

# What are minimal surfaces?

RUKMINI DEY

International Center for Theoretical Sciences (ICTS-TIFR), Bangalore

SWSM, ICTS, Bangalore

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# Topology of compact oriented surfaces – genus

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Sphere, torus, double torus are usual names you may have heard.

Sphere and ellipsoids are abundant in nature – for example a ball and an egg. They are topologically “same” – in the sense if they were made out of plasticine, you could have deformed one to the other.

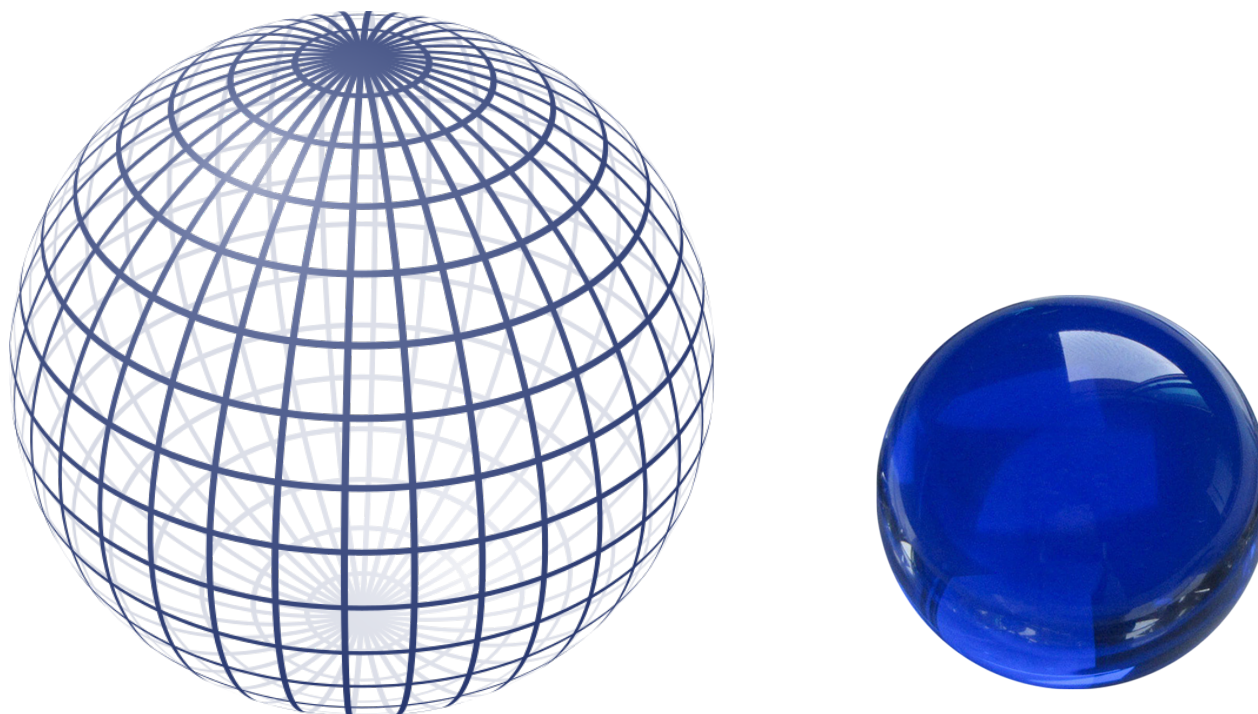


Figure 1: Sphere and a glass ball

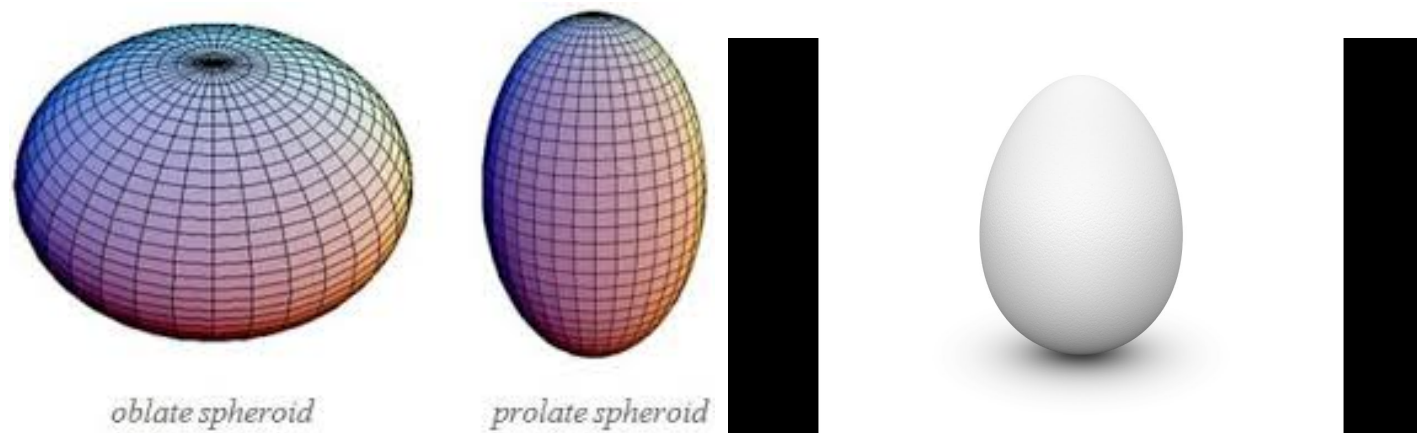


Figure 2: Ellipsoids and an egg

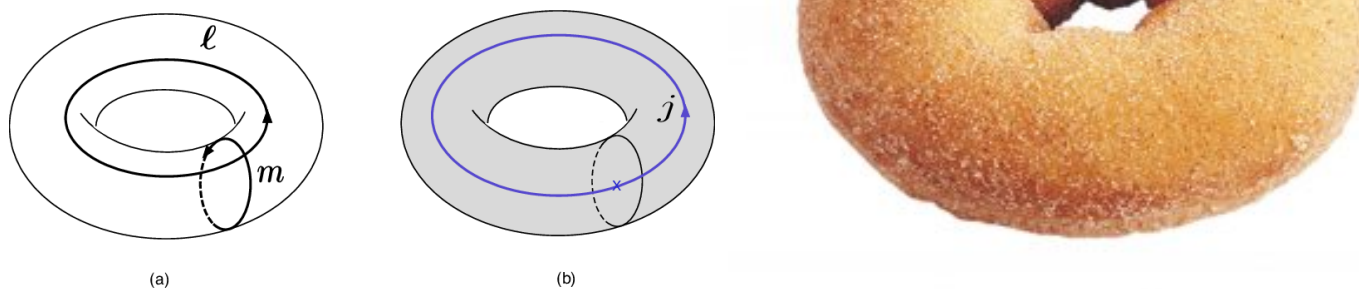


Figure 3: Tori and donut

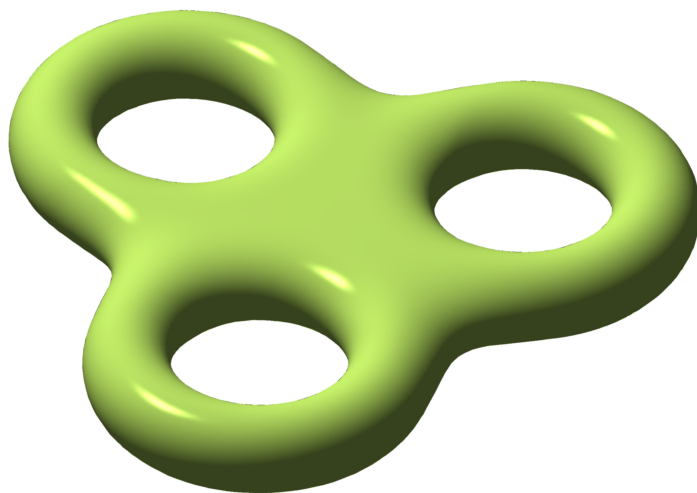


Figure 4: Triple torus and surface of a pretzel



# Handle addition to the sphere –genus

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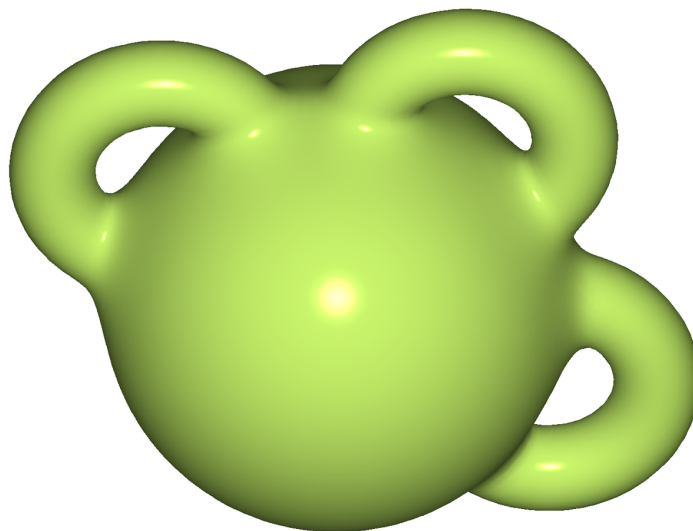


Figure 5: Triple torus

In fact as the above figure suggests one can construct more and more complicated surfaces by adding handles to a sphere – the number of handles added is the genus.

