Constraints on Fluid Dynamics from Equilibrium Partition Functions

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Talk mainly based on

- Constraints on Fluid Dynamics from Equilibrium Partition Functions ,
  N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla, T. Sharma
  arXiv: 1203.3544.
- For related work see
  Towards hydrodynamics without an entropy current,
  K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz
  arXiv:1203.3556.

Introduction

In this talk we shall discuss the constraint imposed on the equations of the relativistic hydrodynamics by two related physical requirement that

- these equations admit a stationary solution on an arbitrarily weakly curved stationary background spacetime.
- the conserved currents (e.g. the stress tensor) on the corresponding solution follow from an equilibrium partition function.

Entropy current

The traditional way of thinking about the fluid dynamics (Landau-Lifshitz) is that the equations are consistent with a local form of the second law of thermodynamics. This in particular imposes

- inequalities on several parameters (like viscosities and conductivities) that appear in the equations of hydrodynamics (Landau-Lifshitz).
- requirement of local entropy increase also yields equalities relating otherwise distinct fluid dynamical parameters, and so reduces the number of free parameters that appear in the equations of fluid dynamics. (see recent AdS/CFT inspired works, Son, Surowka, arXiv:0906.5044, Bhattacharya etall arXiv:1105.3733).

Entropy current vs partition function

- Inequalities corresponds to dissipation. They are not captured by partition function technique.
- In the examples that so far has been checked implies equalities obtained on both the ways are the same.
- Partition function analysis is much simpler and lead to some novel relations between various coefficients which could not be got other wise easily.
- Are these two way of thinking same? No proof yet.

Plan of the talk

- Brief overview of local entropy current method and constraints. Example of charged first order fluid and uncharged second order fluid.
- General discussion on partition function method and constraints.
- Examples: Second order uncharged fluid, Anomalous case....
- Summary.

Charged first order fluid

- $T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} + T^{\mu\nu}_{diss}, \quad J^\mu = qu^\mu + j^\mu_{diss}$
- They satisfy conservation

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu, \quad \nabla_\mu J^\mu = 0. \tag{1}$$

- In order to specify $T^{\mu\nu}_{diss}, \; j^\mu_{diss}$ we need to specify onshell independent first derivative data.
- Tensor: $\sigma_{\mu\nu} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - P_{\alpha\beta} \left(\nabla_\lambda u^\lambda \right) \right)$
- Scalar: $\nabla_\mu u^\mu$.
- Vector:
  $V_1 = -P^{\mu\nu} \partial_\nu \frac{\mu}{T} + \frac{F^{\mu\nu} u_\nu}{T}$
  $V_2 = u^\nu \nabla_\nu u^\mu$
  $V_3 = F^{\mu\nu} u_\nu$
- See arXiv:1105.3733 by J.Bhattacharya, S. Bhattacharyya, S. Minwalla, A. Yarom for details.

One needs to supplement this with entropy current as well.

- $$J_s^\mu = s u^\mu - \frac{1}{T} u_\mu T^{\mu\nu}_{diss} - \frac{\mu}{T} J^\mu_{diss} + s_1 S_1 u^\mu + \sum_{i=1}^{3} v_i V_i^\mu \,. \quad (2)$$
- Schematically, divergence of entropy current will have
- $\partial_\mu J_S^\mu =$ independent two derivative and curvature data + quadratic form in first order data .
- The first term on the right hand side of above equation must vanish while the second term must be tuned to be positive.

