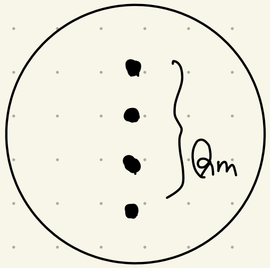
Braids I

ICTS program “Knots through web”



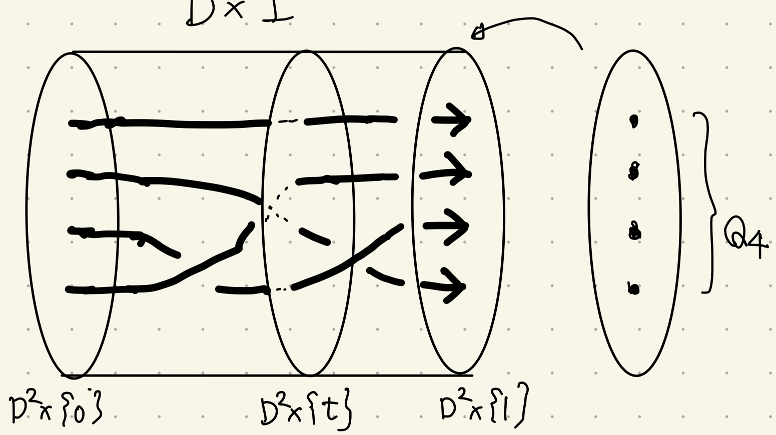
D^2

Q_m is a fixed set of m interior points of D^2

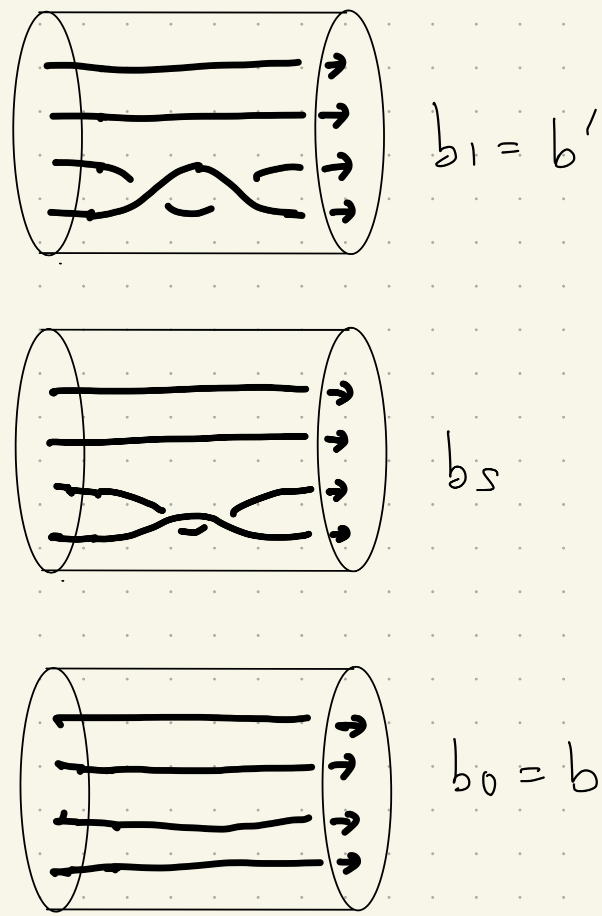
Def A <u>geometric m-braid</u> is the union $b = a_1 \cup \cdots \cup a_m$ of m strings a_i $(i=1,\cdots,m)$ in the cylinder $D^2 \times I$ such that

- for each $t \in I = [0,1]$, b intersects the 2-disk $D^2 \times \{t\}$ transversely in m distinct interior points of $D^2 \times \{t\}$,
- for $t \in \{0,1\}$, the intersection is Q_m.

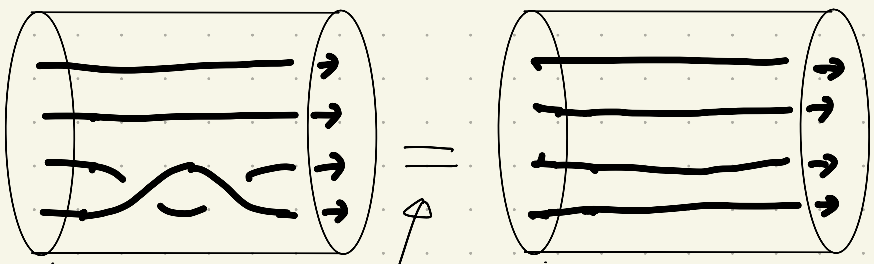
$D^2 \times I$



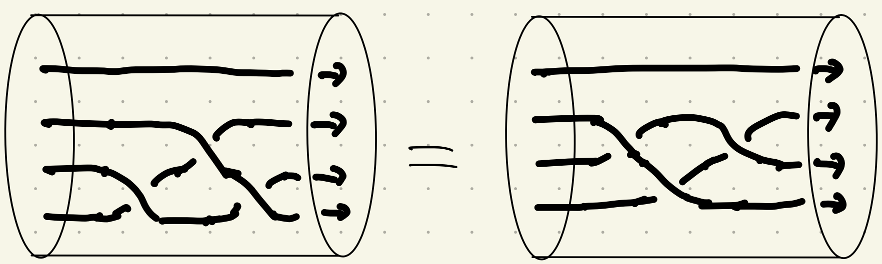
Two geometric m-braids b and b' are <u>equivalent</u> if there exists a continuous sequence of geometric m-braids b_s $(s \in [0,1])$ with $b_0 = b$ and $b_1 = b'$.



An <u>m-braid</u> is an equivalence class of geometric m-braids.

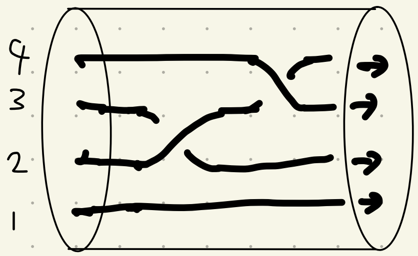


"equal" as a braid
"the same"

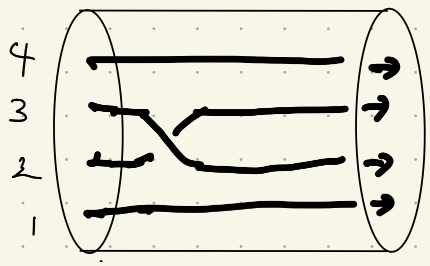


We often use the same symbol b for a geometric braid and the braid represented by it.

<u>Def</u> The <u>m-braid group</u> or the <u>braid group of degree m</u> is the group B_m consisting of m-braids

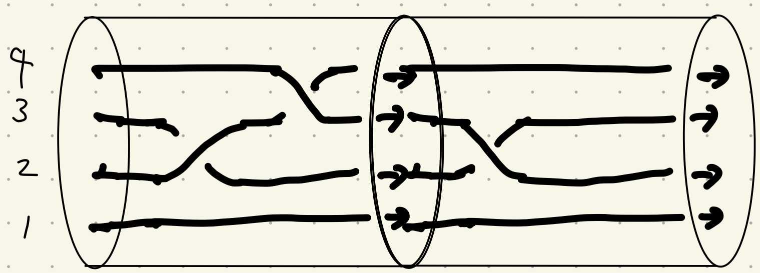


b



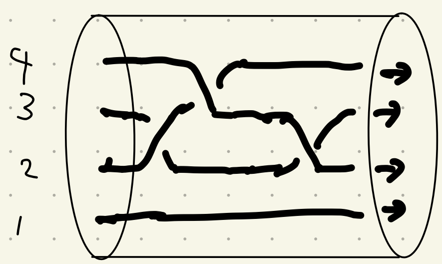
b′

⇓



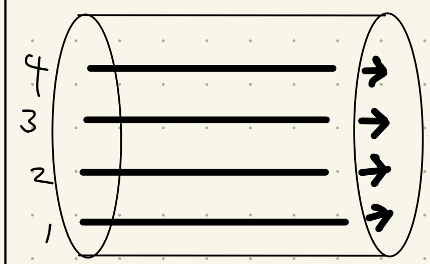
b b′

⇓



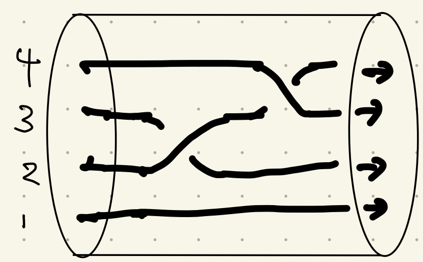
b · b′

The product of b and b′.