# Genus classifies compact oriented surfaces topologically

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Sphere (genus  $g = 0$ ) is not topologically equivalent to torus (genus  $g = 1$ ) and none of them are topologically equivalent to double torus (genus  $g = 2$ ) and so on.

One cannot continuously deform one into the other if the genus-es are different.

## ◆ Euler's formula:

Divide the surface up into faces (without holes) with a certain number of vertices and edges. Let no. of vertices =  $V$ , no. of faces =  $F$ , no. of edges =  $E$ , no. of genus-es =  $g$ .

Then Euler's formula for compact oriented surface of genus  $g$  is

$$V - E + F = 2 - 2g.$$

# Euler's formula for the sphere

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Let us test this formula for the sphere.

For example triangulation of sphere:

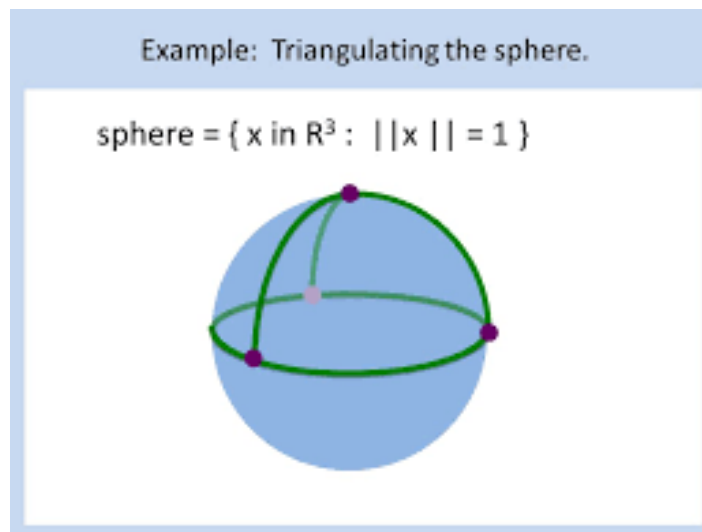
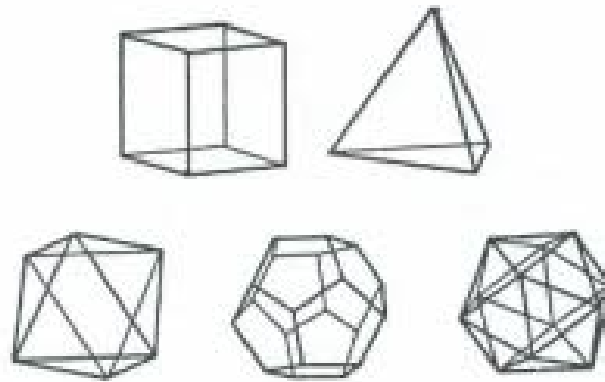


Figure 6: Triangulation of the sphere

It is easy to check that  $V - E + F = 2$  and hence confirm genus  $g = 0$

Polygonal shapes topologically equivalent to the sphere:



**Fig. 1**

Figure 7: Polyhedrons

It is easy to check that  $V - E + F = 2$  and hence confirm genus  $g = 0$

# Euler formula for the torus

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The torus can be obtained from a rectangular sheet of paper with opposite edges identified.

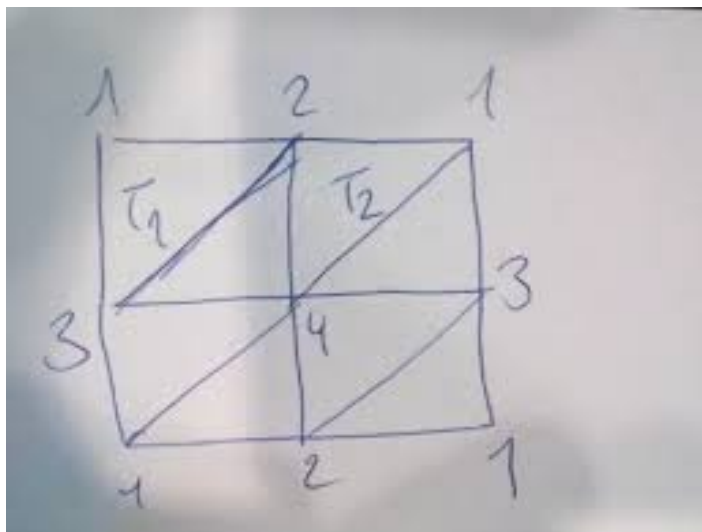


Figure 8: Triangulation of the torus

It is easy to check that  $V - E + F = 0$  confirming genus  $g = 1$ .

# A planar figure

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A triangle or any connected planar figure  $V - E + F = 1$

(Please check)



# An example of a non-oriented surface with a boundary

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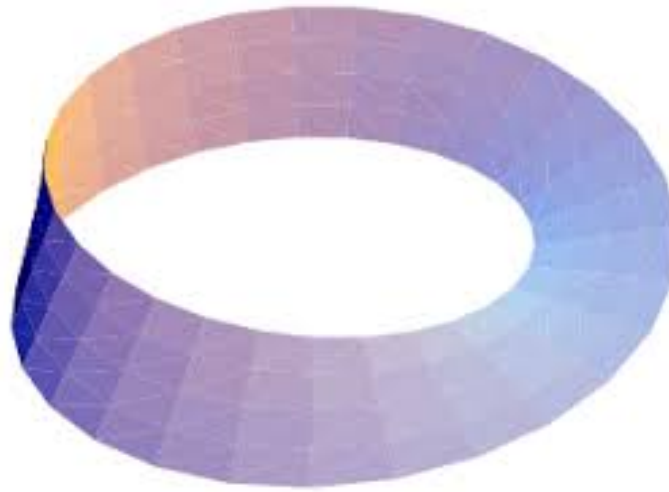


Figure 9: Möbius strip

# Geometry of surfaces

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When we talk of geometry, we talk of lengths of curves on surfaces, area of the surface, Gaussian curvature, mean curvature etc. In this sense, sphere and ellipsoids (though topologically equivalent) are very different. Sphere for example is totally symmetric while the ellipsoid is not!

# Parametrized surfaces in $\mathbb{R}^3$

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Let  $\mathbf{X} = \mathbf{X}(u, v)$  is a surface in  $\mathbb{R}^3$  parametrized by two parameters  $u, v$ , i.e.  $\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v))$  *locally*.

For example, the helicoid:  $z = \tan^{-1}(y/x)$  can be locally parametrized in many ways:

(1)  $\mathbf{X}(u, v) = (u, v, \tan^{-1}(v/u))$

(2)  $\mathbf{X}(r, \theta) = (r \cos \theta, r \sin \theta, \theta)$

(3) Now we introduce a novel parametrization:

Let  $\zeta = u + iv$ ,  $\mathbf{X}(u, v) = \mathbf{X}(\zeta, \bar{\zeta}) = (x(\zeta, \bar{\zeta}), y(\zeta, \bar{\zeta}), z(\zeta, \bar{\zeta}))$  where

$$x(\zeta, \bar{\zeta}) = -\frac{1}{2} \operatorname{Im}(\zeta + \frac{1}{\bar{\zeta}})$$

$$y(\zeta, \bar{\zeta}) = \frac{1}{2} \operatorname{Re}(\zeta - \frac{1}{\bar{\zeta}})$$

$$z(\zeta, \bar{\zeta}) = -\frac{\pi}{2} + \operatorname{Im} \ln(\zeta)$$

This is actually a helicoid!!

This representation fails for  $\zeta = 0$ .

