Statistical Mechanics and Turbulence

Truncated Systems and Time Scales in Turbulence

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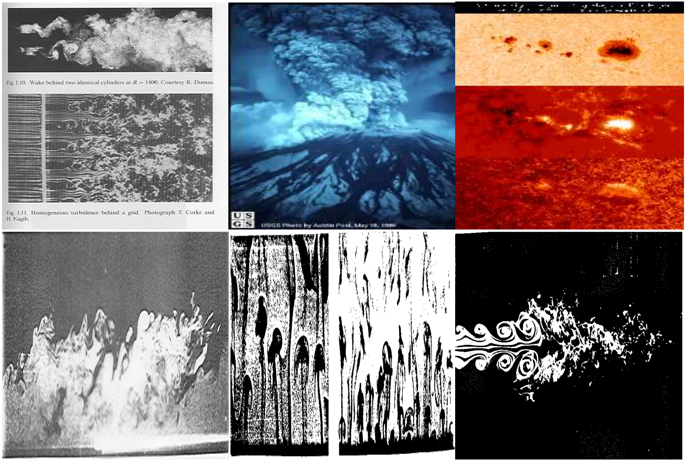
Turbulence in Various Forms



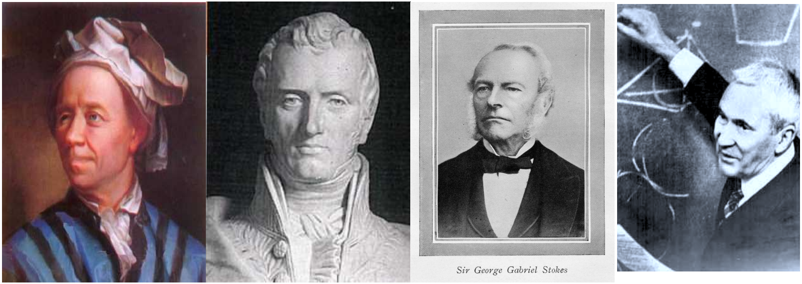
Lessons

- We need a probabilistic description of turbulence.
- Velocity signals from turbulent flows are disorganised.
- They are unpredictable in their detailed behaviour.
- Some average properties of the signals are quite reproducible.

Lessons

- Large spatial scales: contain most of the energy.
- Small scales: Inertial and dissipation ranges.
- Small scales: Homogeneous and isotropic, to a good approximation (far from boundaries, etc.).
- Inertial-range correlation (or structure functions) exhibit power laws with universal exponents (reminiscent of critical phenomena).

Pioneers



Leonhard Euler (1707-1783), Claude-Louis Navier (1785-1836), George Gabriel Stokes (1819-1903), and Andrey Nikolaevich Kolmogorov (1903-1987).

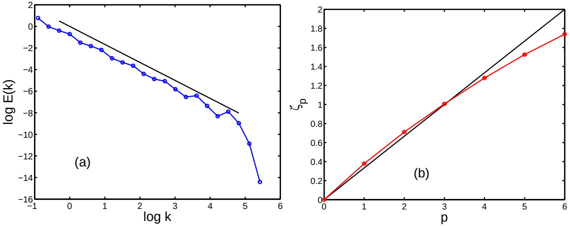
The Equations

- Fluid flows are governed by the Navier–Stokes equation augmented by the incompressibility condition

$$\partial_t \vec{u} + (\vec{u}.\vec{\nabla})\vec{u} = \nu \nabla^2 \vec{u} - \vec{\nabla} p/\rho + \vec{f}/\rho;$$

$$\vec{\nabla}.\vec{u} = 0.$$

$\vec{u}$: Eulerian velocity
p: pressure
ν: kinematic viscosity
ρ: density
$\vec{f}$: external force



Challenges

- Engineers : Characterisation and control of turbulent flows such as flows in pipes or over cars and aeroplanes.
- Mathematicians : Proofs for the smoothness, or lack thereof, of solutions of the Navier-Stokes and related equations.
- Challenges also for fluid dynamicists, astrophysicists, geophysicists, climate scientists, plasma physicists ....

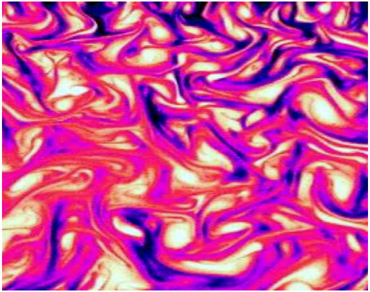
Principal Research Interests

- Statistical studies of different turbulent flows
- Inertial (finite-size) particles in turbulent flows
- Truncated systems and thermalization
- Singularities in the equations of hydrodynamics

Principal Research Interests : Statistical Studies

Principal Collaborators : U. Frisch, D. Mitra, and R. Pandit

- Statistical studies of fluid, magnetohydrodynamic, passive-scalar, and Burgers turbulence.
- Lagrangian (tracer) particles.
- Fundamental questions of scaling behaviour and corrections to scaling in turbulence.

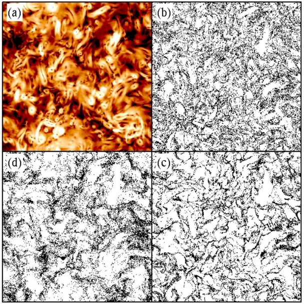


Principal Research Interests : Inertial Particles

Principal Collaborators : J. Bec, E. Bodenschatz, and H. Homann

Experiments, Direct Numerical Simulations, and Theory

- Effect of collisions and hydrodynamic interactions on inertial particles.
- Modelling collision kernels and understanding droplet-growths in clouds.
- *Granular* media.



Principal Research Interests : Truncated Systems

Principal Collaborators : U. Frisch, S. Nazarenko, and I. Procaccia

- The role of statistical mechanics in understanding the physics of turbulent flows.
- Thermalised states in turbulence.

Principal Research Interests : Singularities

Principal Collaborators : C. Bardos, U. Frisch, and E. S. Titi

- Singularities in the equations of hydrodynamics.