The two derivative part of the divergence of the entropy is given by

$$-v_1 P^{\mu\nu}\nabla_\mu \partial_\nu \frac{\mu}{T} + (s_1 + v_2) u^\mu \nabla_\mu \partial_\nu u^\nu + \left(v_3 + \frac{v_1}{T}\right) \nabla_\mu (F^{\mu\nu} u_\nu) .$$

This implies

- $v_1 = v_3 = 0$ and $v_2 = -s_1$.
- One obtains

$$J_S^\mu = s u^\mu - \frac{1}{T} u_\mu T^{\mu\nu}_{diss} - \frac{\mu}{T} J^\mu_{diss} + s_1 \left(S_1 u^\mu - V_2^\mu\right) . \qquad (3)$$

  where s_1 is still an arbitrary function of T and μ.
- Putting system in arbitrarily curved background sets $s_1 = 0$.
- Stress tensor and current are given by

$$T^{diss}_{\mu\nu} = \eta \sigma_{\mu\nu} + P^{\mu\nu} \zeta \nabla_\mu u^\mu, \quad J^\mu_{diss} = \sigma V_1^\mu + \sum_{i=2}^{3} c_i V_i^\mu . \qquad (4)$$

- Now analyzing divergence of entropy current coming from remaining parts gives (well known results)

$$\eta \geq 0, \quad \zeta \geq 0, \quad \sigma \geq 0. \qquad (5)$$

  Other two transport coefficients appearing in current vanishes.

Uncharged second order fluid

Here computation becomes very messy. See for example arXiv:1201.4654, S. Bhattacharyya.

- Symmetry considerations determine the expansion of the hydrodynamical stress tensor up to 15 parity even transport coefficients. It turns out the 7 of the parity even are dissipative that is they satisfy some inequality type relation where as remaining 8 coefficients satisfy 5 equality type relations among themselves.

One can show that the most general symmetry allowed two derivative expansion of the constitutive relations is given by

- $$T^{diss}_{\mu\nu} = T\Big[\tau\ (u.\nabla)\sigma_{\langle\mu\nu\rangle} + \kappa_1 \tilde{R}_{\langle\mu\nu\rangle} + \kappa_2 K_{\langle\mu\nu\rangle} + \lambda_0\ \Theta\sigma_{\mu\nu}$$
$$+ \lambda_1\ \sigma_{\langle\mu}{}^{a}\sigma_{a\nu\rangle} + \lambda_2\ \sigma_{\langle\mu}{}^{a}\omega_{a\nu\rangle} + \lambda_3\ \omega_{\langle\mu}{}^{a}\omega_{a\nu\rangle} + \lambda_4\ \mathfrak{a}_{\langle\mu}\mathfrak{a}_{\nu\rangle}\Big]$$
$$+ TP_{\mu\nu}\Big[\zeta_1(u.\nabla)\Theta + \zeta_2\tilde{R} + \zeta_3\tilde{R}_{00}$$
$$+ \xi_1\Theta^2 + \xi_2\sigma^2 + \xi_3\omega^2 + \xi_4\mathfrak{a}^2\Big] \tag{6}$$

- The equality type relation ship is obeyed by

$$\frac{\Pi_{\mu\nu}}{T} = \kappa_1\tilde{R}_{\langle\mu\nu\rangle} + \kappa_2 K_{\langle\mu\nu\rangle} + \lambda_3\omega_{\langle\mu}{}^{\alpha}\omega_{\alpha\nu\rangle} + \lambda_4\mathfrak{a}_{\langle\mu}\mathfrak{a}_{\nu\rangle}$$
$$+ P_{\mu\nu}(\zeta_2\tilde{R} + \zeta_3\tilde{R}_{00}(u^0)^2 + \xi_3\omega^2 + \xi_4\mathfrak{a}^2) \tag{7}$$

Details

$$
\begin{aligned}
&u^\mu = \text{The normalized four velocity of the fluid}\\
&P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu = \text{Projector perpendicular to } u^\mu\\
&\Theta = \nabla . u = \text{Expansion}, \quad \mathfrak{a}_\mu = (u.\nabla)u_\mu = \text{Acceleration}\\
&\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2} - \frac{\Theta}{3}g_{\alpha\beta}\right) = \text{Shear tensor}\\
&\omega^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\nabla_\alpha u_\beta - \nabla_\beta u_\alpha}{2}\right) = \text{Vorticity}\\
&K^{\mu\nu} = \tilde{R}^{\mu a \nu b}u_a u_b, \quad \tilde{R}^{\mu\nu} = \tilde{R}^{a\mu b\nu}g_{ab} \ \ (\tilde{R}^{abcd} = \text{Riemann tensor})\\
&\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}, \qquad \omega^2 = \omega_{\mu\nu}\omega^{\nu\mu}
\end{aligned}
\tag{8}
$$