# The helicoid

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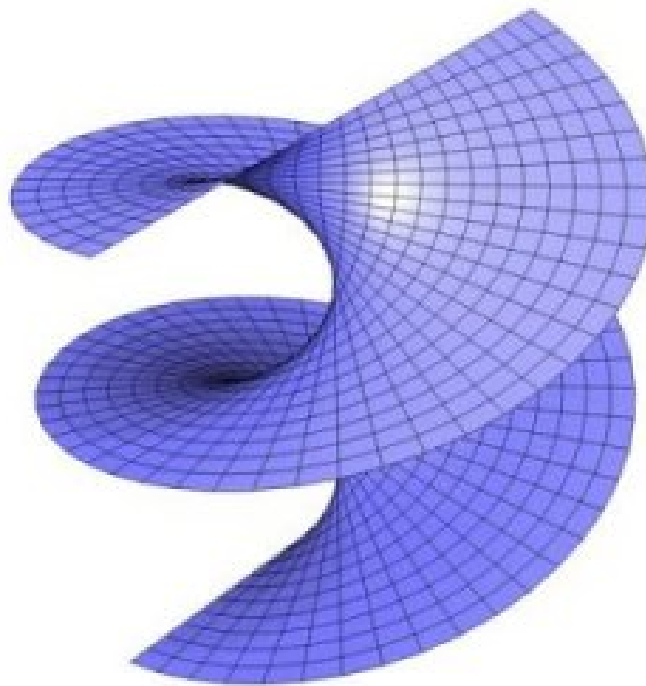


Figure 10: Helicoid: a minimal surface

# Normal and tangent plane to a regular surface at a point on the surface

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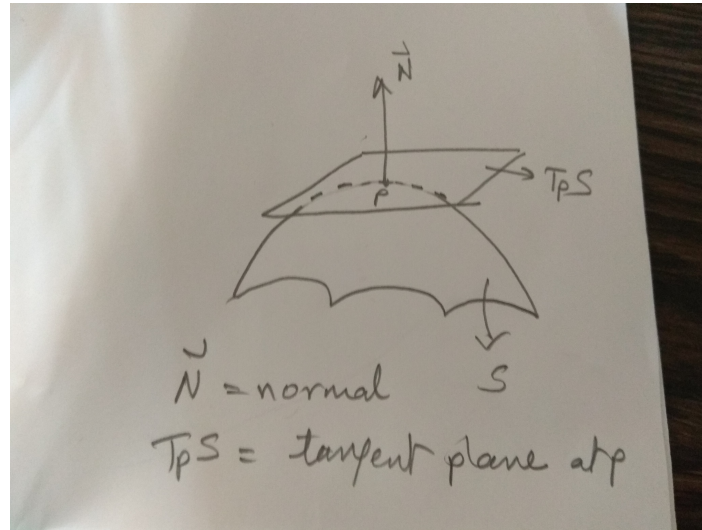


Figure 11: normal and tangent plane to a regular surface at  $p$

- ◆ Suppose  $\mathbf{X} = \mathbf{X}(u, v)$  is a regular surface  $S$  in  $\mathbb{R}^3$  parametrized by two parameters  $u, v$ , i.e.  $\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v))$  *locally*.

Then  $\mathbf{X}_u$  and  $\mathbf{X}_v$  evaluated at  $p$  are linearly independent vectors in  $\mathbb{R}^3$  and span the tangent plane  $T_p S$  at  $p$ .

Then, the unit normal  $\mathbf{N}$  at  $p$  is given by  $\frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}$ , evaluated at  $p$ .

# Normal curvature at a point on the surface in a specified direction

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- ◆ Recall that the curvature of a curve parametrised by  $s$ , namely,

$$\mathbf{l}(s) = (l_1(s), l_2(s), l_3(s))$$

in  $\mathbb{R}^3$  is given by  $k$  where  $|k| = |\mathbf{l}''(s)|$

- ◆ Normal curvature along a given direction: Take any direction  $\mathbf{v}$  in the tangent plane to  $S$  at  $p$ . Take the plane containing the unit normal to the surface,  $\mathbf{N}$ , and  $\mathbf{v}$ . This plane intersects the surface  $S$  in a curve  $\mathbf{l}$ . It is called the normal section at  $p$  along  $\mathbf{v}$ . The curvature  $k$  of this curve is called the normal curvature  $k_n$  in direction  $\mathbf{v}$ .



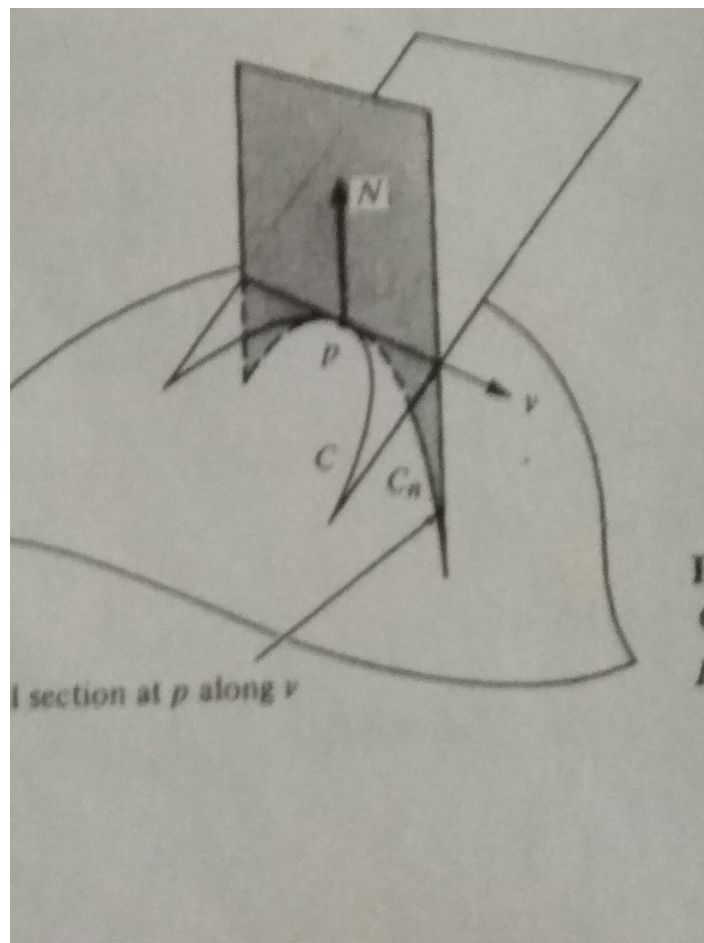


Figure 12: Normal section at  $p$  along  $v$

# Principal curvatures of a surface at a point; Gaussian and Mean curvatures

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Let the direction  $\mathbf{v}$  vary and take all possible directions. One finds that  $k_n$  achieves a maximum and minimum – denoted by  $k_1$  and  $k_2$  – along two orthogonal directions  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . These are called principal curvatures and the two orthogonal directions are called principal directions of curvature.

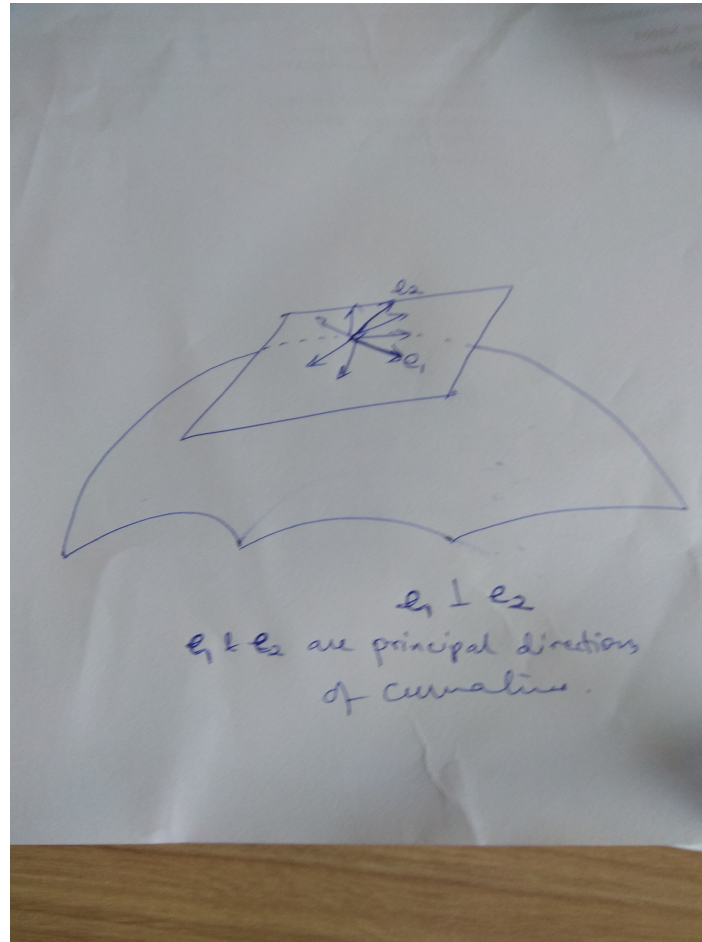


Figure 13: The two principal directions of curvature

The Gaussian curvature is given by  $\mathbf{K} = k_1 k_2$  and the Mean curvature is given by  $\mathbf{H} = k_1 + k_2$

