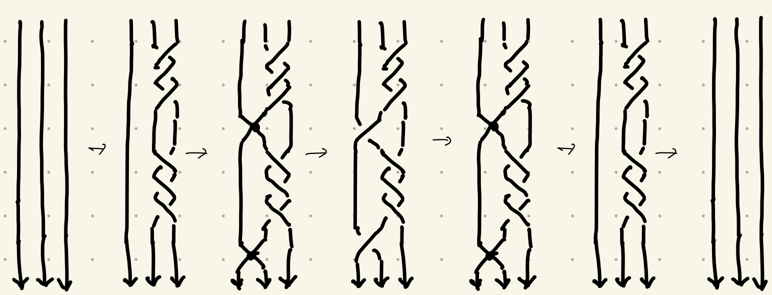


$g \geq 1$ or $S^2 \sqcup S^2$

- $\mathrm{Braid}(F) = 3 \Rightarrow F$ is a ribbon surface link.

Example F: the spun trefoil

$\Rightarrow \mathrm{Braid}(F) = 3$



Thm (K)

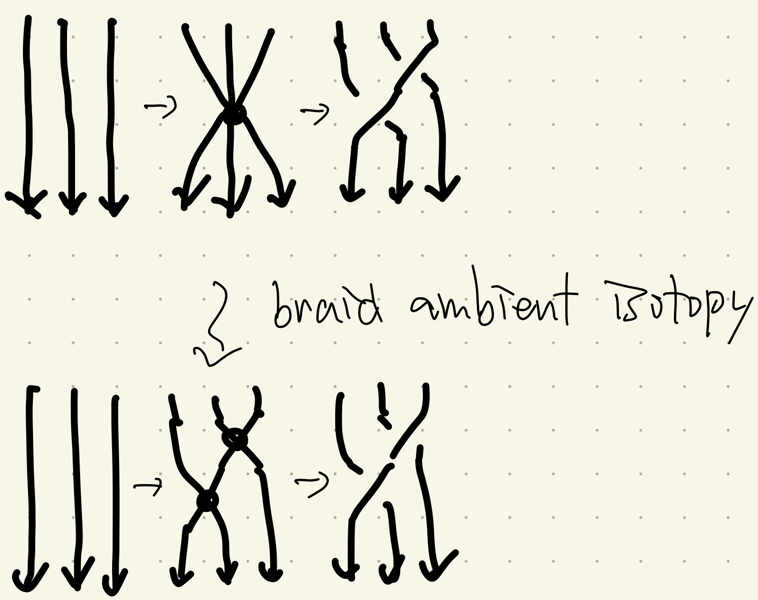
Two 2-dim braids S and S' present the same surface link if and only if they are related by a sequence of

0. braid ambient isotopy
1. conjugations
2. stabilizations
3. destabilizations

braid ambient isotopy

S and S' are braid ambient isotopic if they are ambient isotopic in $D_1^2 \times D_2^2$ keeping the conditions of 2-dim. braids.

A fusion/fission of branch points may occur.