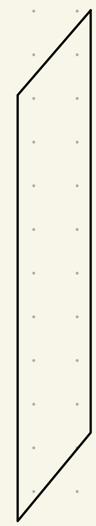


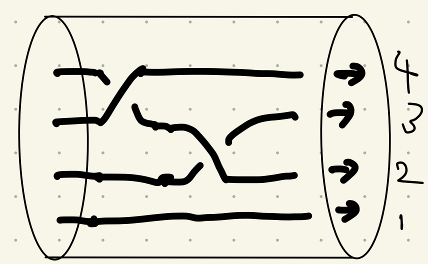
identity of Bm

trivial braid



b





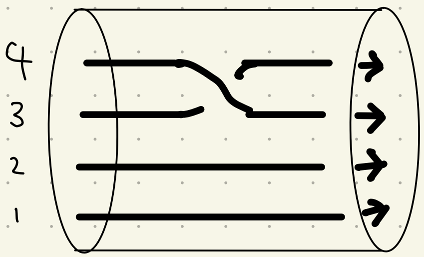
b^{-1}

inverse

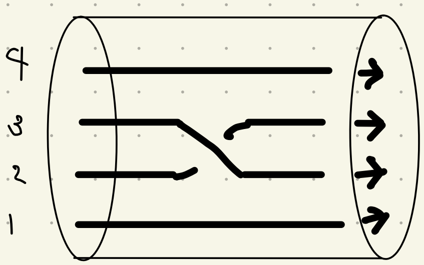
Standard generators .

Artin's generators

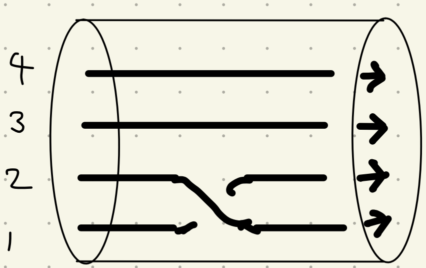
$m=4$



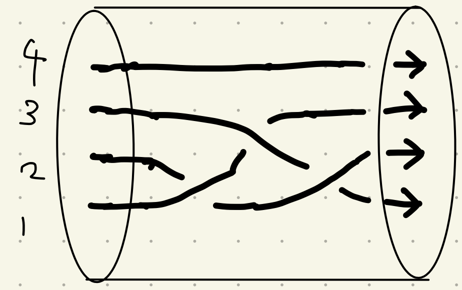
σ_3



σ_2



σ_1



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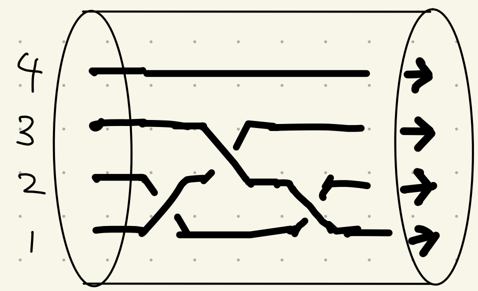
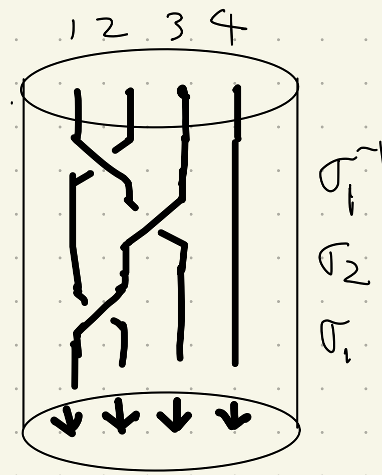
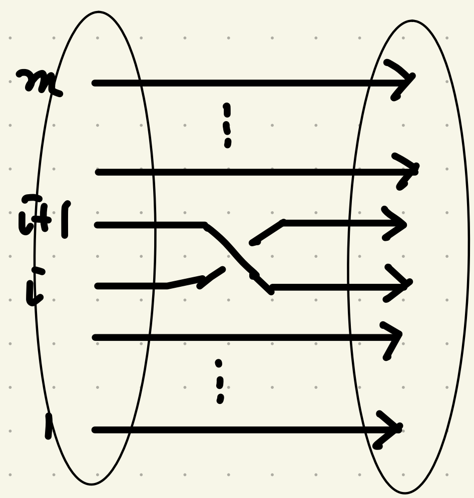


$\sigma_1^{-1}\sigma_2\sigma_1$



B_m is generated by $\sigma_1, \sigma_2, \cdots, \sigma_{m-1}$



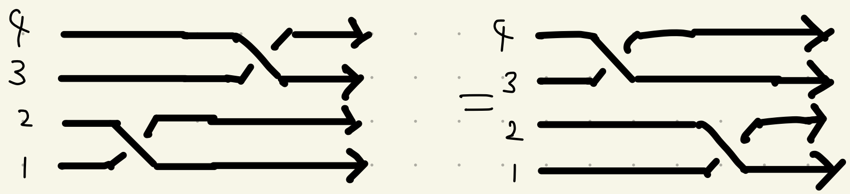
σ_i

Braid relations

If $|i-j|>1$, then $\sigma_i \sigma_j = \sigma_j \sigma_i$

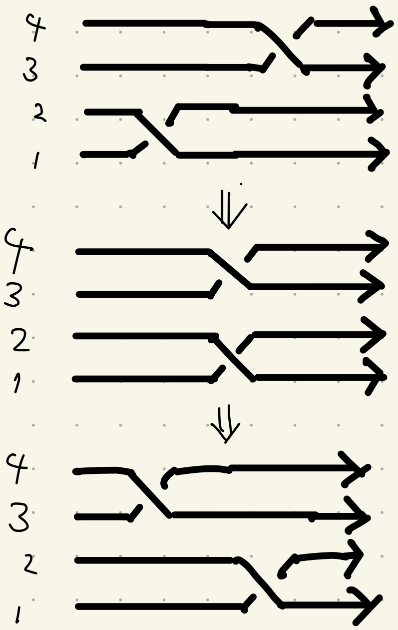
If $|i-j|=1$, then $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$