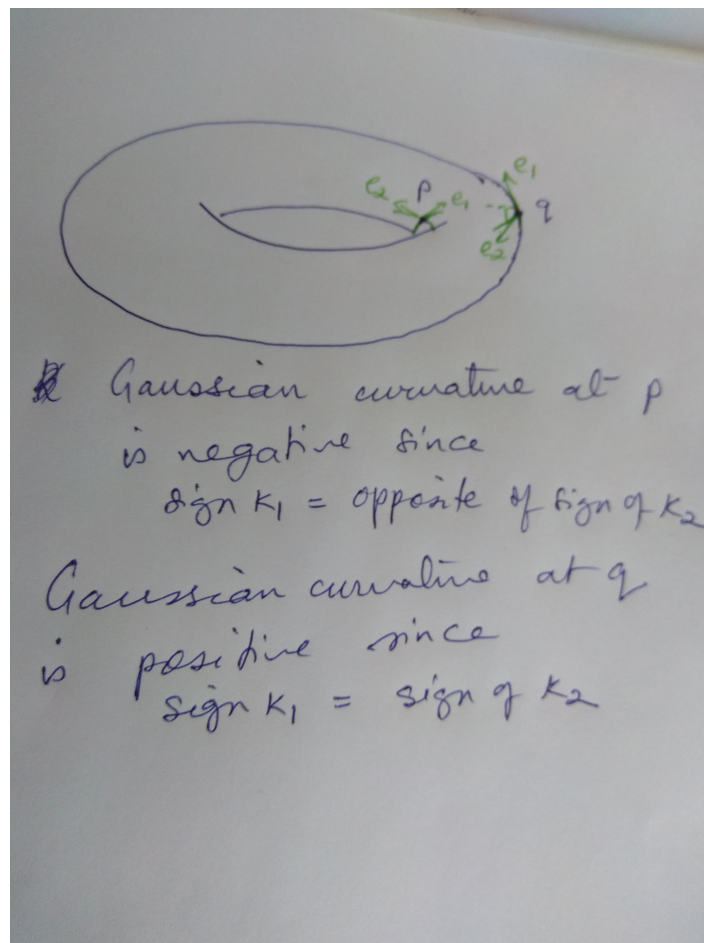


Figure 14: Positive and Negative Gaussian curvatures on a torus

# Minimal Surfaces

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- ◆ We will be talking about surfaces called "**Minimal Surfaces**", which appear in nature as idealised soap-films. They are stationary points of the area functional (i.e. locally they are area minimizing or saddle point of the area functional).

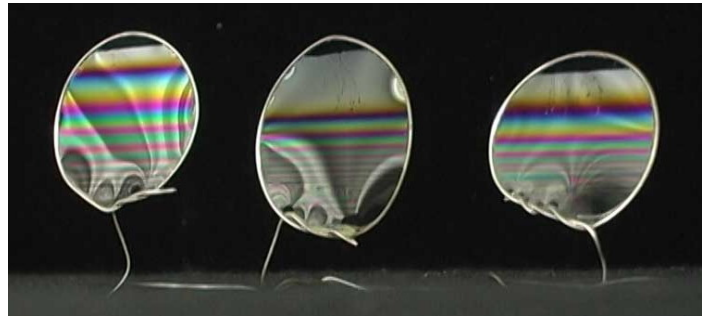


Figure 15: Soap-films spanning a wire-frame

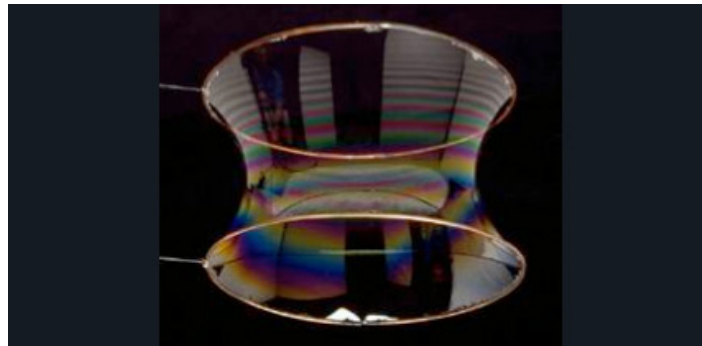


Figure 16: Catenoidal soap-film

- ◆ They have the property that the **mean curvature**  $=0$  at every point on the surface. If the two principal curvatures are  $k_1$  and  $k_2$ , then  $k_1 + k_2 = 0$ . If  $k_1 = -k_2 \neq 0$ , the two principal lines of curvature at every point curve in opposite directions – thus it has the shape of a **saddle** at every non-umbilical point. They have negative or zero **Gaussian curvature**. Umbilical points are where Gaussian curvature is zero.
- ◆ **Caution:** Soap bubbles ( spherical) are not minimal surfaces since they have non-zero mean curvature at every point.



◆ Some Examples:

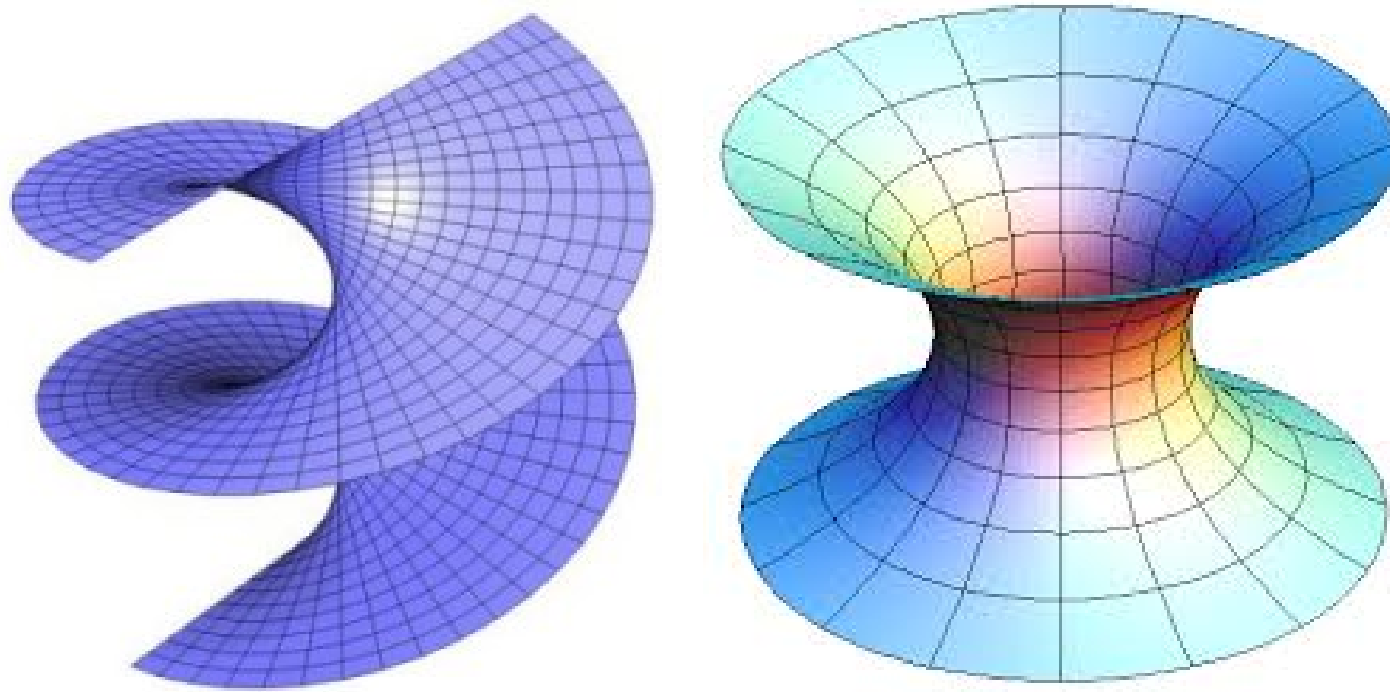


Figure 17: Helicoid and Catenoid: Conjugate minimal surfaces

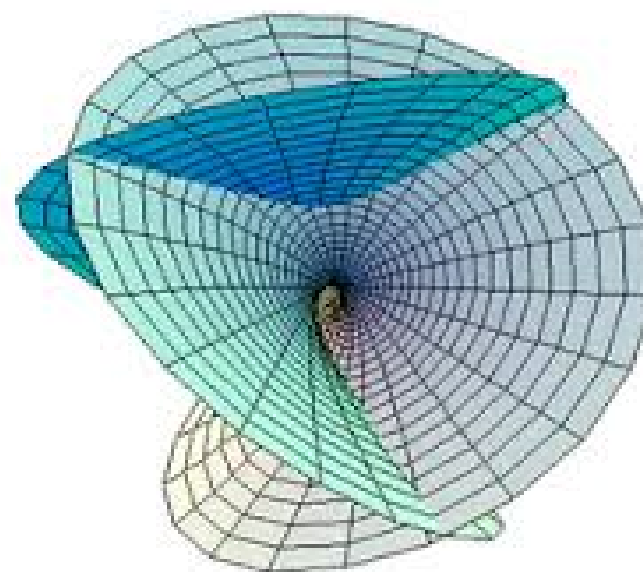
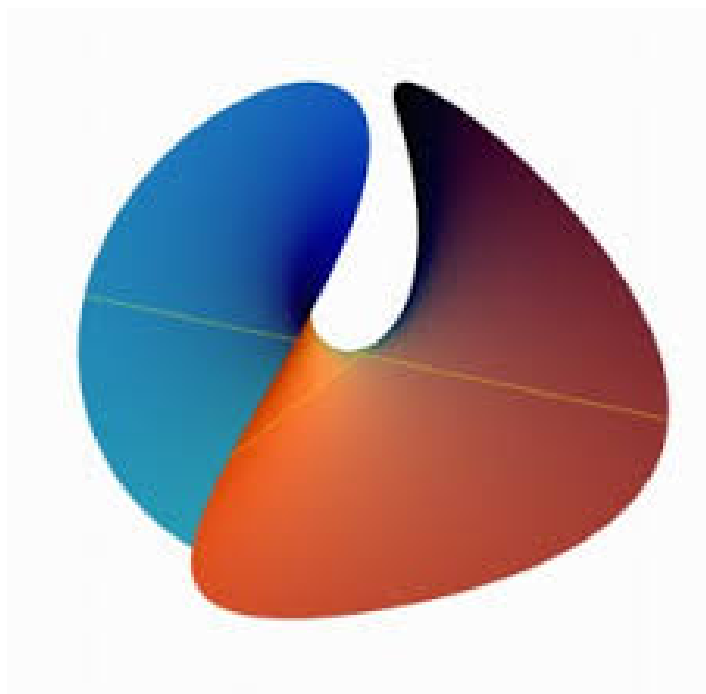


Figure 18: Enneper surface

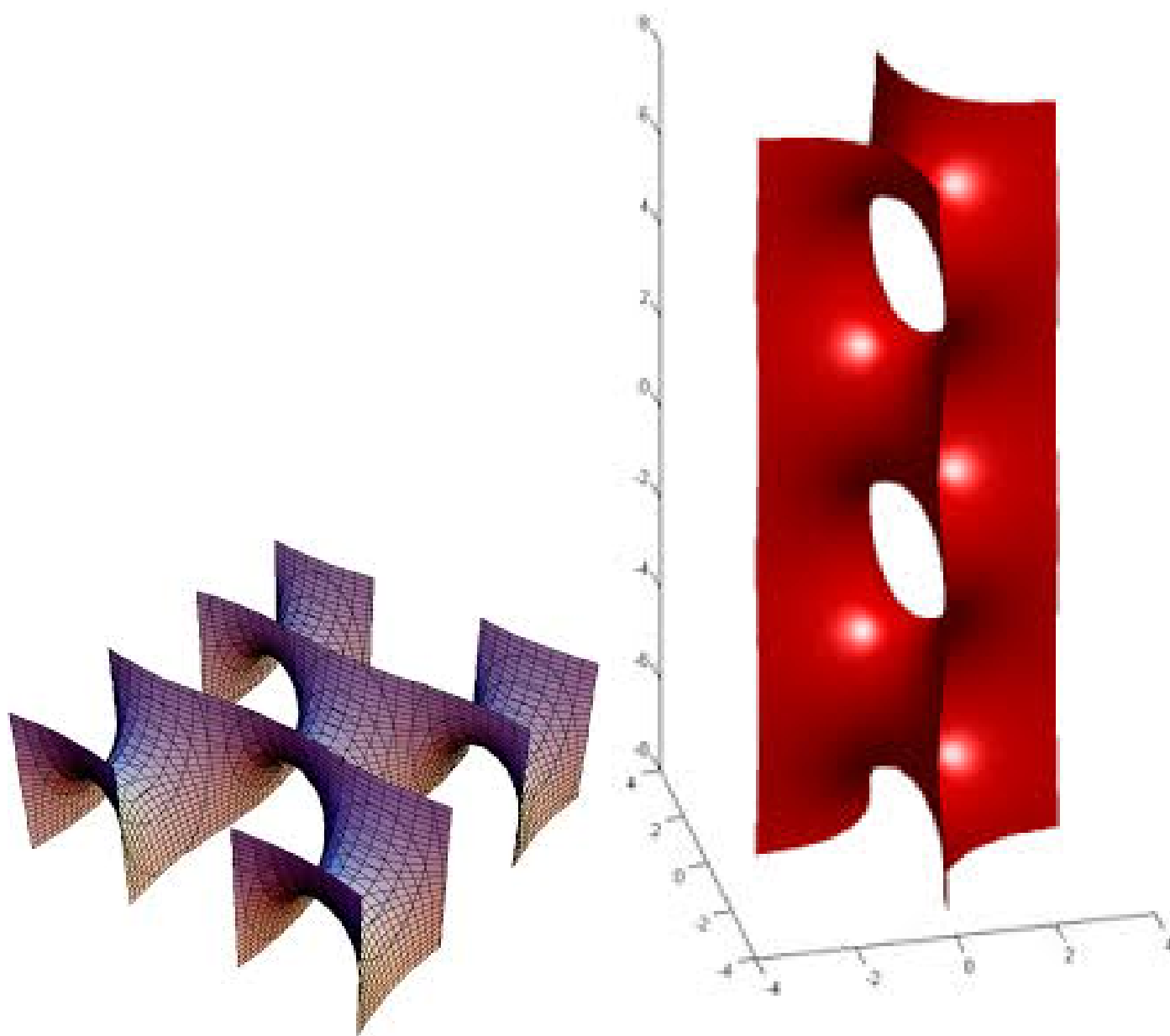


Figure 19: Scherk's surfaces

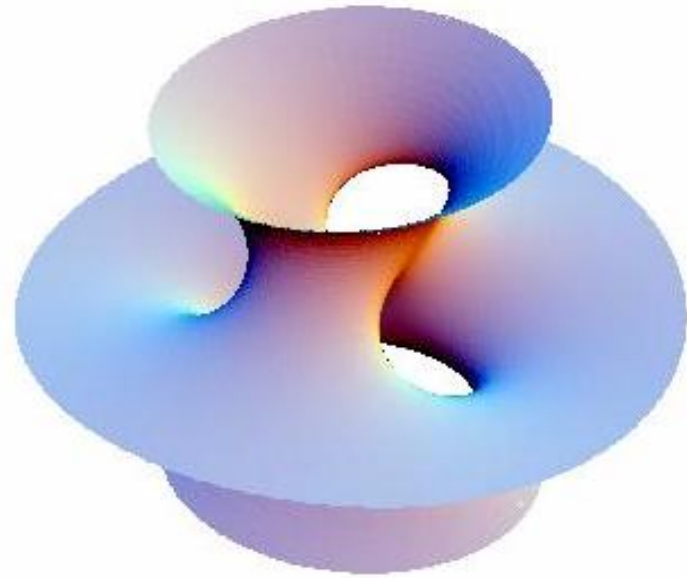


Figure 20: Genus 1 Costa surface: immersion of a torus with 3 punctures!