Outline of my talk

- **Part I** : Truncated Systems – Equilibrium Statistical Mechanics
  - The Tyger Phenomenon in 1D Burgers and 2D Euler Equations
  - Turbulence in Fractal Dimension
- **Part II** : Timescales in Turbulent Flows – Critical Phenomena
  - Dynamic Multiscaling
  - Topological Structures of 2D Flows

Part I : Truncated Systems

References :

1. *Turbulence in Noninteger Dimensions by Fractal Fourier Decimation*,
   U. Frisch, A. Pomyalov, I. Procaccia, and S. S. Ray,
   Physical Review Letters, **108**, 074501 (2012).
2. *Resonance phenomenon for the Galerkin-truncated Burgers and Euler equations*,
   S. S. Ray, U. Frisch, S. Nazarenko, and T. Matsumoto,
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3. *Hyperviscosity, Galerkin truncation and bottlenecks in turbulence*,
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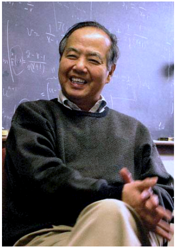
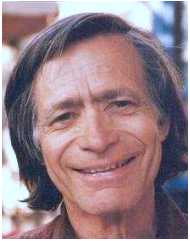
Outline

- Introduction : Statistical Mechanics and Turbulence
- Truncated Systems
  - The Tyger Phenomenon
  - Turbulence in Fractal Dimensions

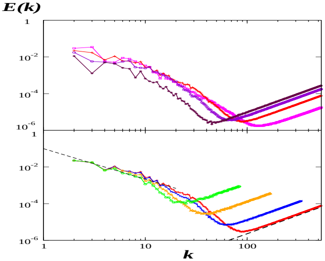
Equilibrium Statistical Mechanics and Turbulence

- Equilibrium statistical mechanics is concerned with conservative Hamiltonian dynamics, Gibbs states, ...
- Turbulence is about dissipative out-of-equilibrium systems.
- In 1952 Hopf and Lee apply equilibrium statistical mechanics to the 3D Euler equation and obtain the equipartition energy spectrum which is very different from the Kolmogorov spectrum.
- In 1967 Kraichnan uses equilibrium statistical mechanics as one of the tools to predict the existence of an inverse energy cascade in 2D turbulence.

Equilibrium Statistical Mechanics and Turbulence

- In 1989 Kraichnan remarks *the truncated Euler system can imitate NS fluid: the high-wavenumber degrees of freedom act like a thermal sink into which the energy of low-wave-number modes excited above equilibrium is dissipated. In the limit where the sink wavenumbers are very large compared with the anomalously excited wavenumbers, this dynamical damping acts precisely like a molecular viscosity.*
- In 2005 Cichowlas, Bonaiti, Debbasch, and Brachet discovered long-lasting, partially thermalized, transients similar to high-Reynolds number flow.



The Galerkin-truncated 1D Burgers equation

- The (untruncated) inviscid Burgers equation :

$$\partial_t u + \partial_x (u^2/2) = 0; \qquad u(x,0) = u_0(x).$$

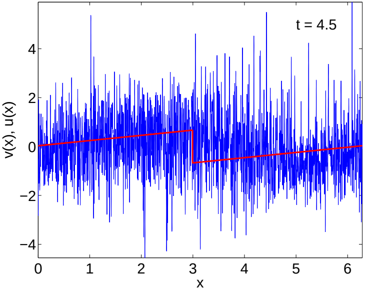
- The associated Galerkin-truncated (inviscid) Burgers equation

$$\partial_t v + \mathrm{P}_{K_\mathrm{G}} \partial_x (v^2/2) = 0; \qquad v_0 = \mathrm{P}_{K_\mathrm{G}} u_0.$$

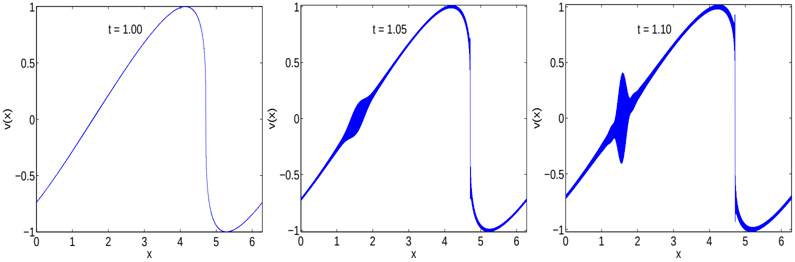
where

$$\mathrm{P}_{K_\mathrm{G}} u(x) = \sum_{|k| \le K_\mathrm{G}} \mathrm{e}^{\mathrm{i}kx} \hat{u}_k.$$

Thermalisation



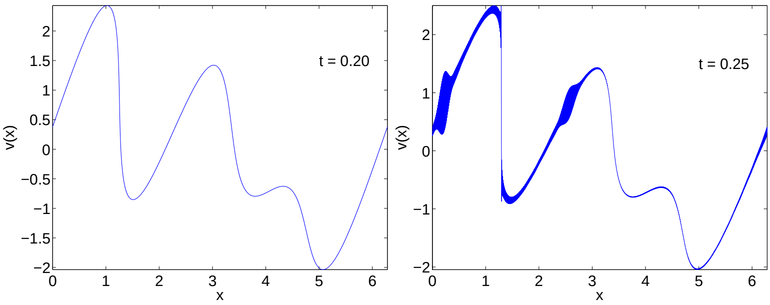
Tygers



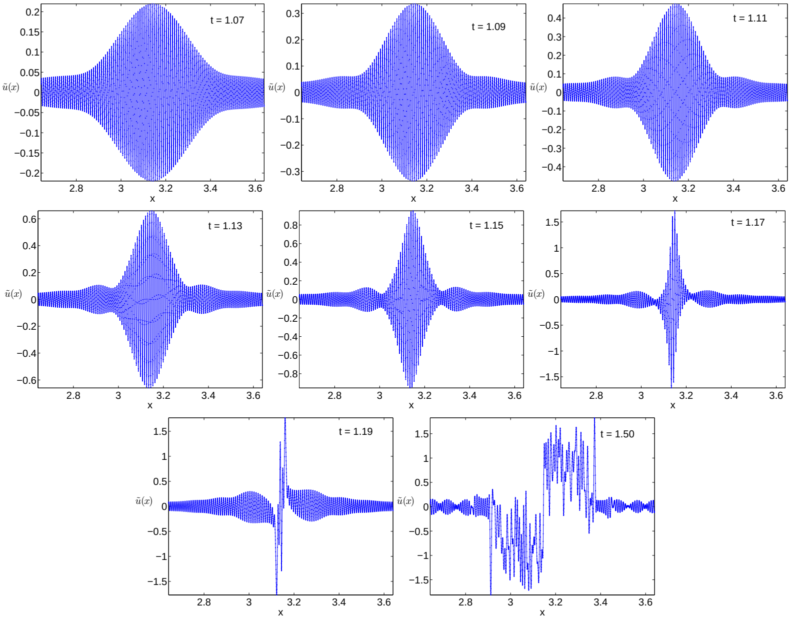
Growth of a tyger in the solution of the inviscid Burgers equation with initial condition $v_0(x) = \sin(x - \pi/2)$. Galerkin truncation at $K_{\mathrm{G}} = 700$. Number of collocation points $N = 16,384$. Observe that the bulge appears far from the place of birth of the shock.