eg $(m=4)$ $\sigma_1 \sigma_3 = \sigma_3 \sigma_1$

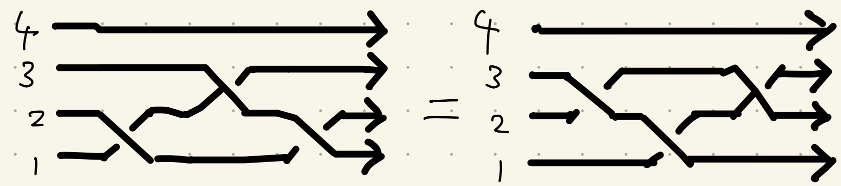


$\sigma_1 \quad \sigma_3 \qquad\qquad \sigma_3 \quad \sigma_1$

pf

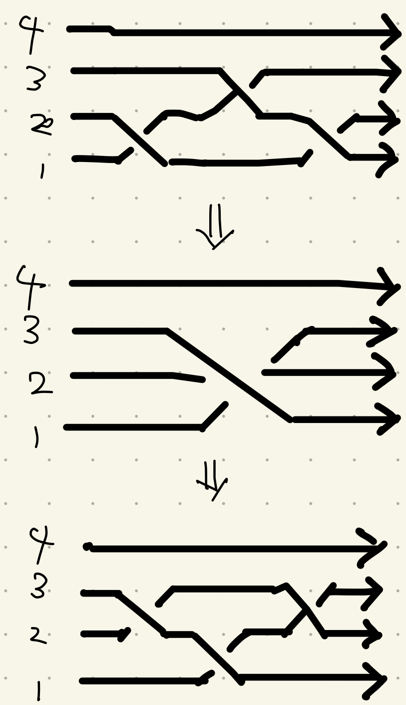


$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \quad (m=4)$

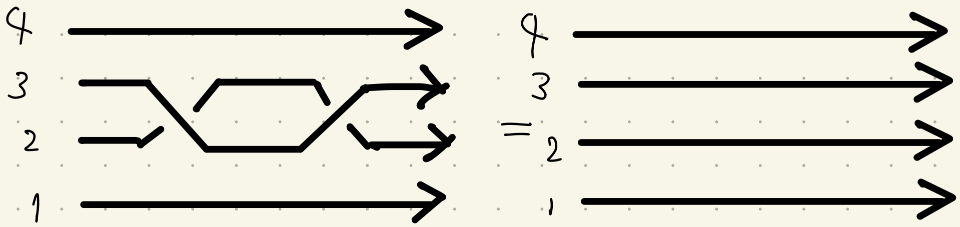


$\sigma_1 \; \sigma_2 \; \sigma_1 \qquad\qquad \sigma_2 \; \sigma_1 \; \sigma_2$

pf



Trivial relations $\begin{cases} \sigma_i \sigma_i^{-1} = 1 \\ \sigma_i^{-1} \sigma_i = 1 \end{cases}$



$\sigma_2 \cdot \sigma_2^{-1}$ $\qquad\qquad$ 1

Thm (Artin)

The braid group B_m has a group presentation as follows:

generating set — standard generators $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$

defining relations — braid relations

$\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j|>1$

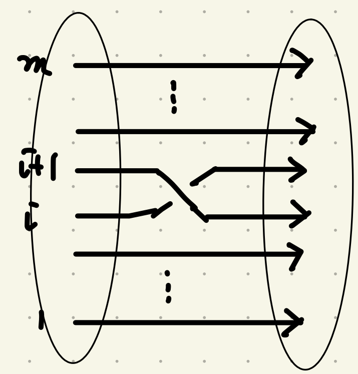
$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for $|i-j|=1$

Algebraic definition of the braid group B_m

$$B_m = \left\langle \sigma_1, \ldots, \sigma_{m-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j|>1 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ for } |i-j|=1 \end{array} \right\rangle$$

Its geometric interpretation is given by

$\sigma_i \longleftrightarrow$

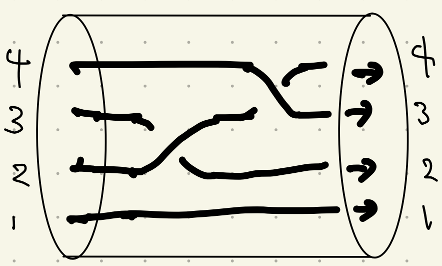


σ_i

Symmetric group and pure braid group

S_m : the symmetric group on m letters
$\quad\|$
the group of permutations of $\{1, 2, \cdots, m\}$

$\pi : B_m \longrightarrow S_m$



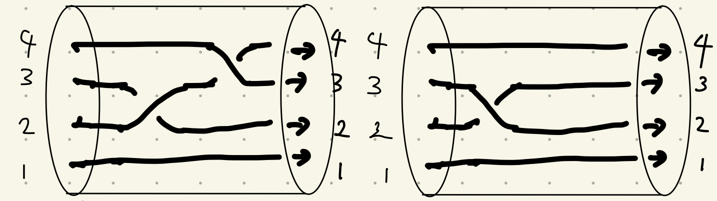
$b \in B_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$\|$$

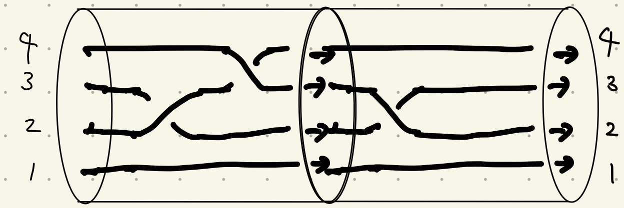
$$\begin{bmatrix} 4 \mapsto 3 \\ 3 \mapsto 2 \\ 2 \mapsto 4 \\ 1 \mapsto 1 \end{bmatrix}$$

$\pi(b) \in S_4$



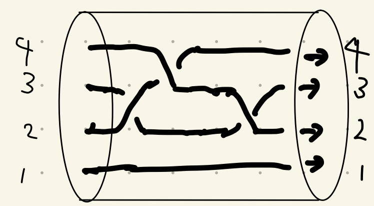
$b \qquad b'$

$$\Downarrow$$



$b \qquad b'$

$$\Downarrow$$



$b \cdot b'$

The product of b and b'.

$\pi(b) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \qquad \pi(b') = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

$$\pi(b \cdot b') = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = \pi(b) \cdot \pi(b')$$

π is a homomorphism.

Def An m-braid $b \in B_m$ is a <u>pure braid</u> if $\pi(b)$ is the identity.

The <u>pure braid group</u> P_m is the subgroup of B_m consisting of pure braids.

By definition, $B_m = \mathrm{Ker}[\pi : B_m \to S_m]$.

We have a short exact sequence

$$1 \longrightarrow P_m \xrightarrow{\iota} B_m \xrightarrow{\pi} S_m \longrightarrow 1.$$

A graphical method of computing braid words

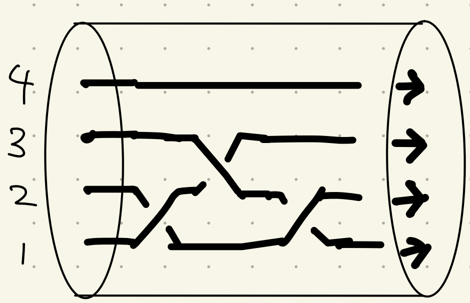
By Artin's theorem, any braid $b \in B_m$ can be presented by a word in the standard generators

$$\sigma_1, \sigma_2, \cdots, \sigma_{m-1}$$

and their inverse elements

$$\sigma_1^{-1}, \sigma_2^{-1}, \cdots, \sigma_{m-1}^{-1}.$$

For example,



$\sigma_1^{-1}\sigma_2\sigma_1$

$\sigma_1^{-1} \cdot \sigma_2 \cdot \sigma_1$

We call such a word a <u>braid word</u>.

Two braid words w and w' present the same m-braid if and only if there is a finite sequence

$$w_0, w_1, \cdots, w_n$$

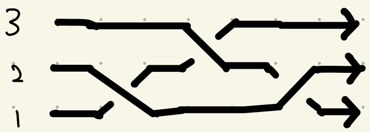
of braid words with $w_0 = w$, $w_n = w'$ such that each w_k $(k=1, \cdots, n)$ is obtained from w_{k-1} by one of the following.