Equality type relation

$$
\begin{aligned}
\kappa_2 &= \kappa_1 + T\frac{d\kappa_1}{dT}, \\
\zeta_2 &= \frac{1}{2}\left[s\frac{d\kappa_1}{ds} - \frac{\kappa_1}{3}\right] \\
\zeta_3 &= \left(s\frac{d\kappa_1}{ds} + \frac{\kappa_1}{3}\right) + \left(s\frac{d\kappa_2}{ds} - \frac{2\kappa_2}{3}\right) + \frac{s}{T}\left(\frac{dT}{ds}\right)\lambda_4 \\
\xi_3 &= \frac{3}{4}\left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\left(T\frac{d\kappa_2}{dT} + 2\kappa_2\right) - \frac{3\kappa_2}{4} + \left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\lambda_4 \\
&\quad + \frac{1}{4}\left[s\frac{d\lambda_3}{ds} + \frac{\lambda_3}{3} - 2\left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\lambda_3\right] \\
\xi_4 &= -\frac{\lambda_4}{6} - \frac{s}{T}\left(\frac{dT}{ds}\right)\left(\lambda_4 + \frac{T}{2}\frac{d\lambda_4}{dT}\right) - T\left(\frac{d\kappa_2}{dT}\right)\left(\frac{3s}{2T}\frac{dT}{ds} - \frac{1}{2}\right) \\
&\quad - \frac{Ts}{2}\left(\frac{dT}{ds}\right)\left(\frac{d^2\kappa_2}{dT^2}\right)
\end{aligned}
\tag{9}
$$

Summary of results obtained from entropy current method

- For charged first order fluid, we only had inequality type relations. No equality type relation for non vanishing term coefficients.
- For uncharged second order system, there are seven inequality type relations and rest of the eight coefficients satisfy 5 equality type relations. So there are three independent non-dissipative transport coefficients.
- Computation for parity odd second order fluid is not yet performed and it'll take considerable amount of effort to reach at the result.
- In both the cases, end results were not very complicated but computations to reach at the results are tedious.

Constraint from equilibrium

Consider a relativistically invariant quantum field theory on a manifold with a time like killing vector. By a suitable choice of coordinates, any such manifold may be put in the form

$$ds^2 = -e^{2\sigma(\vec{x})}\left(dt + a_i(\vec{x})dx^i\right)^2 + g_{ij}(\vec{x})dx^i dx^j \qquad (10)$$

where $i = 1 \ldots p$. ∂_t is the killing vector on this manifold, while the coordinates $\vec{x}$ parametrize spatial slices. Here σ, a_i, g_{ij} are smooth functions of coordinates $\vec{x}$.

Constraint from equilibrium

- In the long wavelength limit the background manifold may be thought of as a union of approximately flat patches, in each of which the system is in a local flat space thermal equilibrium at the locally red shifted temperature

$$T(x) = e^{-\sigma} T_0 + \dots \tag{11}$$

(where T_0 is the equilibrium temperature of the system and the $\dots$ represent derivative corrections).

- It is easy to verify that the equations of perfect fluid hydrodynamics (hydrodynamics at lowest order in the derivative expansion) admit a stationary 'equilibrium' solution in the background (10) given by

$$u^{\mu}_{(0)}(\vec{x}) = e^{-\sigma}(1, 0, \dots, 0), \qquad T_{(0)}(\vec{x}) = T_o e^{-\sigma} \tag{12}$$

Constraints

- It can be shown that just the requirement that the equilibrium solution is consistent with forst orderenergy momentum tensor and charge current give us the same result that we discussed earlier.
- For second order, appropriately correcting the velocity, tempeature at the second order (which comes automatically by demanding consistency of equations) one again obtains same equality type relations.
- Though this is considerably simpler than entropy current method, one still needs to handle differential equations.