Tygers only at regions of positive strain

$$u_0(x) = \sin(x) + \sin(2x + 0.9) + \sin(3x)$$



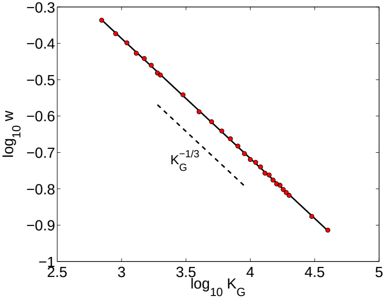
From tygers to thermalization

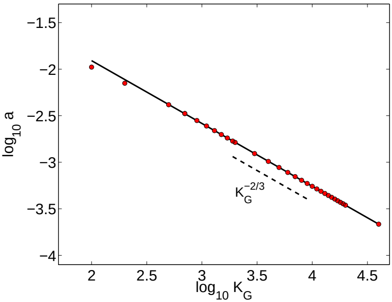


Phenomenological Explanation : 1D Burgers

- A localized strong nonlinearity, such as is present at a preshock or a shock, acts as a source of a *truncation wave*.
- Away from the source this *truncation wave* is mostly a plane wave with wavenumber close to K_{G}.
- Resonant interactions are confined to particles such that $\tau \Delta v \equiv \tau |v - v_{\mathrm{s}}| \lesssim \lambda_{\mathrm{G}}$.
- In a region of negative strain a wave of wavenumber close to K_{G} will be squeezed and thus disappearing beyond the *truncation horizon*.

Scaling properties





$$\begin{aligned} \text{width} \quad &\propto \quad K_{\mathrm{G}}^{-1/3} \qquad \text{(using phase mixing arguments)} \\ \text{amplitude} \quad &\propto \quad K_{\mathrm{G}}^{-2/3} \qquad \text{(using energy conservation arguments)} \end{aligned}$$

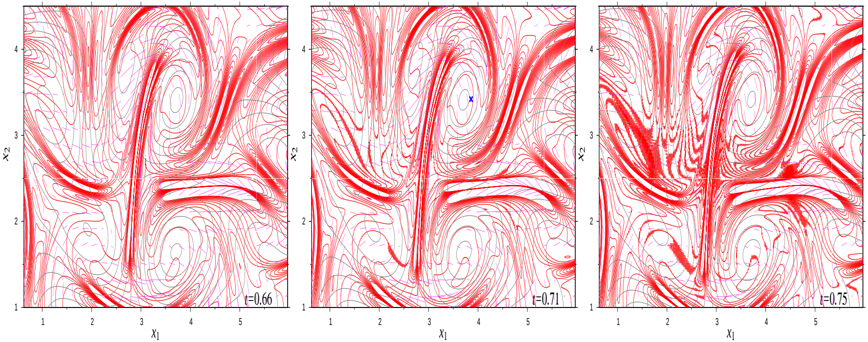
Scaling of Tygers

- Widths :
  - By the time $t_\star$, truncation is significant only for a lapse of time $O({K_{\rm G}}^{-2/3})$.
  - The coherent build up of a tyger only at locations whose velocity differs from that at resonance by an amount $\Delta v \lesssim \frac{2\pi}{{K_{\rm G}}^{-2/3} K_{\rm G}} \propto {K_{\rm G}}^{-1/3}$.
  - Since the velocity v varies linearly with x near the resonance point, the width of the $t_\star$ tyger is itself proportional to ${K_{\rm G}}^{-1/3}$.
- Amplitudes :
  - The Galerkin-truncated Burgers equation conserves energy.
  - The apparent energy loss due to truncation $\sim \int_0^{\lambda_{\rm G}} x^{2/3} dx \sim {K_{\rm G}}^{-5/3}$.
  - Conservation demands that this energy-loss is transferred to the tygers which gives the tyger-amplitude scaling $\propto {K_{\rm G}}^{-2/3}$.

Truncated 2D Euler

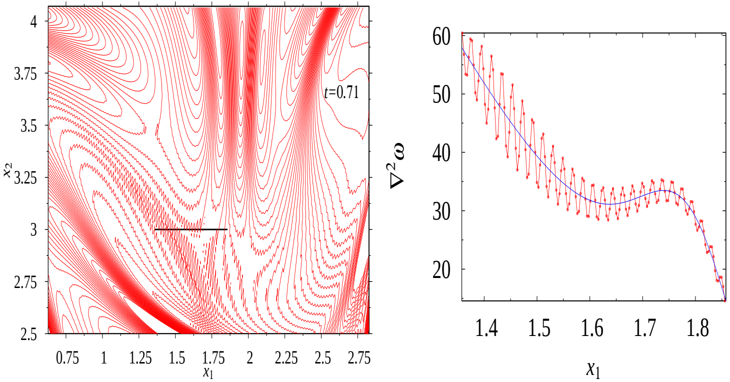
- Numerical integration of the truncated 2D incompressible Euler equation with random initial conditions and resolutions between 512^2 and 8192^2.
- Although for the untruncated solution real singularities are ruled out at any finite time, there is strong enhancement of spatial derivatives of the vorticity.
- The highest values of the Laplacian is found in the straight cigar-like structure.

2D Euler



A 2D tyger: before ($t = 0.66$), early ($t = 0.71$) and later ($t = 0.75$). Figures, moderately zoomed, centered on the main cigar. Contours of the Laplacian of vorticity in red, ranging from -200 to 200 by increments of 25, streamlines in gray, ranging from -1.6 to 1.6 by increments of 2 and positive strain eigendirections in pink segments.