(1) Replace $\sigma_i\sigma_j$ with $\sigma_j\sigma_i$ for $|i-j|>1$

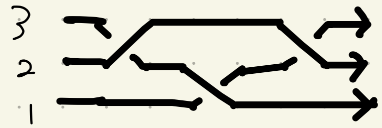
(2) Replace $\sigma_i\sigma_j\sigma_i$ with $\sigma_j\sigma_i\sigma_j$ for $|i-j|=1$

(3) Insert or delete $\sigma_i^{-1}\sigma_i$ or $\sigma_i\sigma_i^{-1}$.

Example $\quad m=3$



$\sigma_1\sigma_2\sigma_1^{-1}$



$\sigma_2^{-1}\sigma_1\sigma_2$

$w_0 = \sigma_1\sigma_2\sigma_1^{-1}$

$\downarrow$

$w_1 = \sigma_2^{-1}\sigma_2\,\sigma_1\sigma_2\sigma_1^{-1}$ insert $\sigma_2^{-1}\sigma_2$

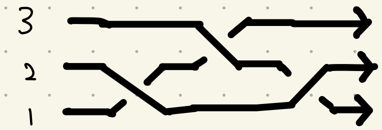
$\downarrow$

$w_2 = \sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_1^{-1}$ replace $\sigma_2\sigma_1\sigma_2$ with $\sigma_1\sigma_2\sigma_1$

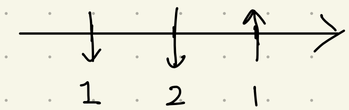
$\downarrow$

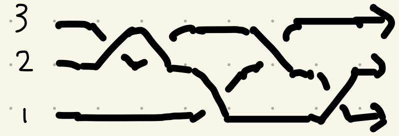
$w_3 = \sigma_2^{-1}\sigma_1\sigma_2$ delete $\sigma_1\sigma_1^{-1}$

We consider this sequence of braid words using a graph called a <u>chart</u>.

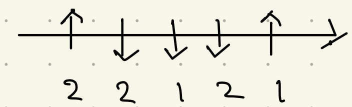


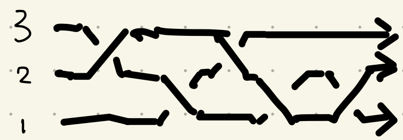
$\sigma_1\ \sigma_2\ \sigma_1^{-1}$



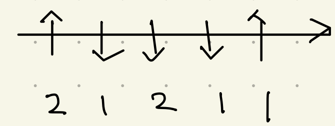


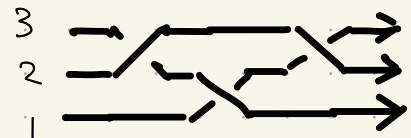
$\sigma_2^{-1}\sigma_2\ \sigma_1\ \sigma_2\ \sigma_1^{-1}$



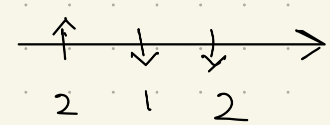


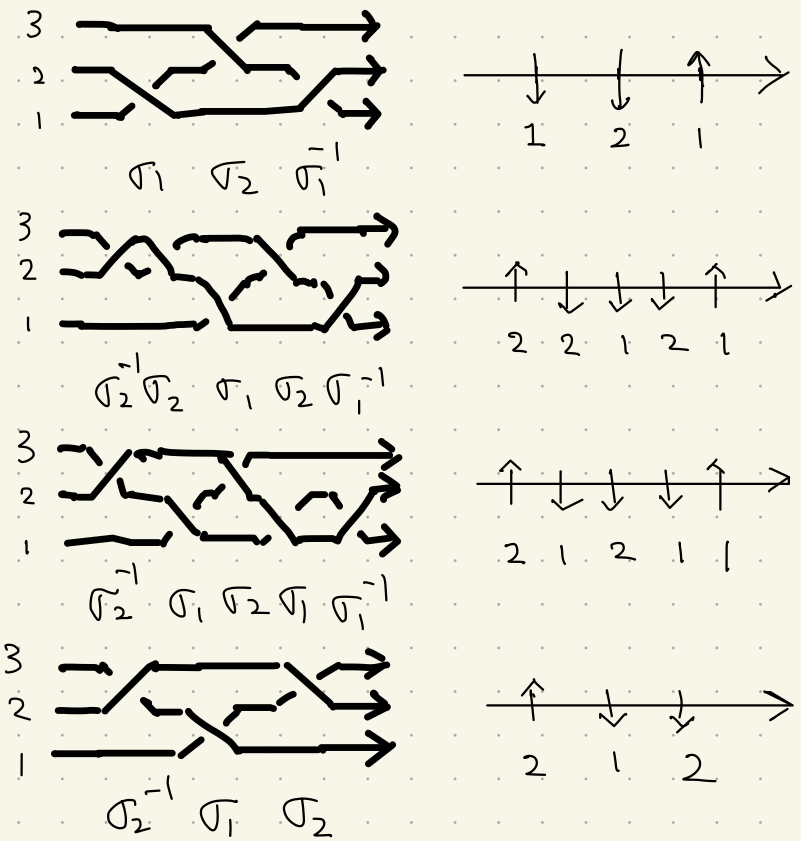
$\sigma_2^{-1}\ \sigma_1\ \sigma_2\ \sigma_1\ \sigma_1^{-1}$

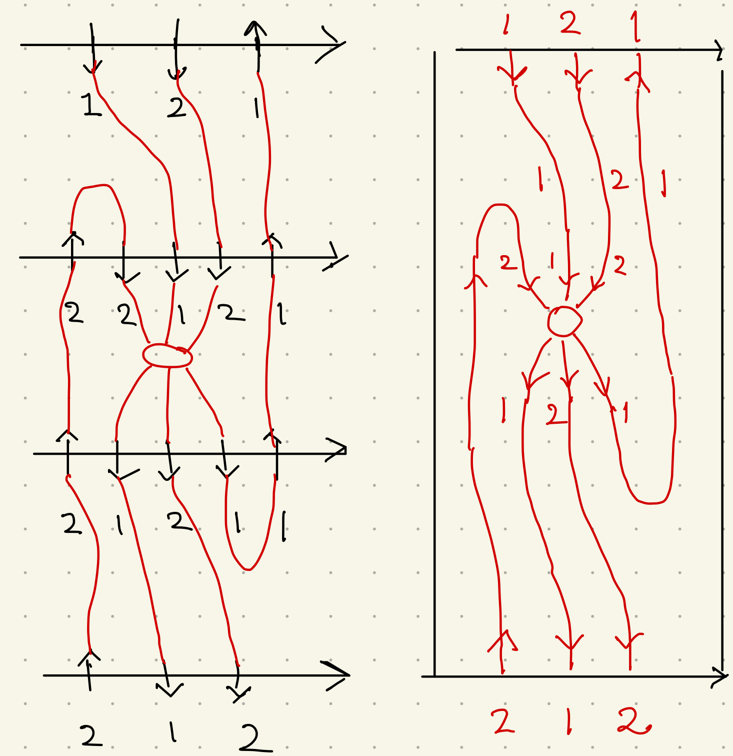




$\sigma_2^{-1}\ \sigma_1\ \sigma_2$

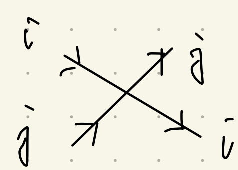
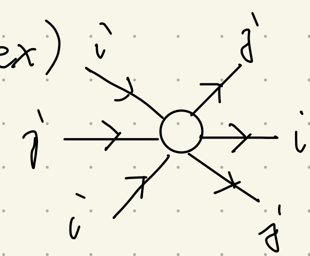






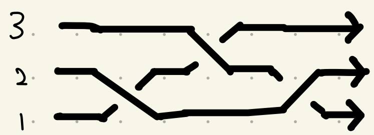
chart

A <u>chart</u> of degree m or an <u>m-chart</u> is a graph in $I \times I$ whose edges are labeled by $\{1, 2, \cdots, m-1\}$ and vertices are as follows:

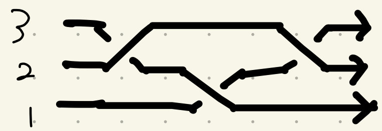
- (crossing)  $|i-j|>1$
- (white vertex)  $|i-j|=1$

Thm Two braid words w and w' present the same m-braid if and only if there is an m-chart in $I \times I$ connecting w and w'.