Partiton function technique

Let H denote the Hamiltonian that generates translations of the time coordinate t.

- Question: What can we say, on general symmetry grounds, about the dependence of the the partition function of the system

$$Z = \mathrm{Tr} e^{-\frac{H}{T_0}}, \tag{13}$$

on σ, g_{ij} and a_i?

- We focus on the long wavelength limit (manifolds whose curvature length scales are much larger than the 'mean free path' of the thermal fluid). In this limit the question formulated above may addressed using the techniques of effective field theory.

Partition function technique

- The form of the metric is preserved by p dimensional spatial diffeomorphism together with redefinitions of time of the form

$$t' = t + \phi(\vec{x}), \quad x' = x. \tag{14}$$

Under coordinate changes of the form (14) the Kaluza Klein gauge field a_i transforms like a connection:

$$a_i' = a_i - \partial_i \phi.$$

- Under this upper spatial indices as well as lower temporal indices are invariant.
- On the other hand lower spatial indices and upper temporal indices transform under the Kaluza Klein gauge transformation (14) according to

$$V_i' = V_i - \partial_i \phi V_0, \quad (V')^0 = V^0 + \partial_i \phi V^i. \tag{15}$$

- Partition function should be three dimensional diffeomorphism invariant as well as KK gauge invariant.
- Consequently the partition function of the system is given by

$$\ln Z = \int d^p x \sqrt{g_p} \frac{1}{T(x)} P(T(x)) + \dots \tag{16}$$

where $P(T)$ is the thermodynamical function that computes the pressure as a function of temperature in flat space. Substituting (11) into (16) we find (zeroth order)

$$\ln Z = \int d^p x \sqrt{g_p} \frac{e^{\sigma}}{T_0} P(T_0 e^{-\sigma}) \tag{17}$$

- Starting from this partition function and evaluating the stress tensor and requiring that it reproduces the zeroth order energy momentum tensor also gives

$$u^{\mu}_{(0)}(\vec{x}) = e^{-\sigma}(1, 0, \dots, 0), \qquad T_{(0)}(\vec{x}) = T_o e^{-\sigma} \tag{18}$$

Partition function at higher order

- At first order one can show that there is no possible scalar consistent with the symmetries that were mentioned in the last slide. Consequently there are no equality (non-dissipative transport coefficient) type relations, which is what also obtained from entropy current method.
- Generalizing this to (parity even) charged system where one has background gauge field turned on (here one extra parameter namely chemical potential), one can easily conclude that there is no possible scalar which is consistent with gauge symmetry and previous discussed symmetries.
- So just by this simple analysis we again reach at the conclusion that for first order charged fluid, apart from conductivity (shear and bulk viscosities as well) there are no other transport coefficients. The terms containing viscosities and conductivities as coefficients vanishes at equilibrium.

Second order uncharged fluid

- First thing to notice: Seven coefficients vanish giving us remaining 8 nondissipative transport coefficients.
- At second order, the partition function (parity odd as well) consistent with symmetry takes the form

$$\log Z = \int d^p x \sqrt{g_p} \left(P_1(\sigma) R + T_0^2 P_2(\sigma)(\partial_i a_j - \partial_j a_i)^2 + P_3(\sigma)(\nabla\sigma)^2 \right) \tag{19}$$

  where $P_1(\sigma)$, $P_2(\sigma)$ and $P_3(\sigma)$ are arbitrary functions.
- The most general fluid dynamical partition function, on the other hand, is given in terms of three functions of σ.

- Before doing any computation we conclude
- Among eight parity even symmetry allowed nondissipative transport coefficients, only three are independent. To find precise relations between them we need to do more work. But even to reach at this conclusion one needs to do lots of computation in the other way.