2D Euler tygers : Physical space



Left: zoomed version of contours of the Laplacian of vorticity at $t = 0.71$.

Right: plot of the Laplacian of vorticity along the horizontal segment near $x_2 = 3$, shown in the left panel.

Take Home Message : Truncated Inviscid Systems

- Tygers provide a clue as to the onset of thermalization.
- Tygers do not modify shock dynamics but modify the flow elsewhere because the tygers induce Reynolds stresses on scales much larger than the Galerkin wavelength; hence the weak limit of the Galerkin-truncated solution as $K_{\mathrm{G}} \to \infty$ is NOT the inviscid limit of the untruncated solution.
- There is good evidence that the key phenomena associated to tygers are also present in the two-dimensional incompressible Euler equation and also perhaps in three dimensions.
- It is clear that complex-space singularities approaching the real domain within one Galerkin wavelength are the triggering factor in both the 2D Euler and the 1D Burgers case.
- Can we "purge tygers away" and thereby obtain a subgrid-scale method which describes the inviscid-limit solution right down to the Galerkin wavelength?

Generalised Galerkin truncation : **Decimation**

- The forced incompressible Navier-Stokes equation

$$\begin{aligned} \partial_t \mathbf{u} &= B(\mathbf{u},\mathbf{u}) + \mathbf{f} + \Lambda \mathbf{u} \ , \\ B(\mathbf{u},\mathbf{u}) &= -\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla} p \ , \quad \Lambda = \nu \nabla^2 \end{aligned}$$

- Define a Fourier decimation operator P_D

$$\text{If} \quad \mathbf{u} = \sum_{\mathbf{k}\in\mathcal{Z}^2} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\mathbf{u}}_{\mathbf{k}}, \quad \text{then} \quad P_D \mathbf{u} = \sum_{\mathbf{k}\in\mathcal{Z}^2} e^{i\mathbf{k}\cdot\mathbf{x}} \theta_{\mathbf{k}} \hat{\mathbf{u}}_{\mathbf{k}} \ .$$

$$\theta_{\mathbf{k}} = \begin{cases} 1 \text{ with probability } h_k \\ 0 \text{ with probability } 1 - h_k \ , \quad k \equiv |\mathbf{k}| \ . \end{cases}$$

- To obtain D-dimensional dynamics we choose

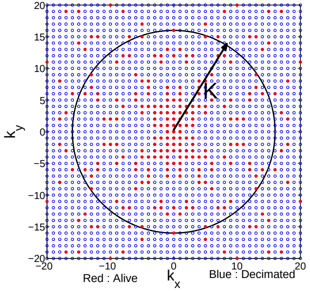
$$h_k = C(k/k_0)^{D-2} \ , \quad 0 < D \leq 2, \quad 0 < C \leq 1 \ .$$

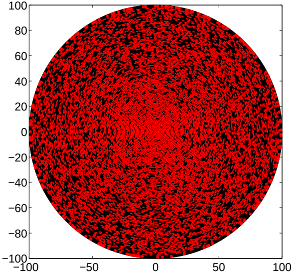
- The *decimated Navier–Stokes equation*

$$\partial_t \mathbf{v} = P_D B(\mathbf{v},\mathbf{v}) + P_D \mathbf{f} + P_D \Lambda \mathbf{v} \ .$$

Are we able to do this numerically?

- We discovered fractal decimation appropriate for hydrodynamics, in which one keeps a randomly selected set of Fourier modes in such a way that the number of modes in a ball of radius k centered at wave vector zero varies as k^D for large k.
- We shall do our decimation starting from $d = 2$.





Digression : Two-dimensional Turbulence

- Study of high-Reynolds-number solution of the incompressible Navier-Stokes equations:

$$\begin{aligned} D_t \mathbf{u} &= -\nabla p + \nu \nabla^2 \mathbf{u} \\ \nabla . \mathbf{u} &\equiv 0 \end{aligned}$$

- Energy conservation in the inviscid, unforced limit.

$$\begin{aligned} \partial_t E &= -2\nu\Omega \\ E &= 1/2 \int_{\mathbf{x}\in R^3} |\mathbf{u}|^2 \\ \Omega &= 1/2 \int_{\mathbf{x}\in R^3} |\omega|^2 \end{aligned}$$

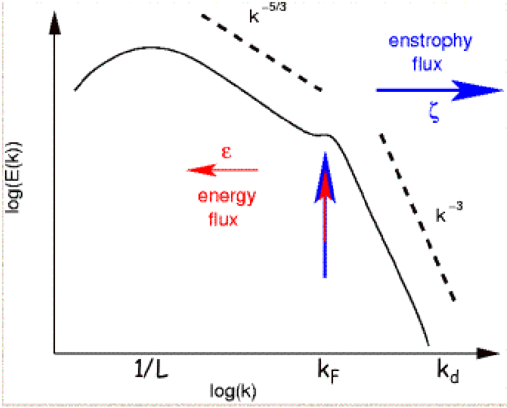
- Enstrophy conservation in the inviscid, unforced limit.

$$\begin{aligned} \partial_t \Omega &= -2\nu P \\ P &= 1/2 \int_{\mathbf{x}\in R^3} |\nabla \times \omega|^2 \end{aligned}$$

Cascades

[Kraichnan, Phys. Fluids, **10**, (1967*a*), Batchelor, Phys. Fluids Suppl. *II*, **12**, (1969)]

- Energy injected at a length scale l_{inj} will inverse-cascade to large length scales with $E(k) \sim k^{-5/3}$.
- Energy injected at a length scale l_{inj} will forward-cascade to small length scales with $E(k) \sim k^{-3}$.



Turbulence in non-integer dimensions less than 2

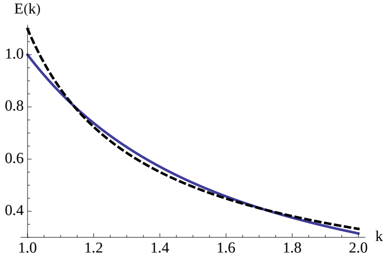
- In theoretical physics a number of interesting results have been obtained by extending the dimension to non-integer values.
- The same kind of extension can be carried out for turbulence using the Kraichnan–Wyld diagramatic formalism.
- A difficulty appears for $d < 2$: the energy spectrum $E(k)$ can become negative in some band of wave numbers k, so that this kind of extension lacks probabilistic realizability.
- Nevertheless, if there exists a way of doing the extension below dimension 2 in which the nonlinearity conserves energy and enstrophy, then an interesting phenomenon should happen in dimension $4/3$.