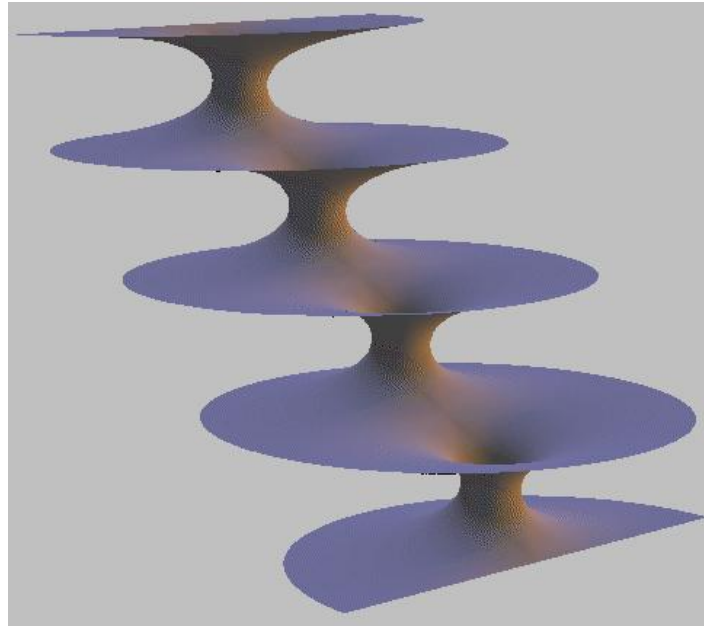


Figure 21: The Riemann Staircase

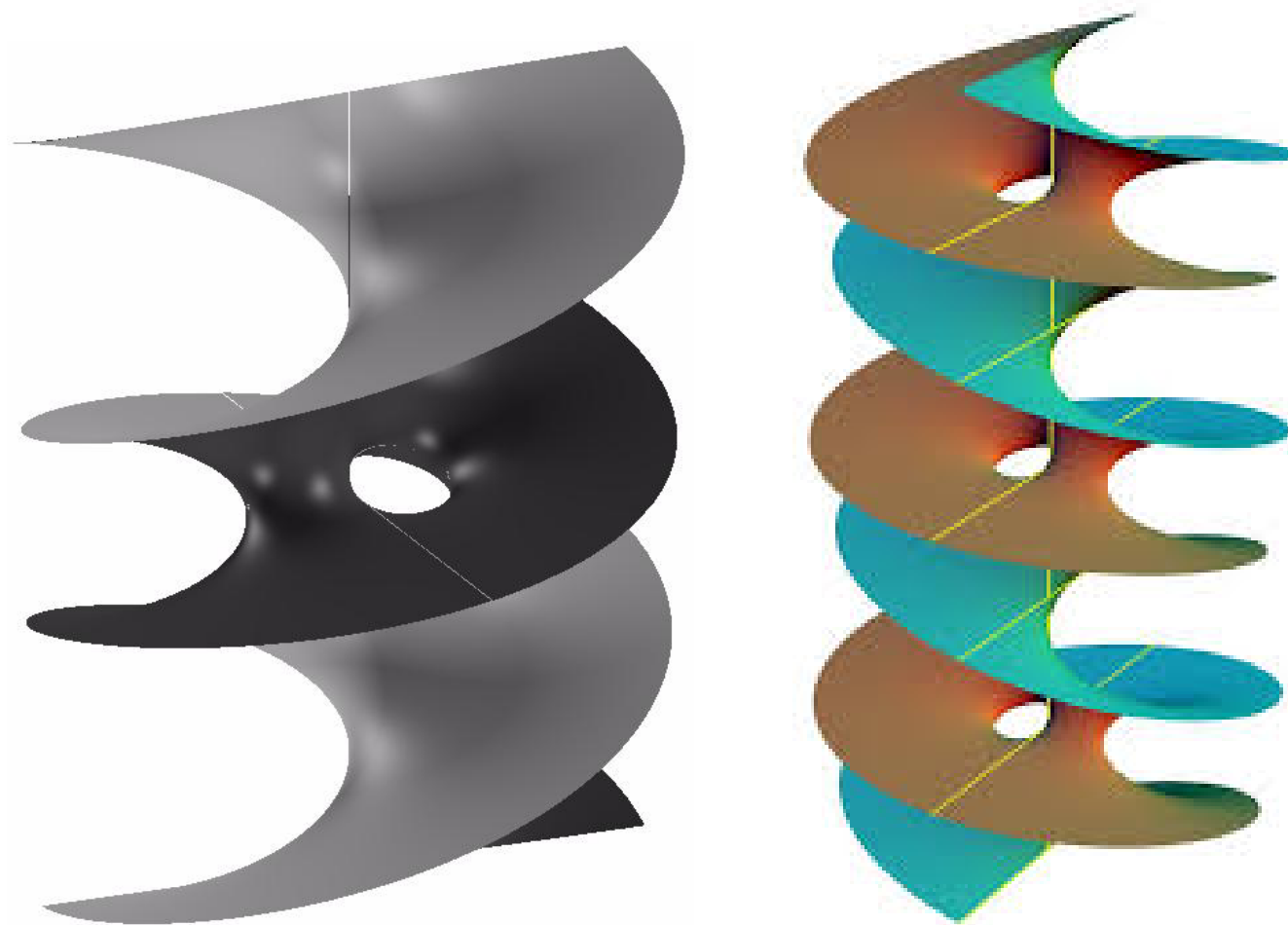


Figure 22: Genus 1 and Multiple genus helicoid



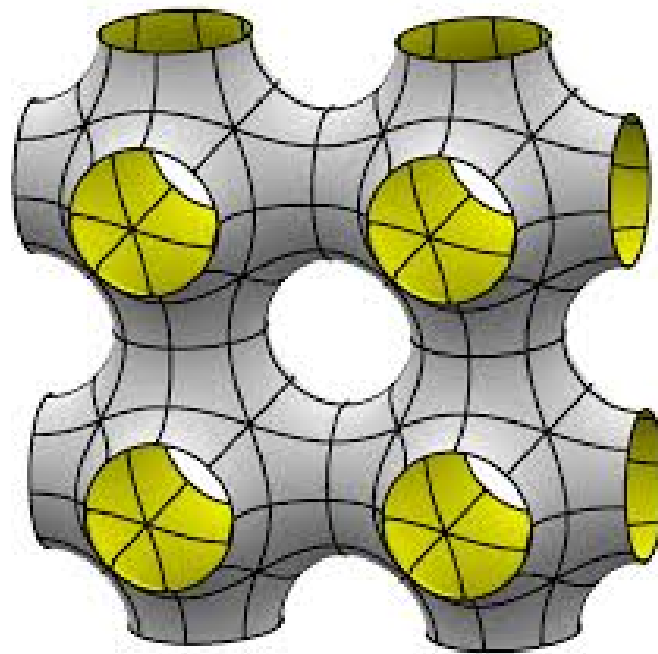


Figure 23: Triply periodic minimal surface

# The Plateau's problem for the layman

- ◆ Given a small closed wire frame, if you dip it in soap solution, is there a soap film spanning it and if so how many? **Douglas** gave answer to the mathematical formulation of this question in the affirmative and subsequently this has been an active field of research. The question of how many minimal surfaces span a given curve has been studied by **Tromba** using bifurcation theory.

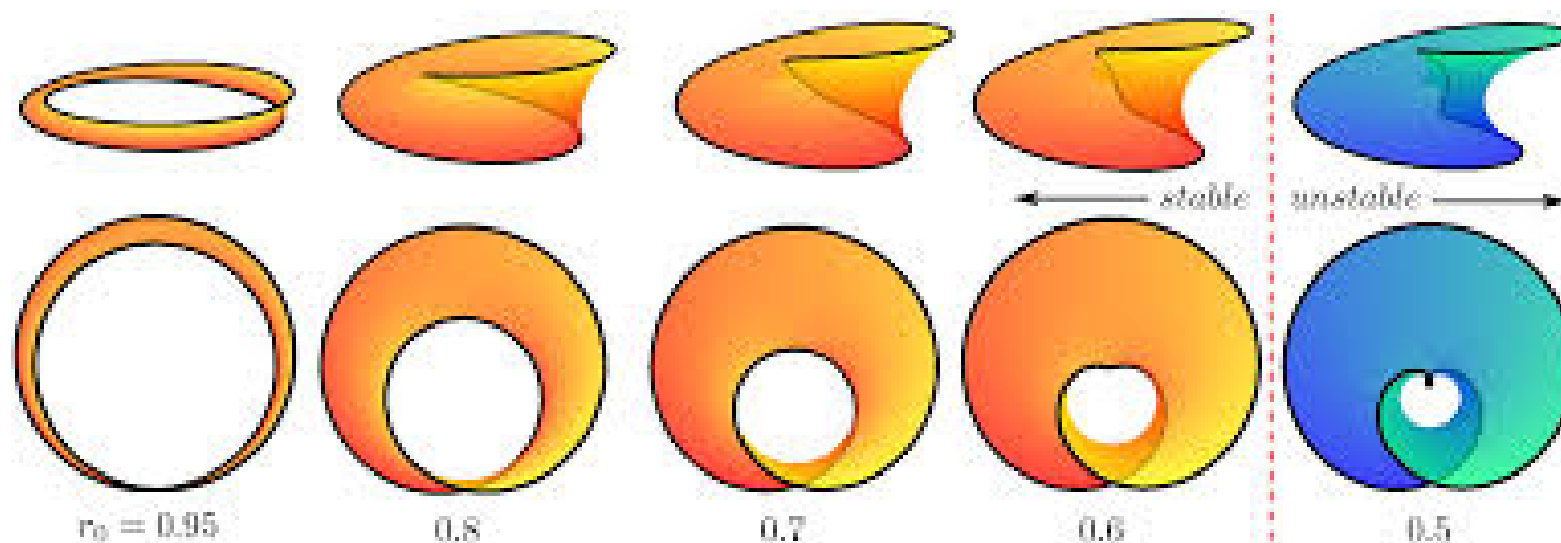


Figure 24: Two types of minimal surfaces spanning a mobius curve

# Harmonicity of coordinates in Isothermal parameters

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- ◆ Locally, if the minimal surface can be written as a graph of a function of two variables, namely  $(x, y, \phi = \phi(x, y))$ , then mean curvature = 0 implies minimal surfaces are solutions of the following non-linear equation:

$$(1 + \phi_x^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + (1 + \phi_y^2)\phi_{xx} = 0.$$

The surface is locally given by  $X(x, y) = (x, y, \phi(x, y))$ .

- ◆ However, for minimal surfaces (and in fact all smooth surfaces) there exists certain special parameters, called **isothermal (or conformal) parameters**.