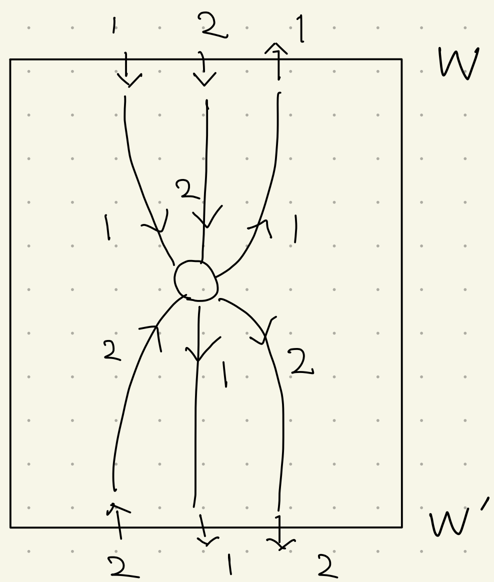
Example $m=3$



$W = \sigma_1 \sigma_2 \sigma_1^{-1}$



$W' = \sigma_2^{-1} \sigma_1 \sigma_2$



Proof ($\Rightarrow$) Suppose w and w' present the same m-braid.

$\exists$ $W = W_0, W_1, \cdots, W_n = W'$

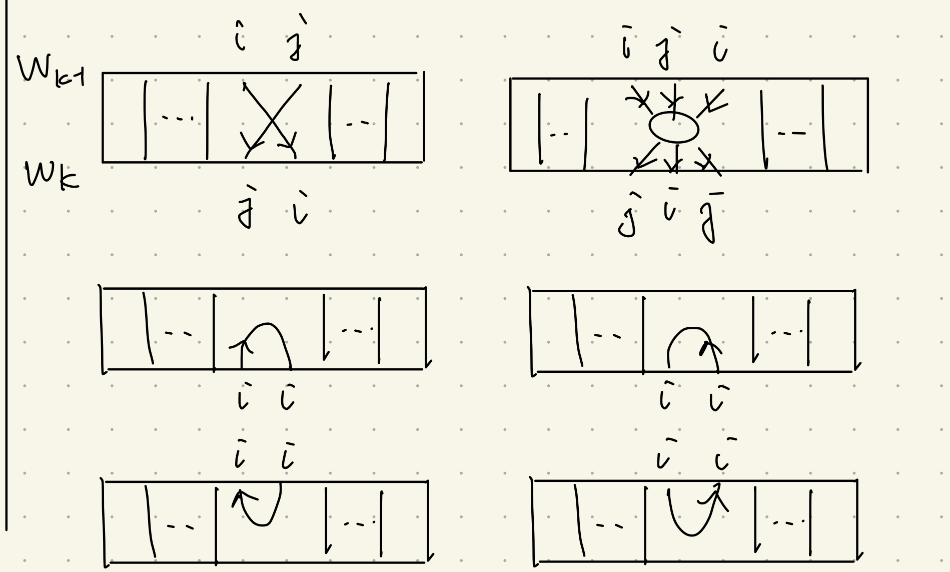
such that each W_k $(k=1,\cdots,n)$ is obtained from W_{k-1} by

(1) Replace $\sigma_i \sigma_j$ with $\sigma_j \sigma_i$ for $|i-j|>1$

(2) Replace $\sigma_i \sigma_j \sigma_i$ with $\sigma_j \sigma_i \sigma_j$ for $|i-j|=1$

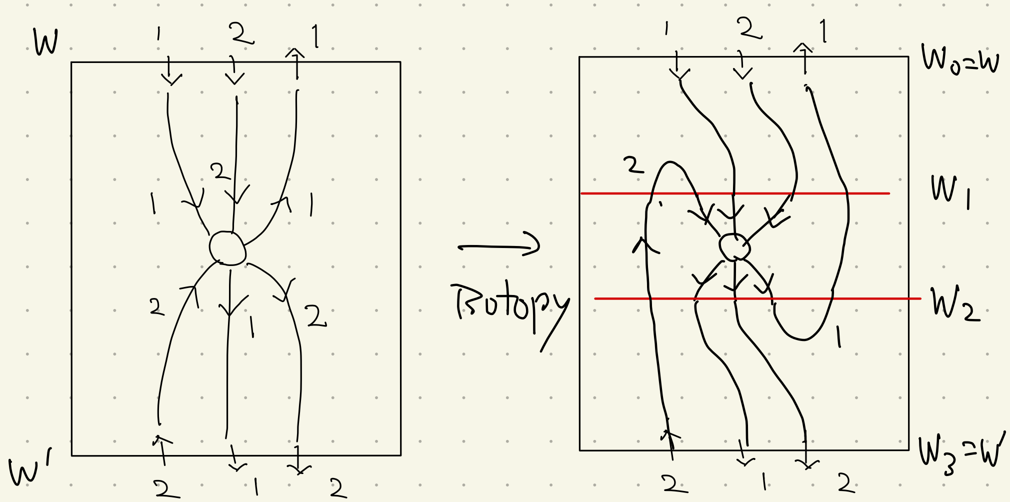
(3) Insert or delete $\sigma_i^{-1}\sigma_i$ or $\sigma_i \sigma_i^{-1}$.

Consider a chart connecting W_{k-1} and W_k according to (1) ~ (3).



Combining these pieces, we obtain a chart connecting W and W'.

($\Leftarrow$) Suppose that $\exists$ a chart conneting W and W'. By Rotopy of $I \times I$, deform it so that it is a combination of the foundamental pieces.



Then we obtain a sequence

$$W = W_0, W_1, \cdots, W_n = W'$$

such that each W_k $(k=1,\cdots,n)$ is obtained from W_{k-1} by

(1) Replace $\sigma_i\sigma_j$ with $\sigma_j\sigma_i$ for $|i-j|>1$

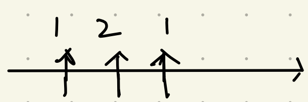
(2) Replace $\sigma_i\sigma_j\sigma_i$ with $\sigma_j\sigma_i\sigma_j$ for $|i-j|=1$

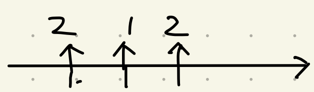
(3) Insert or delete $\sigma_i^{-1}\sigma_i$ or $\sigma_i\sigma_i^{-1}$.