Decimated Systems and Statistical Mechanics

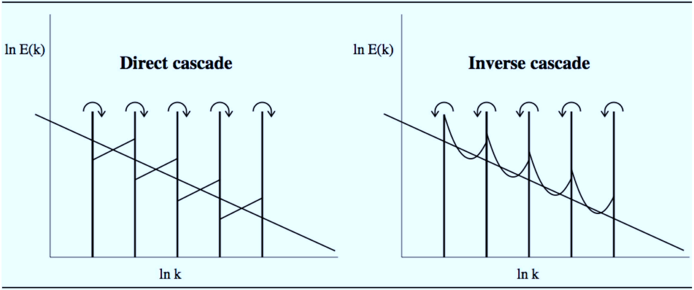
- The Galerkin-truncated and the decimated inviscid Navier–Stokes satisfies a Liouville theorem.
- This implies the existence of (statistically) invariant Gibbs states for which the probability is a Gaussian, proportional to $e^{-(\alpha E+\beta\Omega)}$.
- For such Gibbs states the corresponding energy spectrum $E(k)=\frac{k^{D-1}}{\alpha+\beta k^2}; \quad \beta>0, \quad \alpha>-\beta$.
- For *enstrophy equipartition*: $\alpha=0 \implies E(k)\propto k^{D-3}$.
- This equilibrium spectrum coincides with the Kolmogorov 1941 $k^{-5/3}$ spectrum at the critical dimension $D_c=4/3$.

Kolmogorov Spectrum

- Energy transfer is dominated by triads of wavenumbers with comparable magnitudes.
- Decompose the energy inertial range into bands of fixed relative width, say one octave.
- Pure intraband dynamics would lead to thermalization.
- Perform a thermodynamic thought experiment : Starting from a $k^{-5/3}$ spectrum, prevent the various bands from interacting. In each band, the modes will then thermalize and achieve a Gibbs state with the constraint that the total band energy and enstrophy be the same as for the $-5/3$ spectrum.

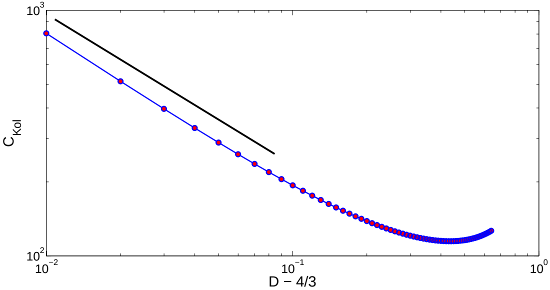


Cascade Direction



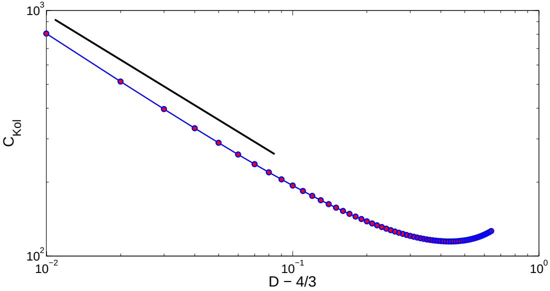
Kolmogorov Constant Blow-up

- Close to $D_c = 4/3$, use perturbation techniques and obtain for the upper-to-lower-band energy transfer $0.009(D - 4/3)$ to leading order.
- In the K41 inertial range, the energy spectrum and the energy flux Π_E are related by $E(k) = C_{\mathrm{Kol}}|\Pi_E|^{2/3}k^{-5/3}$, where C_{Kol} is the Kolmogorov constant, we infer that the Kolmogorov constant diverges as $(D - 4/3)^{-2/3}$.

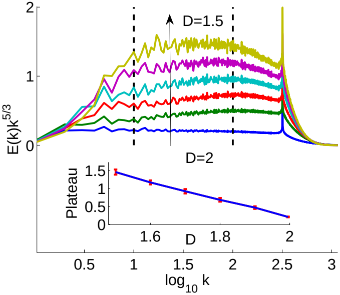


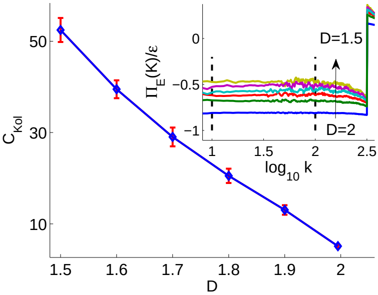
Closure Model

$$\partial_t E_k + 2\nu k^2 E_k = \frac{8S_1}{S_2} \int \frac{dpdq}{\sin\alpha} \frac{k^2}{pq} \left(\frac{1}{k_0}\right)^{D-2} \left(\frac{k_0}{p}\right)^{D-2} \left(\frac{k_0}{q}\right)^{D-2} \theta_{kpq} \left[a^{(2)}_{kpq} k^{D-1} E_p E_q - b^{(2)}_{kpq} p^{D-1} E_q E_k\right]$$



Direct Numerical Simulations





Take Home Message : D-Dimensional Turbulence

- We have shown that for $4/3 < D \leq 2$, when the energy spectrum is prescribed to be $E(k) = k^{-5/3}$ over the inertial range, there is a *negative energy flux* Π_E, vanishing linearly with $D - 4/3$ near the critical dimension $D_c = 4/3$.
- We finally observe that the fractal Fourier decimation procedure — that allows numerical experimentation by spectral simulation — can be started from any integer dimension and can be applied to a large class of problems in compressible and incompressible hydrodynamics and MHD.

Part II : Timescales in Turbulent Flows

References :

1. *Dynamic Multiscaling in Two-dimensional Turbulence*,
   S. S. Ray, D. Mitra, P. Perlekar, and R. Pandit,
   Physical Review Letters, **107**, 184503 (2011).
2. *The Persistence Problem in Two-Dimensional Fluid Turbulence*,
   P. Perlekar, S. S. Ray, D. Mitra, and R. Pandit,
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3. *The Universality of Dynamic Multiscaling in Homogeneous, Isotropic Navier-Stokes and Passive-Scalar Turbulence*,
   S. S. Ray, D. Mitra, and R.Pandit,
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4. *Dynamic Multiscaling in Turbulence*,
   R. Pandit, S. S. Ray, and D. Mitra,
   European Physics Journal B 64, 463 (2008).