$X(u, v) = (x(u, v), y(u, v), z(u, v))$ ,  $u, v$  are the conformal parameters. (To be noted: each minimal surface has its own special conformal parameters)

- ◆ In these coordinates, a miracle happens! The minimal surface equation becomes,

$$\Delta_{uv}X = 0 \text{ (harmonicity of the coordinates)}$$

and the isothermality condition, namely,  $|X_u| = |X_v|$  and  $\langle X_u, X_v \rangle = 0$ .

- ◆ Harmonicity of the coordinates makes it possible to relate the theory of minimal

surfaces to complex analysis!! Real and imaginary parts of holomorphic functions are harmonic!

(Recall: holomorphic functions are such that their real and imaginary parts satisfy the Cauchy-Riemann equations).

♦ Complete minimal surfaces cannot be compact!!

# The Weierstrass-Enneper Representation of Minimal surfaces

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- ◆ Let  $\zeta = u + iv$  and  $\bar{\zeta} = u - iv$ . For every non-planar minimal surface  $X(u, v) = X(\zeta, \bar{\zeta}) = (x(\zeta, \bar{\zeta}), y(\zeta, \bar{\zeta}), \phi(\zeta, \bar{\zeta}))$ ,  $\zeta \in \Omega$ , a simply connected domain in  $\mathbb{C}$ , there is a holomorphic function  $f$  and a meromorphic function  $g$  in  $\Omega$  with  $f, g$  not identically zero, such that  $fg^2$  is holomorphic in  $\Omega$  and that

$$x(\zeta, \bar{\zeta}) = x_0 + \operatorname{Re} \int_{\zeta_0}^{\zeta} \frac{1}{2} f(w)(1 - g^2(w))dw$$

$$y(\zeta, \bar{\zeta}) = y_0 + \operatorname{Re} \int_{\zeta_0}^{\zeta} \frac{i}{2} f(w)(1 + g^2(w))dw$$

$$\phi(\zeta, \bar{\zeta}) = \phi_0 + \operatorname{Re} \int_{\zeta_0}^{\zeta} f(w)g(w)dw$$

# Re-writing of Weierstrass-Enneper Representation of Minimal surfaces

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- ◆ In a neighbourhood of a non-umbilical point (i.e. where the Gaussian curvature  $\neq 0$ ), any minimal surface can be represented in terms of a non-vanishing meromorphic function  $R(\omega)$ :

$$\begin{aligned}x(\zeta, \bar{\zeta}) &= x_0 + \operatorname{Re} \int_{\zeta_0}^{\zeta} (1 - w^2) R(w) dw \\y(\zeta, \bar{\zeta}) &= y_0 + \operatorname{Re} \int_{\zeta_0}^{\zeta} i(1 + w^2) R(w) dw \\\phi(\zeta, \bar{\zeta}) &= \phi_0 + \operatorname{Re} \int_{\zeta_0}^{\zeta} 2w R(w) dw\end{aligned}$$

◆ Examples:

1.  $R(w) = 1$  corresponds to Enneper surface
2.  $R(w) = \frac{k}{2w^2}$ ,  $k$  real, corresponds to Catenoid
3.  $R(w) = \frac{ik}{2w^2}$ ,  $k$  real, corresponds to Helicoid
4.  $R(w) = \frac{ke^{i\alpha}}{2w^2}$ , corresponds to the General Helicoid
5.  $R(w) = \frac{2}{1-w^4}$ , corresponds to the one of the Scherk's minimal surface

which can be written in the graph form as

$$z = \ln\left(\frac{\cos(y)}{\cos(x)}\right), \text{ (valid in some domain for } (x, y)\text{)}.$$

6.  $R(w) = \frac{-2a \sin(2\alpha)}{1+2w^2 \cos 2\alpha + w^4}$ ,  $0 < \alpha < \pi/2$ ,  $a > 0$  corresponds to the other type of Scherk's surface.

It is given by  $\tanh\left(\frac{z}{a}\right) = \tan\left(\frac{x}{a \cos \alpha}\right) \tan\left(\frac{y}{a \sin \alpha}\right)$ .

Later the formula that will be important for us is when the above is re-written as follows:

$$\frac{x}{a \cos \alpha} = \tan^{-1}\left(\tanh\left(\frac{z}{a}\right) \cot\left(\frac{y}{a \sin \alpha}\right)\right).$$

7.  $R(w) = 1 - w^{-4}$ , and (substituting  $-y$  for  $y$ ) lead to Henneberg's minimal surface.

8.  $R(w) = ia(w^2 - 1)/w^3 - ib/2w^2$ ,  $a$  and  $b$  real, and setting  $w = e^{-i\gamma/2}$  leads to another type of Enneper's minimal surface, and in particular for  $a = 1, b = 0$  to Catalan's surface.

One can generate all examples this way away from the umbilical points.

Gaussian curvature of the surface is  $K = -4|R(w)|^{-2}(1 + |w|^2)^{-4}$ .

(See [Nitsche](#), Lectures on Minimal Surfaces ).



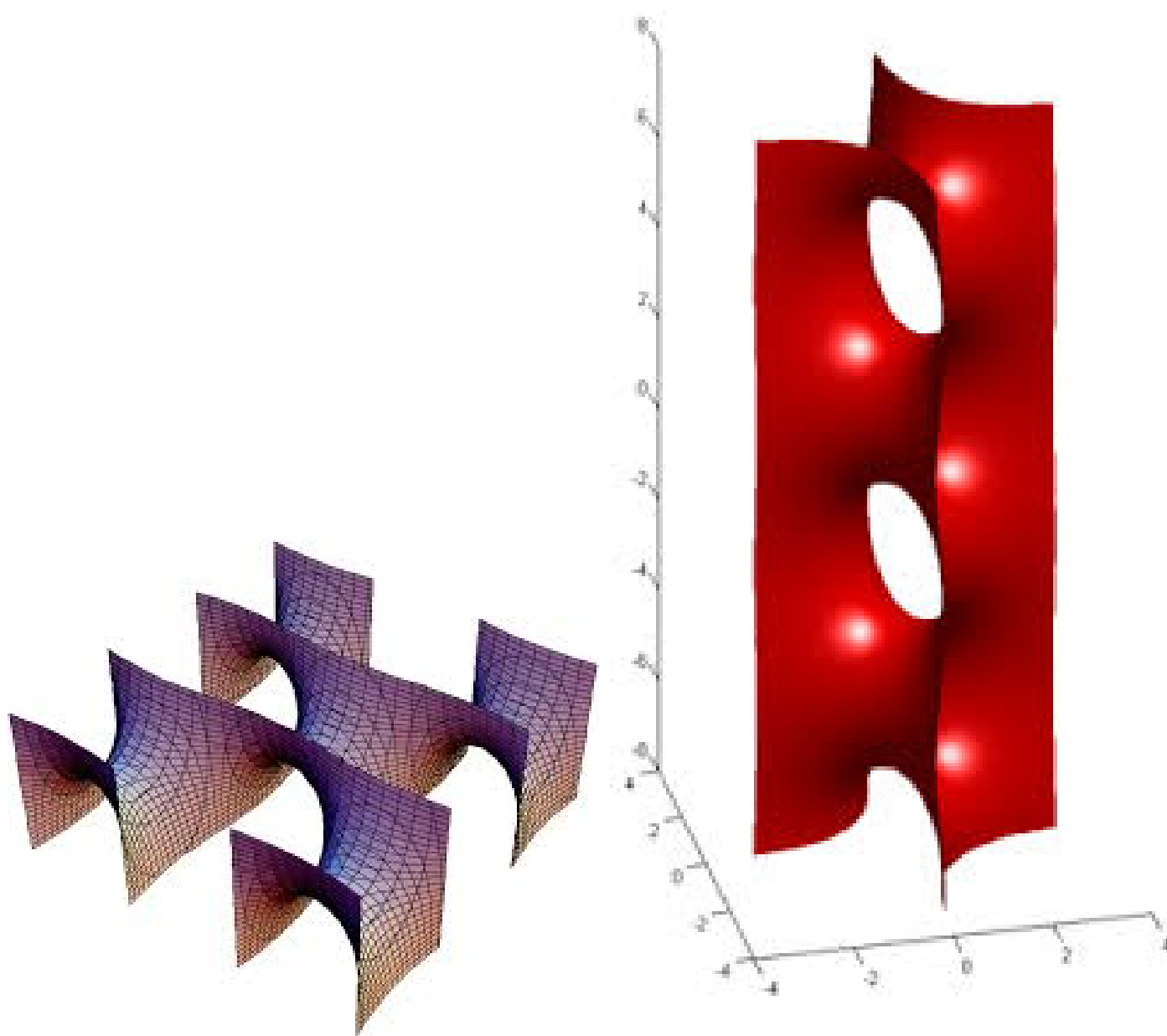


Figure 25: Scherk's surfaces

- ◆ W-E representation for Catenoid,  $\phi = \cosh^{-1}(\sqrt{x^2 + y^2})$ :  
 $R(w) = \frac{1}{2w^2}$ , corresponds to Catenoid.

$$\begin{aligned}x(\zeta, \bar{\zeta}) &= \frac{1}{2} \operatorname{Re}\left(\zeta + \frac{1}{\zeta}\right) \\y(\zeta, \bar{\zeta}) &= \frac{1}{2} \operatorname{Im}\left(\zeta - \frac{1}{\zeta}\right) \\\phi(\zeta, \bar{\zeta}) &= -\operatorname{Re} \ln(\zeta)\end{aligned}$$

This representation breaks down at  $\zeta = 0$ .