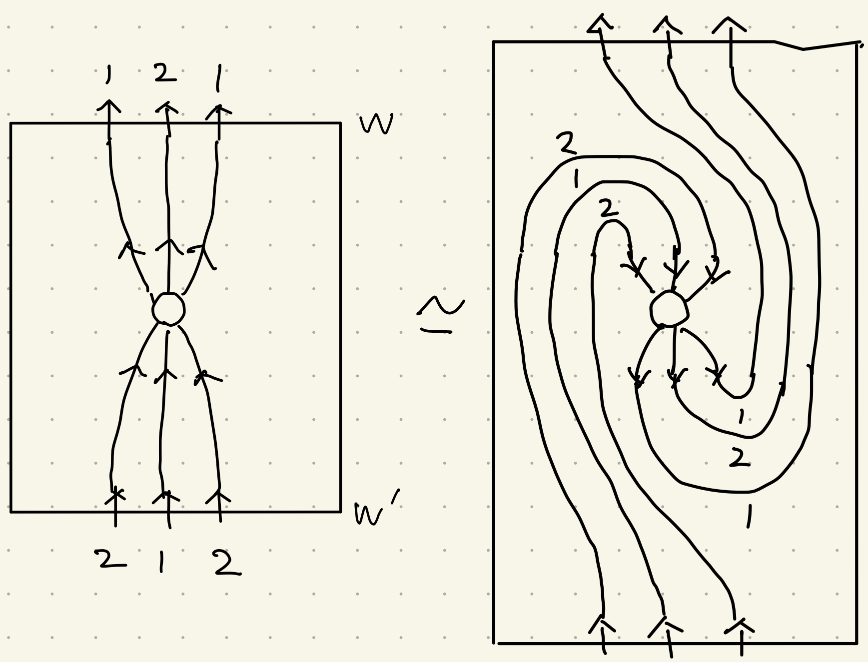
$\square$

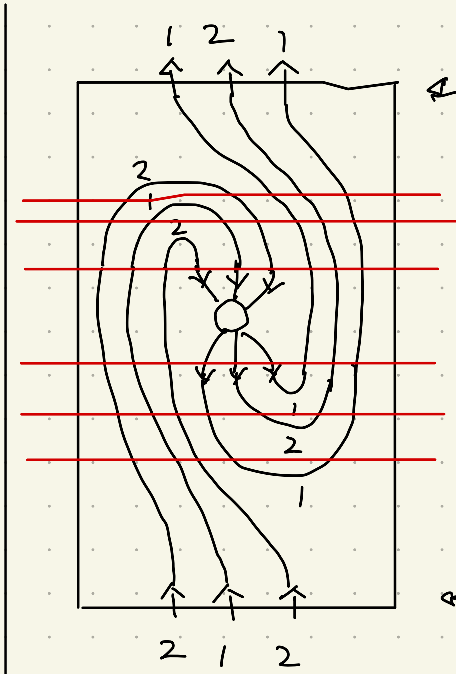
Example $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$

Find a sequence of braid words connecting $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$.

$\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} =$ 

$\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} =$ 





$\bar{1}\,\bar{2}\,\bar{1}$

$\bar{2}\,2\,\bar{1}\,\bar{2}\,\bar{1}$

$\bar{2}\,\bar{1}\,1\,2\,\bar{1}\,\bar{2}\,\bar{1}$

$\bar{2}\,\bar{1}\,\bar{2}\,2\,1\,2\,\bar{1}\,\bar{2}\,\bar{1}$

$\bar{2}\,\bar{1}\,\bar{2}\,1\,2\,1\,\bar{1}\,\bar{2}\,\bar{1}$

$\bar{2}\,\bar{1}\,\bar{2}\,1\,2\,\bar{2}\,\bar{1}$

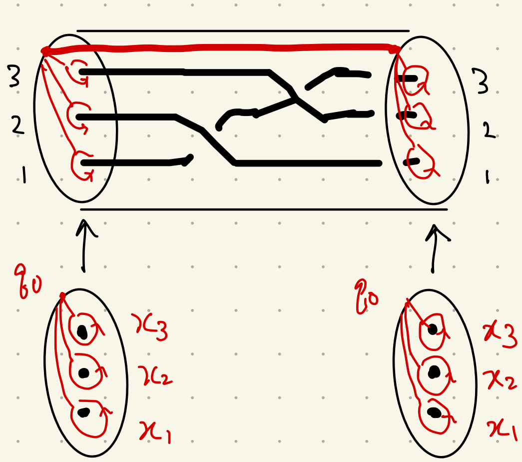
$\bar{2}\,\bar{1}\,\bar{2}\,1\,\bar{1}$

$\bar{2}\,\bar{1}\,\bar{2}$

Artin's automorphism

$F_m = \langle x_1, x_2, \dots, x_m \rangle$ the free group

$b \in B_m \mapsto \mathrm{Artin}(b): F_m \to F_m$



$x_1 \mapsto x_1 x_2 x_3 x_2^{-1} x_1^{-1}$

$x_2 \mapsto x_1$

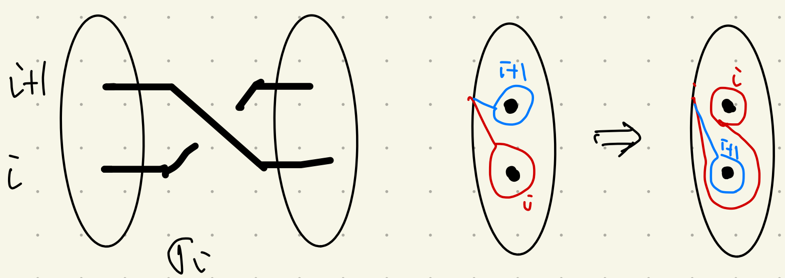
$x_3 \mapsto x_2$

Artin(b) is an automorphism of F_m, which is called <u>Artin's automorphism of b</u>

(1) $\mathrm{Artin}(\sigma_i)$: $x_i \mapsto x_i x_{i+1} x_i^{-1}$

$x_{i+1} \mapsto x_i$

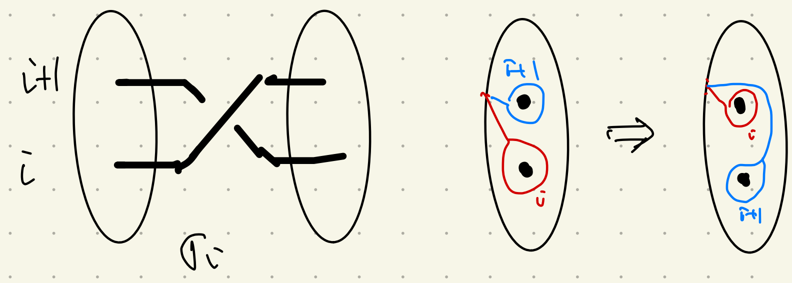
$x_k \mapsto x_k \quad (k \neq i, i+1)$



(2) $\mathrm{Artin}(\sigma_i^{-1})$: $x_i \mapsto x_{i+1}$

$x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}$

$x_k \mapsto x_k \quad (k \neq i, i+1)$



(3) $\mathrm{Artin}(b_1 b_2) = \mathrm{Artin}(b_2) \circ \mathrm{Artin}(b_1)$

By (1), (2), (3), we can compute Artin (b) algebraically when b is presented by standard generators.