Outline

- Timescales
- Two-dimensional turbulence in soap films
  - Dynamic Multiscaling
  - Persistence and Flow Topology

Critical Phenomena

$\Gamma(r,t,h) \approx \frac{1}{r^{d-2+\eta}}\mathcal{F}(t^{\nu}\xi, h/t^{\Delta})$

- r: separation between the spins in d dimensions
- $t \equiv (T - T_c)/T_c$
- $h \equiv H/k_B T_c$
- k_B: Boltzmann constant
- T: temperature
- T_c: critical temperature
- H: magnetic field
- ξ: correlation length (diverges at criticality)
- η, ν and Δ: static critical exponents
- $\mathcal{F}$: universal scaling function

Critical Phenomena

In Fourier space

$\tilde{\Gamma}(q,t,h) \approx \frac{1}{q^{2-\eta}}\mathcal{F}(t^{\nu}\xi, h/t^{\Delta});$

$\vec{q}$: wave vector with magnitude q

Dynamic scaling for time-dependent correlation functions in the vicinity of a critical point.

$\tilde{\Gamma}(q,\omega,t,h) \approx \frac{1}{q^{2-\eta}}\mathcal{G}(q^{-z}\omega, t^{\nu}\xi, h/t^{\Delta});$

- z: dynamic critical exponent
- ω: frequency
- $\mathcal{G}$: a scaling function

Relaxation time τ diverges as

$$\tau \sim \xi^{z}.$$

Equal-Time Structure Functions

- Order-p, equal-time, structure functions:

$$\mathcal{S}_p(r) \equiv \langle [\delta u_{\parallel}(\vec{x}, \vec{r}, t)]^p \rangle \sim r^{\zeta_p}$$

$$\delta u_{\parallel}(\vec{x}, \vec{r}, t) \equiv [\vec{u}(\vec{x} + \vec{r}, t) - \vec{u}(\vec{x}, t)] \cdot \frac{\vec{r}}{r}$$

  η_d: Kolmogorov dissipation scale;
  L: large length scale at which energy is injected into the system.

- Experiments favour multiscaling: ζ_p a nonlinear, convex monotone increasing function of p.
- Simple-scaling prediction of Kolmogorov: $\zeta_p^{K41} = p/3$.

Time-Dependent Structure Functions

- The order-p, time-dependent longitudinal structure function:

$$\mathcal{F}_p(r, \{t_1, \ldots, t_p\}) \equiv \langle [\delta u_{\parallel}(\vec{x}, t_1, r) \ldots \delta u_{\parallel}(\vec{x}, t_p, r)] \rangle$$

For simplicity we consider $t_1 = t$ and $t_2 = \ldots = t_p = 0$.

- Given $\mathcal{F}(r, t)$, different ways of extracting time scales yield different exponents that are defined via dynamic-multiscaling ansätze:

$$\mathcal{T}_p(r) \sim r^{z_p}.$$

Time Scales

- From the longitudinal, time-dependent, order-p structure functions, the order-p, degree-M, integral time scale is defined as,

$$\mathcal{T}^{I}_{p,M}(r) \equiv \left[\frac{1}{\mathcal{S}_p(r)}\int_0^\infty \mathcal{F}_p(r,t)t^{(M-1)}dt\right]^{(1/M)} \sim r^{z^{I}_{p,M}}.$$

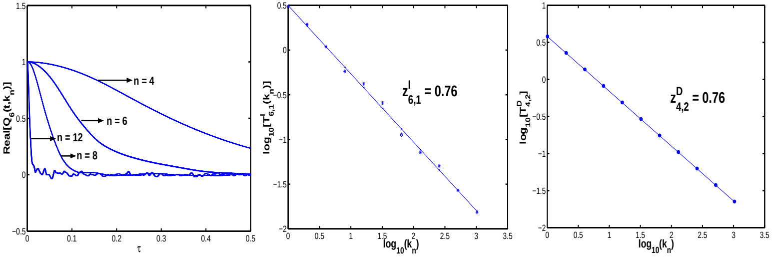
- Similarly, the order-p, degree-M derivative time scale is defined as

$$\mathcal{T}^{D}_{p,M}(r) \equiv \left[\frac{1}{\mathcal{S}_p(r)}\frac{\partial^M \mathcal{F}_p(r,t)}{\partial t^M}\right]^{(-1/M)} \sim r^{z^{D}_{p,M}}.$$

- The multifractal model predicts the following bridge relations:

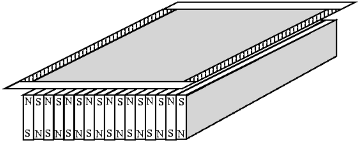
$$z^{I}_{p,M} = 1 + \frac{[\zeta_{p-M} - \zeta_p]}{M}; \quad z^{D}_{p,M} = 1 + \frac{[\zeta_p - \zeta_{p+M}]}{M}.$$

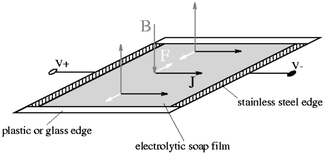
Timescales from Shell Models of 3D Turbulence



Back to 2D : Electromagnetically forced soap films

[M. Rivera, Ph.D. Thesis, arXiv:physics/010305v1]





- Soap film: 400ml distilled water + 40ml glycerol + 5ml commercial liquid detergent,
- The soap film is suspended on a rectangular frame,
- The magnetic array produces a Kolmogorov forcing $F_x = F_0 sin(k_y y)$.

Modelling soap films: Incompressible limit

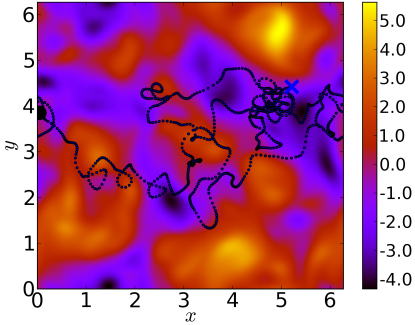
[Chomaz et al., PRA, **41**, (1990), Chomaz, JFM, (2001), P. Fast, arXiv:physics/0511175v1, (2005).]

