Non-interacting fermions, strings, and the *1/N* expansion

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Introduction

One leitmotiv in the work of Spenta has been the interplay between non-interacting fermions, strings, and the $1/N$ expansion. In this talk I will present a new example of this interplay

$$N \text{ non-interacting fermions} \xrightarrow[1/N \text{ expansion}]{} \text{string theory}$$

Quantum Mechanics, operators, spectra

(Enumerative) geometry, combinatorics

Similar examples: $c \leq 1$ strings, 2d YM theory

Topological strings

Topological strings on non-compact Calabi-Yau (CY) manifolds are solvable models of string theory with many applications. They engineer *N*=2 supersymmetric gauge theories. They have a rich enumerative content in terms of holomorphic maps from Riemann surfaces to the CY target.

Their genus g free energies $F_g(\lambda)$ can be computed recursively, as a function of the geometric moduli of the CY [BCOV, BKMP]. They can be expanded in terms of worldsheet instanton contributions

$$F_g(\lambda) = \sum_{d \geq 1} N_{g,d} \, \mathrm{e}^{-d\lambda}$$

Gromov-Witten invariants

(Topological) string perturbation theory diverges

However, for fixed moduli, the free energies grow (doubly) factorially, as other string theories [Gross-Periwal, Shenker]

$$F_g(\lambda) \sim (2g)! \qquad g \gg 1$$

This means that the *total* free energy is an asymptotic series

$$F(\lambda, g_s) \sim \sum_{g \geq 0} F_g(\lambda) g_s^{2g-2}$$

Question (non-perturbative completion): is there an underlying, well-defined function, which has the above series as its asymptotic expansion?

CYs and algebraic curves

To answer this question, we have to be more explicit about the CY geometry. In the non-compact case, a CY X can be encoded in an *algebraic curve*, of the form

$$W_X(\mathrm{e}^x, \mathrm{e}^p) = 0$$

The CY threefold can then be described by the equation

$$uv = W_X(\mathrm{e}^x, \mathrm{e}^p)$$

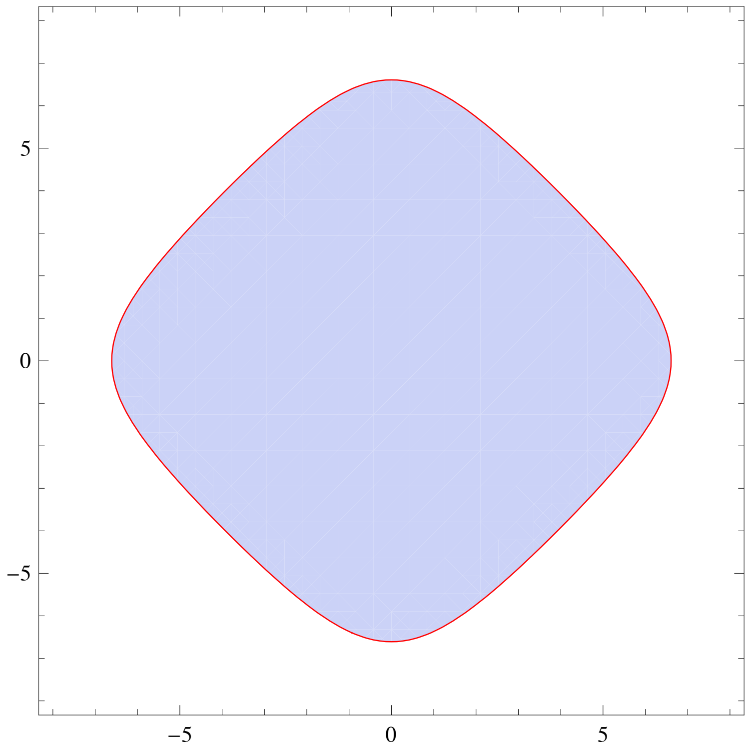
Examples: $W_X(\mathrm{e}^x, \mathrm{e}^p) = \mathrm{e}^x + \mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-p} + \kappa$

$$W_X(\mathrm{e}^x, \mathrm{e}^p) = \mathrm{e}^x + \mathrm{e}^p + \mathrm{e}^{-x-p} + \kappa$$