- ◆ W-E representation for the Helicoid,  $\phi = \tan^{-1}(y/x)$ :  
 where  $R(w) = \frac{i}{2w^2}$ ,

$$x(\zeta, \bar{\zeta}) = -\frac{1}{2}\text{Im}\left(\zeta + \frac{1}{\zeta}\right)$$

$$y(\zeta, \bar{\zeta}) = \frac{1}{2}\text{Re}\left(\zeta - \frac{1}{\zeta}\right)$$

$$\phi(\zeta, \bar{\zeta}) = -\frac{\pi}{2} + \text{Im}\ln(\zeta)$$

# Isometry between Conjugate Minimal surfaces

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- ◆ Suppose  $X = (x_1(\zeta_1, \zeta_2), y_1(\zeta_1, \zeta_2), z_1(\zeta_1, \zeta_2))$  and  $Y = (x_2(\zeta_1, \zeta_2), y_2(\zeta_1, \zeta_2), z_2(\zeta_1, \zeta_2))$  are two minimal surfaces in isothermal coordinates  $\zeta_1, \zeta_2$  such that  $X$  and  $Y$  are harmonic conjugates of each other (component-wise).

In other words,  $X + iY$  is an analytic function (each coordinate wise).

Then for all  $\theta \in [0, \pi/2]$ ,  $\cos(\theta)X + \sin(\theta)Y$  is a minimal surface again!! (This is a well known fact).

- ✦ For example, the Helicoid and the Catenoid are harmonic conjugates of each other and  $\cos(\theta)X_h + \sin(\theta)Y_c$  is again a minimal surface for each  $\theta$ . It is in fact an isometric deformation!!

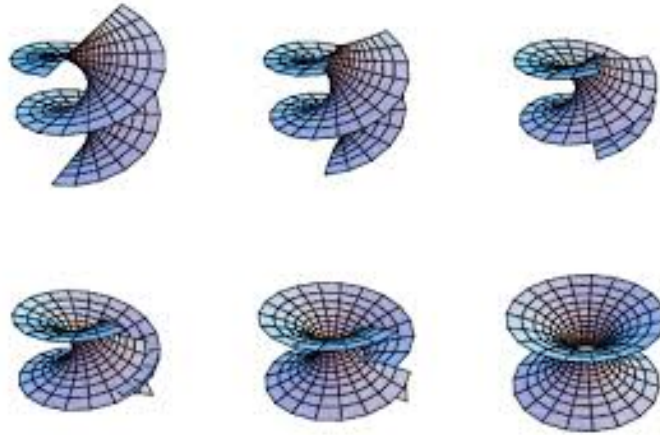


Figure 26: Isometric deformation from helicoid to catenoid

# Geometrical interpretation of an identity of Ramanujan

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As a consequence of a Ramanujan's identity:

$$\tan^{-1}(\tanh(y)\cot(x)) = \sum_{k=-\infty}^{k=\infty} \tan^{-1}\left(\frac{y}{x + k\pi}\right).$$

The left hand side is the height function of Scherk's first surface and the right hand side is sum of height functions of shifted helicoids – all minimal surfaces!

Said differently,  $z = \tan^{-1}(\tanh(y)\cot(x))$  is the Scherk's first surface and  $z = \tan^{-1}\left(\frac{y}{x+k\pi}\right)$  is a shifted helicoid.

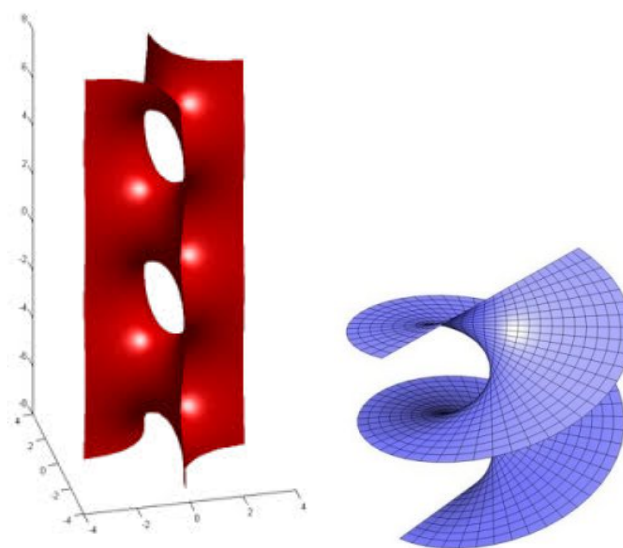


Figure 27: Scherk's first surface and helicoid

This last fact was discovered and used by a condensed matter physicist and his collaborators –Randall Kamien et. al. , U . Penn.

Using Euler-Ramanujan identities of the above kind and W-E representation of minimal surfaces, we obtain a plethora of identities.

# Interpolation of two real analytic curves by a minimal surface

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- ◆ Given two real analytic curves “close enough” there is a connected minimal surface which contain both of them. This follows from Douglas’ work, but we show it using the Bjorling-Schwartz problem and the inverse function theorem in infinite dimensional setting.



# Maximal surfaces in Lorentz-Minkowski space

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- ◆ A graph  $(x, y, f(x, y))$  in Lorentz-Minkowski space  $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$  is maximal if it satisfies

$$(1 - f_x^2)f_{yy} + 2f_x f_y f_{xy} + (1 - f_y^2)f_{xx} = 0, \quad (1)$$

for some smooth function  $f(x, y)$  satisfying  $f_x^2 + f_y^2 < 1$ .

It is the **zero mean curvature** equation in Lorentz-Minkowski space.

This equation is known as maximal surface equation.

- ◆ The Weierstrass-Enneper representation for a maximal surface  $(x, y, z)$ , in Lorentz-Minkowski space  $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ , whose Gauss map is one-one is given by

$$x(\zeta) = x_0 + \operatorname{Re}(\int_{\zeta_0}^{\zeta} M(\omega)(1 + \omega^2)d\omega),$$

$$y(\zeta) = y_0 + \operatorname{Re}(\int_{\zeta_0}^{\zeta} iM(\omega)(1 - \omega^2)d\omega),$$

$$z(\zeta) = z_0 + \operatorname{Re}(\int_{\zeta_0}^{\zeta} -2M(\omega)\omega d\omega)$$

where  $\zeta = u + iv$  belongs to a small domain and  $M(\omega)$  is a meromorphic function.

## ◆ Examples

### 1. Spacelike Elliptic Catenoid:

$$X(u, v) = (\cos(u)\sinh(v), \sin(u)\sinh(v), v)$$

### 2. Spacelike Parabolic Catenoid:

$$X(u, v) = (v + \frac{v^3}{3} - u^2v, 2uv, v - \frac{v^3}{3} + u^2v)$$

### 3. Spacelike Hyperbolic Catenoid:

$$X(u, v) = (v, \sin(v)\sinh(u), \sin(v)\cosh(u))$$

There are corresponding **timelike catenoids**, **spacelike** and **timelike helicoids**.



Figure 28: Elliptic, Parabolic and Hyperbolic catenoids

Maximal surfaces have many properties analogous to minimal surfaces, for instance:

1. W-E representation which can be obtained by hodographic coordinates (Rahul K. Singh)
2. Relation to complex Born-Infeld solitons (equations can be obtained by Wick rotation) (R.K. Singh + R.D.)
3. Some identities were derived from Euler-Ramanujan identities and W-E representation of minimal surfaces (R.D.). Some more identities derived from Euler-Ramanujan's identities those which are slightly different from those obtained from minimal surfaces. (R.K.Singh + R.D.)
4. Dr. Pradip Kumar is currently studying the problem of interpolation of two arbitrary spacelike real analytic curves by singular maximal surfaces.

# Talk is partly based on:

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- ❖ [D. Lovett](#): Demonstrating Science with Soap Films; Institute of Physics Publishing.
- ❖ [R. Osserman](#): A Survey of Minimal Surfaces; Dover Publications, New York, 1986.
- ❖ [U. Dierkes, S. Hildebrandt, A. Küster, O. Wohlrab](#): Minimal Surfaces I; Springer Verlag, 1991.
- ❖ [Nitsche J.C.C.](#): Lectures on Minimal surfaces, Volume 1, Cambridge University Press, 1989.
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Thank You