- Mach Number $M_e \equiv u_{rms}/c$, where c is the speed of the sound in the soap films. For the experiments with electromagnetically forced soap films $M_e \sim 0.06$.
- To leading order soap-film behaviour is governed by the Navier–Stokes (NS) equations in two dimensions + an air drag

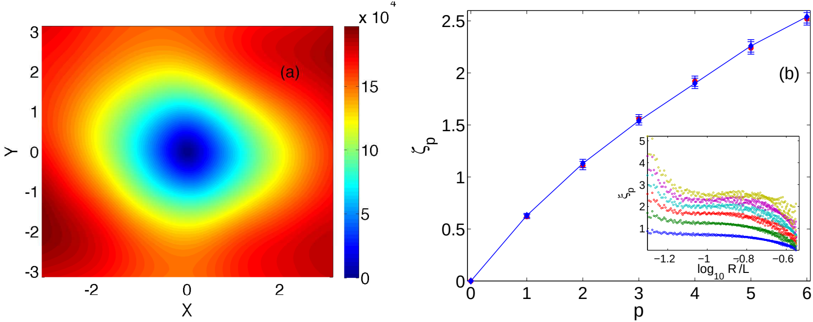
$$
\begin{aligned}
D_t\mathbf{u} &= \nu\nabla^2\mathbf{u} - \nabla p - \alpha\mathbf{u} \\
\nabla\cdot\mathbf{u} &= 0
\end{aligned}
$$

Multiscaling in 2D Flows

- Multiscaling in equal-time, Eulerian vorticity structure functions.
- Investigating dynamic-multiscaling in time-dependent, quasi-Lagrangian vorticity structure functions.
- Tracking a single particle in a $2D$ flow with friction to generate quasi-Lagrangian fields.

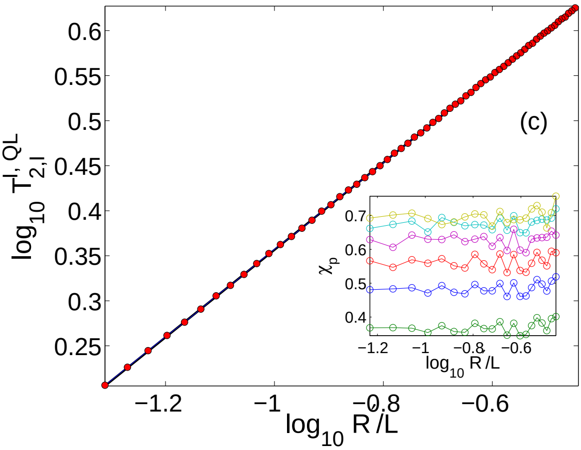


Equal-time Structure Functions



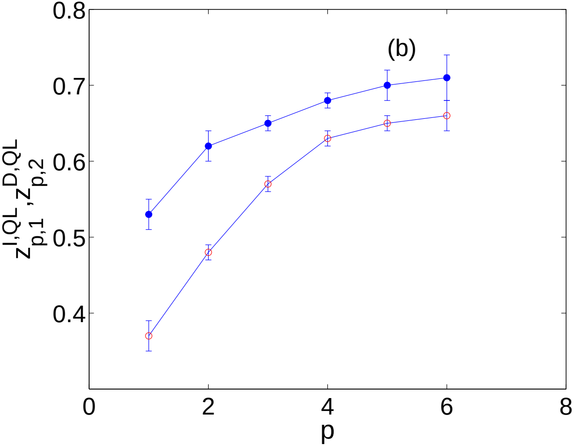
- Left : $S_3^\omega(\mathbf{R})$ for the quasi-Lagrangian field, obtained by averaging over the centers r_c.
- Right : Scaling exponents for equal-time, vorticity structure functions, for both the Eulerian and quasi-Lagrangian fields.

Time-dependent Structure Functions



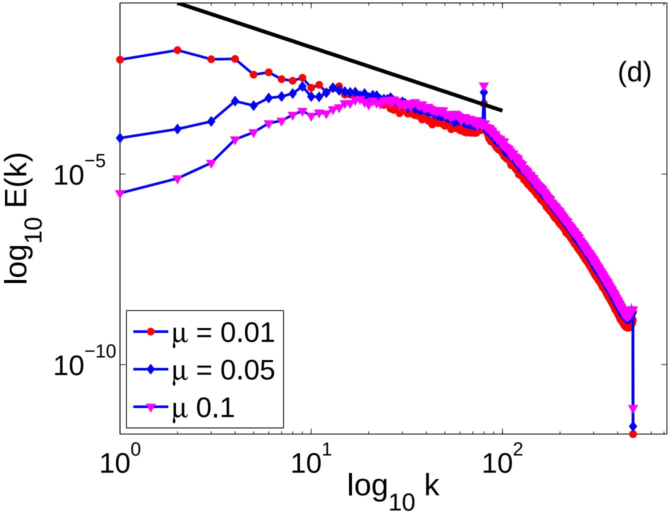
A loglog plot of $T^I_{2,1}$ versus the separation r; the data points are shown by open red circles and the straight black line shows the line of best fit in the inertial range.

Time-dependent multiscaling exponents



Plots of the vorticity, dynamic-multiscaling, quasi-Lagrangian exponents $z^{I,\mathrm{QL}}_{p,1}$ (open red circles) and $z^{D,\mathrm{QL}}_{p,2}$ (full blue circles) versus p with the error bars

Effect of Friction



A log-log plot of the energy spectrum versus the wavevector k for various values of μ.

Take Home Message : Dynamic Multiscaling

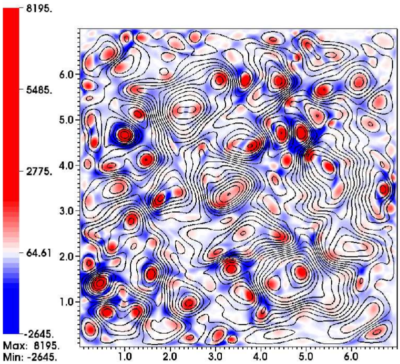
- Dynamic multiscaling exponents z_p depend on how $\mathcal{T}_p(r)$ is extracted.
- z_p is related to the equal-time exponents via bridge relations.
- In 2D, we find that friction also suppresses sweeping effects so, with such friction, even Eulerian vorticity structure functions exhibit dynamic multiscaling with exponents that are consistent with their quasi-Lagrangian counterparts.
- For passive-scalar, dynamic multiscaling is obtained only if the advecting velocity is intermittent.
- Simple dynamic scaling is obtained for a simple version of the passive-scalar problem in which the advecting velocity field is Gaussian, even though equal-time structure functions display multiscaling in this model.