Second order uncharged fluid: Parity odd

Here we shall do simple counting

- Symmetry considerations determine the expansion of the hydrodynamical stress tensor up to 5 parity odd transport coefficients.
- 2 of the odd terms vanish in equilibrium. In other words, on symmetry grounds our system has 2 parity odd dissipative coefficients.
- So we have 3 parity odd non dissipative coefficients.
- It turns out that the partition function (only probable term is $f(\sigma)\epsilon^{ijk}\partial_i\sigma f_{jk}$, which is total derivative) has no parity odd contribution at second order. So it implies, all the three non-dissipative transport coefficients must vanish. Again in order to verify (computation not being dome yet using entropy positivity approach) this statement one needs to do lots of work.

Details of partition function technique

In order to proceed further, we need following ingredients

- The stress tensor is defined as

$$T_{\mu\nu} = -2T_0 \frac{\delta \ln Z}{\delta g^{\mu\nu}} \tag{20}$$

- By application of the chain rule to the above expression for stress tensor, we find

$$\begin{aligned} T_{00} &= -\frac{T_0 e^{2\sigma}}{\sqrt{-g_{(p+1)}}} \frac{\delta W}{\delta \sigma}, \quad T_0^i = \frac{T_0}{\sqrt{-g_{(p+1)}}} \frac{\delta W}{\delta a_i}, \\ T^{ij} &= -\frac{2T_0}{\sqrt{-g_{(p+1)}}} g^{il} g^{jm} \frac{\delta W}{\delta g^{lm}}, \end{aligned} \tag{21}$$

where, for instance, the derivative w.r.t σ is taken at constant a_i, g^{ij}.

steps of computation

- Write down most general symmetry allowed two derivative stress tensor.
- Write down most general symmetry allowed two derivative equilibrium velocity and temperature configuration at equilibrium
- Evaluate stress tensor up to second order at equilibrium (including correction to zeroth order piece coming from velocity and temperature correction).
- Using expression for stress tensor in terms of the partition function, compute the stress tensor and compare with above.

steps of computation

- One obtains relation between transport coefficients, velocity coefficients and temperature coefficients.
- Eliminate the velocity and temperature coefficients to obtain relation between transport coefficients.
- One also obtains equilibrium velocity and temperature up to second order in derivative expansion.

Relations

One obtains same relation as was obtained from entropy positivity method.

$$
\begin{aligned}
\kappa_2 &= \kappa_1 + T\frac{d\kappa_1}{dT}, \quad \zeta_2 = \frac{1}{2}\left[s\frac{d\kappa_1}{ds} - \frac{\kappa_1}{3}\right] \\
\zeta_3 &= \left(s\frac{d\kappa_1}{ds} + \frac{\kappa_1}{3}\right) + \left(s\frac{d\kappa_2}{ds} - \frac{2\kappa_2}{3}\right) + \frac{s}{T}\left(\frac{dT}{ds}\right)\lambda_4 \\
\xi_3 &= \frac{3}{4}\left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\left(T\frac{d\kappa_2}{dT} + 2\kappa_2\right) - \frac{3\kappa_2}{4} + \left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\lambda_4 \\
&+ \frac{1}{4}\left[s\frac{d\lambda_3}{ds} + \frac{\lambda_3}{3} - 2\left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\lambda_3\right] \\
\xi_4 &= -\frac{\lambda_4}{6} - \frac{s}{T}\left(\frac{dT}{ds}\right)\left(\lambda_4 + \frac{T}{2}\frac{d\lambda_4}{dT}\right) - T\left(\frac{d\kappa_2}{dT}\right)\left(\frac{3s}{2T}\frac{dT}{ds} - \frac{1}{2}\right) \\
&- \frac{Ts}{2}\left(\frac{dT}{ds}\right)\left(\frac{d^2\kappa_2}{dT^2}\right)
\end{aligned}
\tag{22}
$$

Summary

To summarize, we have shown an alternate way to constrain the fluid dynamics which gives similar results as the entropy current method. This is remains to be proven or dis proven that both methods will always give rise to same constraint.