<u>Example</u> $m=3$, $b = \sigma_1 \sigma_2$

$\mathrm{Artin}(\sigma_1): \begin{cases} x_1 \mapsto x_1 x_2 x_1^{-1} \\ x_2 \mapsto x_1 \\ x_3 \mapsto x_3 \end{cases}$ $\quad \mathrm{Artin}(\sigma_2): \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_2 x_3 x_2^{-1} \\ x_3 \mapsto x_2 \end{cases}$

$$\mathrm{Artin}(\sigma_1\sigma_2): \begin{cases} x_1 \overset{\sigma_1}{\mapsto} x_1 x_2 x_1^{-1} \overset{\sigma_2}{\mapsto} x_1 x_2 x_3 x_2^{-1} x_1^{-1} \\ x_2 \overset{\sigma_1}{\mapsto} x_1 \overset{\sigma_2}{\mapsto} x_1 \\ x_3 \overset{\sigma_1}{\mapsto} x_3 \overset{\sigma_2}{\mapsto} x_2 \end{cases}$$

Theorem (Artin)

Let b and $b' \in B_m$.

$$b = b' \iff \mathrm{Artin}(b) = \mathrm{Artin}(b').$$

In particular,

$$b = 1 \iff \mathrm{Artin}(b) = \mathrm{id}$$

Rem $\mathrm{Aut}^R(F_m)$: the group of automorphism of F_m, where

$$gh := h \circ g$$

Then
$$\begin{array}{rccc} \mathrm{Artin} : & B_m & \to & \mathrm{Aut}^R(F_m) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & b & \mapsto & \mathrm{Artin}(b) \end{array}$$

is an injective homomorphism.

Rem

Define $\mathrm{artin}(b) : F_m \to F_m$ by $\mathrm{Artin}(b)^{-1}$,

① $\mathrm{artin}(\sigma_i)$: $x_i \mapsto x_{i+1}$

$x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}$

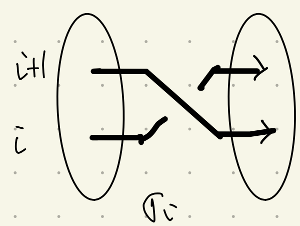
$x_k \mapsto x_k \quad (k \neq i, i+1)$

② $\mathrm{artin}(\sigma_i^{-1})$: $x_i \mapsto x_i x_{i+1} x_i^{-1}$

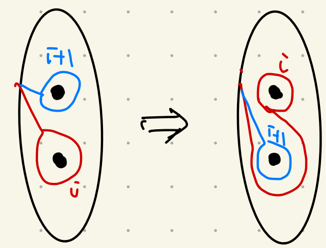
$x_{i+1} \mapsto x_i$

$x_k \mapsto x_k \quad (k \neq i, i+1)$

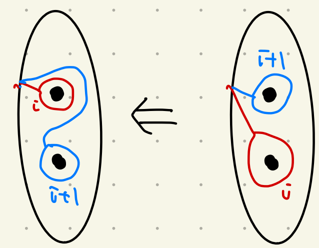
③ $\mathrm{artin}(b_1 b_2) = \mathrm{artin}(b_1) \circ \mathrm{artin}(b_2)$



$\mathrm{Artin}(\sigma_i)$



$\mathrm{artin}(\sigma_i)$



Then $\mathrm{artin} : B_m \to \mathrm{Aut}^L(F_m)$ is an injective homomorphism.

Racks and Quandles

Def A set with a binary operator $*$ is called a <u>rack</u> or a <u>quandle</u>, resp. if the following conditions (2) and (3), or (1), (2) and (3), are satisfied.

(1) $\forall a \in X$, $a*a=a$

(2) $\forall a, b \in X$, $\exists! c \in X$, $c*b=a$

(3) $\forall a, b, c \in X$, $(a*b)*c=(a*c)*(b*c)$

(right self-distributivity)

Example (dihedral quandle)

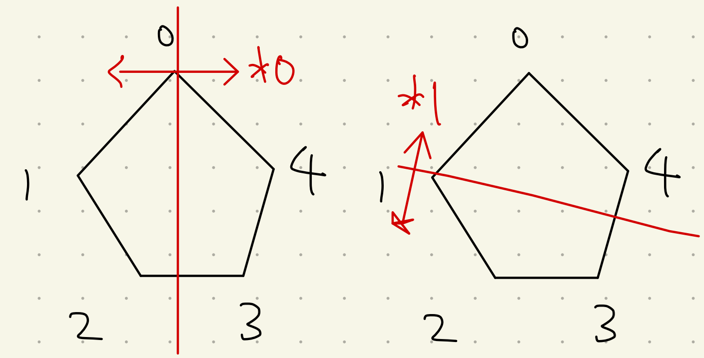
$X = \mathbb{Z}/m\mathbb{Z}$

$a*b = 2b-a$

$m=5$

$0*0=0$
$1*0=4$
$2*0=3$
$3*0=2$
$4*0=1$

$*$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4



Example (conjugation quandle)

$X = G$ a group

$a*b = b^{-1}ab$ (or $a*b = bab^{-1}$)

The dual operation

$X = (X, *)$: a quandle

Define $\overline{*} : X \to X$ by $a \overline{*} b = c$ if (and only if) $c*b=a$

Then $(X, \overline{*})$ is also a quandle.

We call $\overline{*}$ the dual of $*$, and $(X, \overline{*})$ the dual of $(X, *)$.

Def A <u>kei</u> (圭) or an <u>involutory quandle</u> is a quandle with $(a*b)*b=a$. (or $a*b = a\overline{*}b$).

Free rack and free quandle

Let $\mathbb{X} = \{x_1, x_2, \cdots, x_m\}$

The free rack $FR(\mathbb{X})$ (or free quandle $FQ(\mathbb{X})$) is a unique rack (quandle) generated by $\mathbb{X}$ satisfying the following universal property.

- For any rack (quandle) Y and for any mapping $f: \mathbb{X} \to Y$, there is a rack (quandle) homomorphism $\tilde{f}: FR(\mathbb{X}) \to Y$ ($\tilde{f}: FQ(\mathbb{X}) \to Y$) with $\tilde{f}(x_i) = f(x_i)$ for $i = 1, \cdots, m$.

$$\begin{tikzcd} \mathbb{X} \arrow[r] \arrow[dr, "f"'] & FR(\mathbb{X})\ (FQ(\mathbb{X})) \arrow[d, "\tilde{f}"] \arrow[dl, phantom, "\circlearrowright"] \\ & Y \end{tikzcd}$$

$E = \{$ all expressions on $\mathbb{X}\}$

e.g. $(x_3 * x_2) \bar{*} ((x_1 \bar{*} x_5) * x_7)$

$FR(\mathbb{X}) = E /$ rack relations

$$(A * B) \bar{*} B = A$$
$$(A \bar{*} B) * B = A$$
$$(A * B) * C = (A * C) * (B * C)$$

$FQ(\mathbb{X}) = E /$ quandle relations $\quad A * A = A$

- FR-model of $FR(\mathbb{X})$ Fenn–Rourke

  $FR(\mathbb{X}) = \mathbb{X} \times FG(\mathbb{X})$

  $(x, w) * (y, u) := (x, w u^{-1} y u)$

- A model of $FQ(\mathbb{X})$

  $FQ(\mathbb{X}) = \mathbb{X} \times FG(\mathbb{X}) / \sim$

  $[(x, w)] = [x, x^n w] \quad (n \in \mathbb{Z})$

  $[(x, w)] * [(y, u)] = [x, w u^{-1} y u]$

- Another model

  $FQ(\mathbb{X}) = \{$ conjugates of $x_i \mid i = 1, 2, \cdots, m\} \subset FG(\mathbb{X})$

  $w^{-1} x_i w * u^{-1} x_j u = (u^{-1} x_j u)^{-1} w^{-1} x_i w (u^{-1} x_j u)$

Artin like automorphism

$b \in B_m \rightsquigarrow \mathrm{Artin}(b) \in \mathrm{Aut}(FR(X))$
or $\mathrm{Aut}(FQ(X))$

It is defined algebraically by the following

① $\mathrm{Artin}(\sigma_i)$: $\begin{cases} x_i \mapsto x_{i+1} \bar{*} x_i \\ x_{i+1} \mapsto x_i \\ x_k \mapsto x_k \end{cases}$