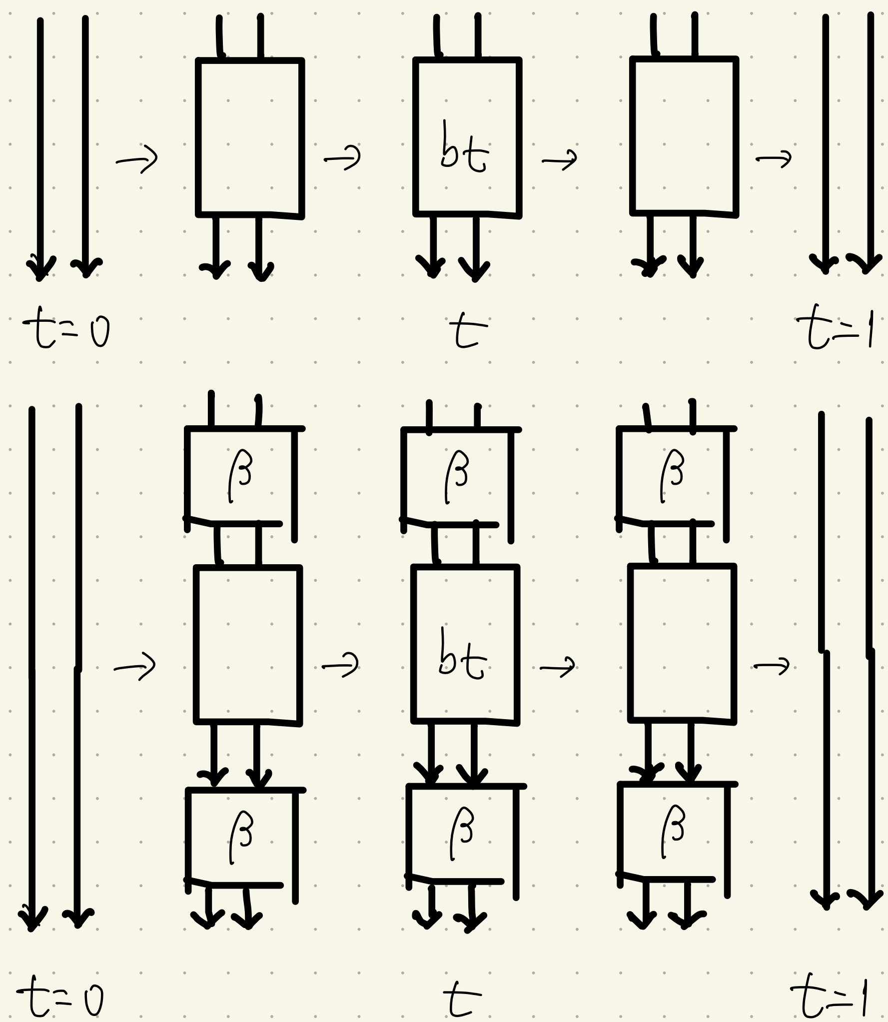


Conjugations

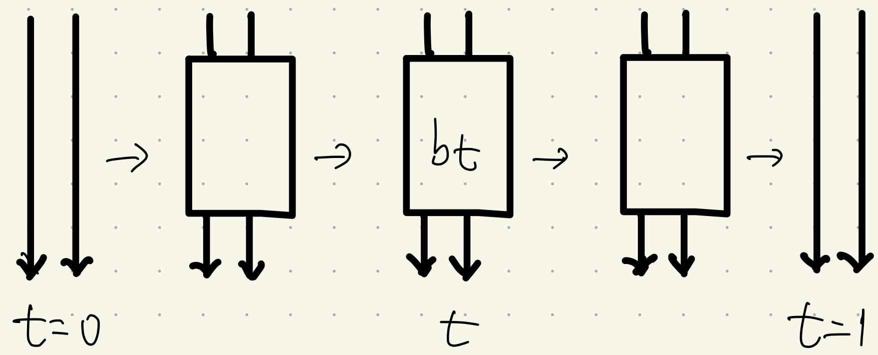
S: a 2-dim m-braid

β: a classical m-braid

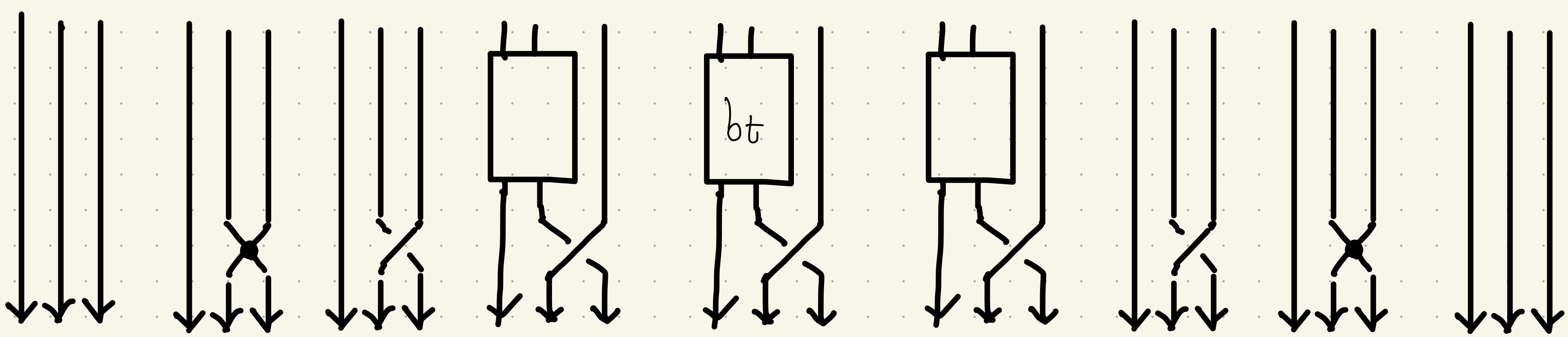
$\rightsquigarrow$ the conjugate of S by β



Stabilization



$\big\downarrow$ stabilization



<u>Thm</u> $\{$ 2-dim braids $\}/\underset{\text{Markov}}{\sim} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{oriented} \\ \text{surface links} \end{array} \right\}$