I want to thank organizers Suvrat Raju, Aninda Sinha, Sujoy Ashok for invitation and opportunity to present the work. I also want to thank my collaborators specially Prof. Shiraz Minwalla for creating extremely enjoyable atmosphere to work.

Anomalous transport

Here one has

- $\nabla_\mu T^{\mu\nu} = F^\nu_\alpha J^\alpha, \quad \nabla_\mu J^\mu = c * (F \wedge F)$
-

$$T^{\mu\nu}_{diss} = -\zeta\theta\mathcal{P}_{\mu\nu} - \eta\sigma_{\mu\nu}$$
$$J^\mu_{diss} = \sigma\left(E_\mu - T\mathcal{P}^\alpha_\mu\partial_\alpha\nu\right) + \alpha_1 E^\mu + \alpha_2\mathcal{P}^{\mu\alpha}\partial_\alpha T + \xi_\omega\omega^\mu + \xi_B B^\mu \tag{23}$$

- Most general entropy current is

$$J^\mu_S = su^\mu - \nu J^\mu_{diss} + D_\theta\Theta u^\mu + D_c\left(E^\mu - T\mathcal{P}^{\mu\alpha}\partial_\alpha\nu\right) + D_E E^\mu + D_a\mathfrak{a}^\mu + D_\omega\omega^\mu + D_B B^\mu + h\epsilon^{\mu\nu\lambda\sigma}\mathcal{A}_\nu\partial_\lambda\mathcal{A}_\sigma \tag{24}$$

where h is a constant

The constraints that one obtains by demanding positivity is given by

$$
\begin{aligned}
\xi_\omega &= C\nu^2 T^2\big(1 - \frac{2q}{3(\epsilon + P)}\nu T\big) \\
&+ T^2\big[(4\nu C_0 - 2C_2) - \frac{qT}{\epsilon + P}(4\nu^2 C_0 - 4\nu C_2 + 4C_1)\big], \\
\xi_B &= C\nu T\big(1 - \frac{q}{2(\epsilon + P)}\nu T\big) + T\big(2C_0 - \frac{qT}{\epsilon + P}(2\nu C_0 - C_2)\big), \\
\alpha_1 &= \alpha_2 = 0 \qquad (25)
\end{aligned}
$$

here c_1 and c_2 are integration constants. So free parameter three.

Partition function method

Here one can easily write down the partition function up to first order in derivative

$$
\begin{aligned}
\ln Z &= W^0 + W^1_{inv} + W^1_{anom} \\
W^0 &= \int \sqrt{g_3}\frac{e^{\sigma}}{T_0} P\left(T_0 e^{-\sigma}, e^{-\sigma} A_0\right) \\
W^1_{inv} &= \frac{C_0}{2}\int AdA + \frac{T_0^2 C_1}{2}\int ada + \frac{T_0 C_2}{2}\int Ada \\
W^1_{anom} &= \frac{C}{2}\left(\int \frac{A_0}{3T_0} AdA + \frac{A_0^2}{6T_0} Ada\right)
\end{aligned}
\tag{26}
$$

So here naturally three parameters appear. One can follow same procedure as discussed above to reach at the same result as obtained above.

In two dimensions one has

$$\nabla_\mu J^\mu = c_s \epsilon^{\mu\nu} F_{\mu\nu}, \quad \nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu + c_g \epsilon^{\mu\nu} \nabla_\nu R \tag{27}$$

Most general current in two dimensions can be written as

$$J^\mu = q u^\mu + c_1 \epsilon^{\mu\nu} u_\nu. \tag{28}$$

It turns out that there is a relation between c_g and c_1. Notice the derivative order mixing. Recently, using the frame work of partition function this is shown to be hold. See arXiv:1207.5824, Jensen, Loganayagam, Yarom.