② $\mathrm{Artin}(\sigma_i^{-1})$, $\begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i * x_{i+1} \\ x_k \mapsto x_k \end{cases}$

③ $\mathrm{Artin}(b_1 b_2) = \mathrm{Artin}(b_2) \circ \mathrm{Artin}(b_1)$

Theorem

Let b and $b' \in B_m$.
Let $\mathrm{Artin}(b)$, $\mathrm{Artin}(b') \in \mathrm{Aut}(FR(X))$
or $\mathrm{Aut}(FQ(X))$

$$b = b' \iff \mathrm{Artin}(b) = \mathrm{Artin}(b').$$

In particular,

$$b = 1 \iff \mathrm{Artin}(b) = id$$

$b \in B_m \mapsto \mathrm{artin}(b) \in \mathrm{Aut}(FR(X))$
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Hurwitz action of B_m

G: a group

$G^m = \{(g_1, \cdots, g_m) \mid g_1, \cdots, g_m \in G\}$

$b \in B_m$, $\quad P(b): G^m \to G^m$

① $P(\sigma_i)$: $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_m)$

$$\downarrow$$

$(g_1, \cdots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \cdots, g_m)$

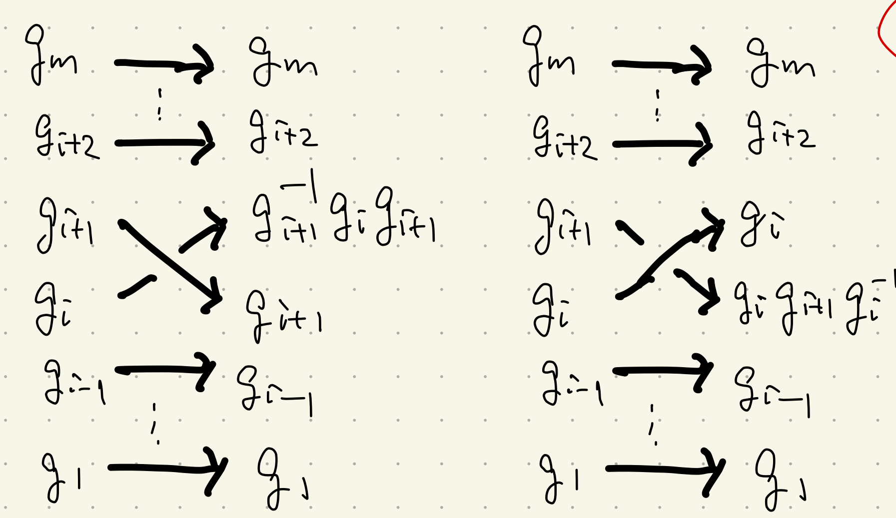
② $P(\sigma_i^{-1})$: $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_m)$

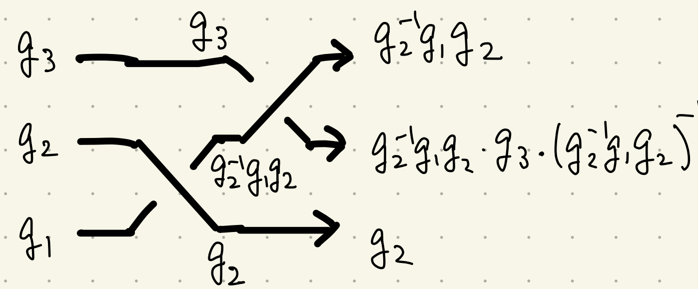
$$\downarrow$$

$(g_1, \cdots, g_i g_{i+1} g_i^{-1}, g_i, \cdots, g_m)$

③ $\quad P(b_1 b_2) = P(b_2) \circ P(b_1)$

By P, the braid group B_m acts on G^m from the right.





① $\mathrm{artin}(\sigma_i)$: $x_i \mapsto x_{i+1}$

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③ $\mathrm{artin}(b_1 b_2) = \mathrm{artin}(b_1) \circ \mathrm{artin}(b_2)$

Hurwitz action of B_m

X : a quandle

$X^m = \{(g_1, \cdots, g_m) \mid g_1, \cdots, g_m \in X\}$

$b \in B_m$, $\quad P(b) : X^m \to X^m$

① $P(\sigma_i)$: $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_m)$

$\qquad\qquad \downarrow$

$(g_1, \cdots, g_{i+1}, g_i * g_{i+1}, \cdots, g_m)$

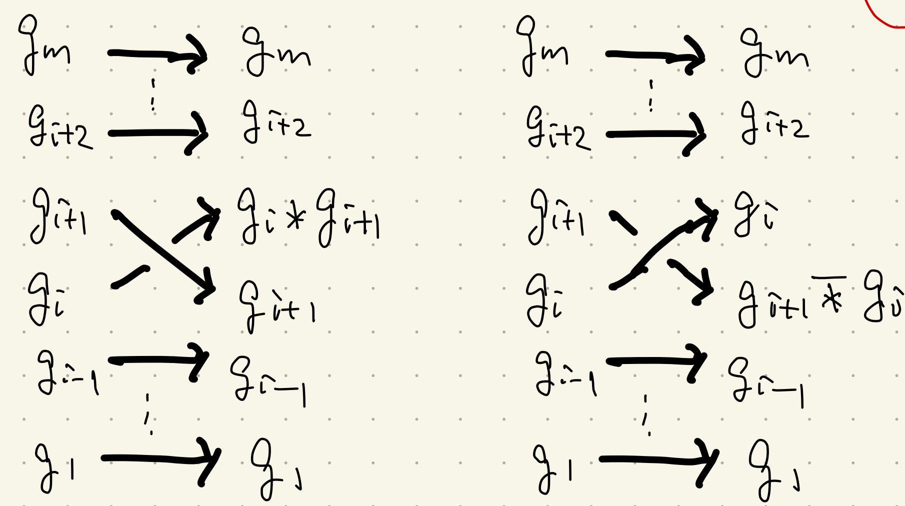
② $P(\sigma_i^{-1})$: $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_m)$

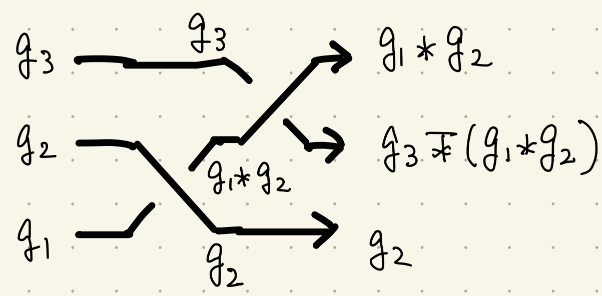
$\qquad\qquad \downarrow$

$(g_1, \cdots, g_{i+1} \overline{*} g_i, g_i, \cdots, g_m)$

③ $\quad P(b_1 b_2) = P(b_2) \circ P(b_1)$

By P, the braid group B_m acts on X^m from the right.





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③ $\mathrm{artin}(b_1 b_2) = \mathrm{artin}(b_1) \circ \mathrm{artin}(b_2)$

Thm

$b, b' \in B_m$

The following are equivalent

① $b = b'$

② $P(b) = P(b')$ for every group G

③ $P(b) = P(b')$ for $G = FG(\mathbb{X})$

$\mathbb{X} = x_1, \cdots, x_m$

In particular

① $b = 1$

② $P(b) = \mathrm{id}$ for every group G

③ $P(b) = \mathrm{id}$ for $G = FG(\mathbb{X})$

Thm

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③ $P(b) = \mathrm{id}$ for $X = FQ(\mathbb{X})$

In the next talk, we use the Hurwitz action of B_m for computation of the knot quandle and quandle colorings.