Flow Topology : Okubo-Weiss parameter

A. Okubo, Deep-Sea Res. **17**, 17 (1970),
J. Weiss, Physica, **48D**, 273 (1991).

- From the velocity-gradient tensor $\mathcal{A}$, with components $A_{ij} \equiv \partial_i u_j$, we obtain the Okubo-Weiss parameter Λ, the discriminant of the characteristic equation for $\mathcal{A}$.
- If Λ is positive (negative) then the flow is vortical (extensional).
- In an incompressible flow in two dimensions $\Lambda = \det\mathcal{A}$; and the PDF of Λ has been shown to be asymmetrical about $\Lambda = 0$ (vortical regions are more likely to occur than strain-dominated ones). Note $\langle\Lambda\rangle = 0$.

ψ and Λ



- Contours of ψ overlayed on the pseudocolor plot of Λ.
- $\Lambda > 0$(centers)
- $\Lambda < 0$(saddles)

The Persistence Problem

- Satya N. Majumdar, Persistence in Nonequilibrium Systems, Curent Science, **77**, 370 (1999); cond-mat/9907407v1
  Let $\phi(x,t)$ be a nonequilibrium field fluctuating in space and time according to some dynamics. Persistence is simply the probability $P_0(t)$ that, at a fixed point in space, the quantity sgn$[\phi(x,t) - \langle\phi(x,t)\rangle]$ does not change upto time t.
  - $P^{\phi}(\tau) \sim \tau^{-\beta}$ as $\tau \to \infty$, where β is the *persistence exponent*.
- We ask :
  - How long does a Lagrangian particle stay in region where $\Lambda > 0$ (center) or where $\Lambda < 0$ (saddle)?
  - How long does the Λ field not change sign at a position (x, y) i.e., persistence time of a center or a saddle?

Persistence in Two-dimensional Turbulence

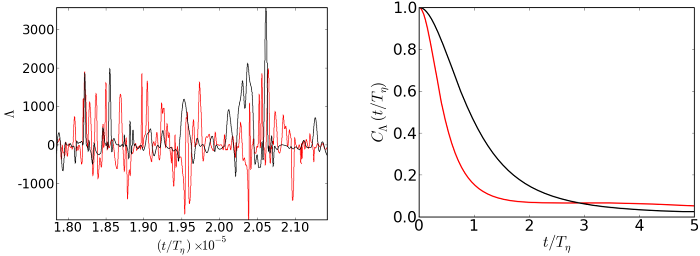
- Lagrangian persistence: We follow N_p particles and evaluate Λ along their trajectories.
- Eulerian persistence: We monitor the time evolution of Λ at N positions in the simulation domain.
- For both the cases we find the time-intervals τ over which $\Lambda > 0$ or $\Lambda < 0$. The PDF of these intervals characterizes the analog of persistence in two dimensional turbulence.

Persistence-time PDF

- We denote the persistence-time PDFs by P; the subscripts E and L on these PDFs signify Eulerian and Lagrangian frames, respectively; and the superscripts $+$ or $-$ distinguish PDFs from vortical points from those from extensional ones.
- To find out the persistence-time PDF $P_E^+(\tau)$ [resp., $P_E^-(\tau)$] we analyse the time-series of Λ obtained from each of the N_p Eulerian points and construct the PDF of the time-intervals τ over which Λ remains positive (resp., negative).
- The same method applied to the time series of Λ, obtained from each of the N_p Lagrangian particles, yields $P_L^+(\tau)$ [resp., $P_L^-(\tau)$].

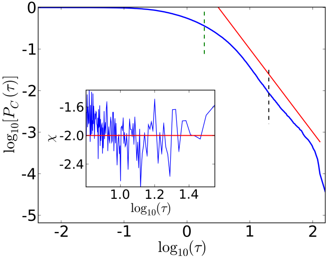
Time series of Λ

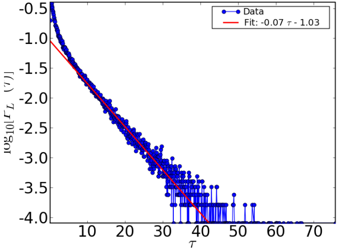
Lagrangian versus Eulerian frame



- Lagrangian Λ tracks (red) show rapid fluctuations in comparison to the corresponding Eulerian tracks (black).
- Autocorrelation $C_\Lambda = \langle \Lambda(t_0)\Lambda(t_0+t)\rangle$ decays faster for the Lagrangian case.

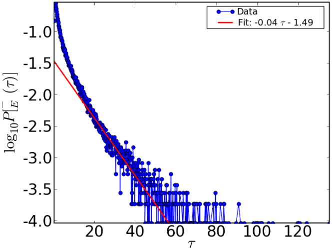
Persistence: particle in a vortex

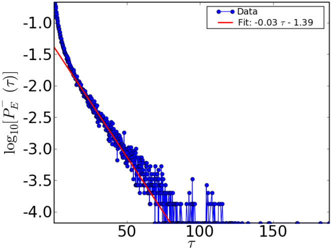




- $P^C(\tau) = \tau^{-(\beta-1)}$, $\beta = 2.9 \pm 0.2$.
- Independent of Re, k_{inj}, and α

Persistence: Eulerian Field





- Lin-log plot of the persistence time of the region of vorticity at position (x, y).
- Lin-log plot of the persistence time of the region of strain at position (x, y).

Take Home Message : Flow Topology

- The Okubo-Weiss parameter provides us with a natural way of formulating and studying the persistence problem in two-dimensional fluid turbulence.
- The persistence-time PDF of Lagrangian particles in vortical and strain-dominated regions are different.
- The persistence-time PDF of Lagrangian particles in vortical regions show a power-law tail with an exponent $\beta = 2.9$.
- The persistence-time PDF of Lagrangian particles in strain-dominated regions shows an exponential tail.