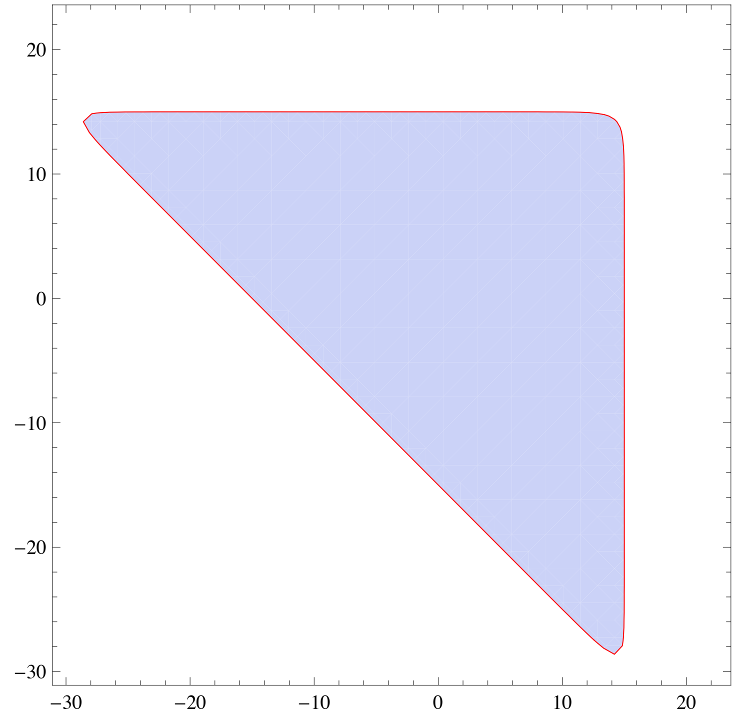
modulus

From curves to fermions

Main idea: think about these curves as "Fermi droplets" in phase space *(x,p)*



$$e^x + e^{-x} + e^p + e^{-p} = -\kappa = e^E$$



$$e^x + e^p + e^{-x-p} = -\kappa = e^E$$

This is of course a classical picture. We should *quantize* these curves (i.e. quantize the string target geometry)

Operators from curves

We just promote *x, p* to canonically conjugate Heisenberg operators

$$[\mathsf{x}, \mathsf{p}] = \mathrm{i}\hbar \qquad \hbar \in \mathbb{R}_{>0}$$

Weyl quantization of the curve produces a *self-adjoint operator* on $L^2(\mathbb{R})$

$$W_X(\mathrm{e}^x, \mathrm{e}^p) \to \mathsf{O}_X$$

Examples: $$\mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}}$$

$$\mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{x}-\mathsf{p}}$$

What is the nature of these operators?

Theorem

[Grassi-Hatsuda-M.M., Kashaev-M.M., Laptev-Schimmer-Takhtajan]

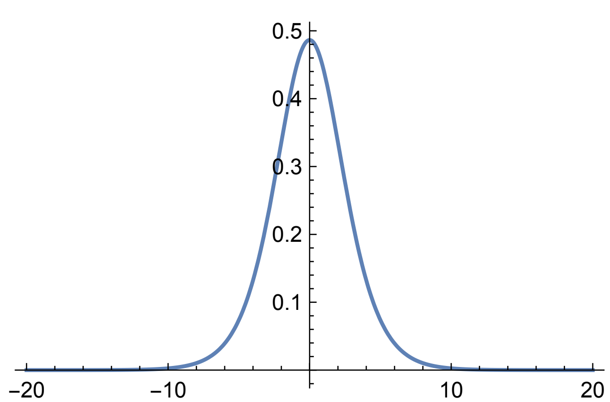
> The operators $\rho_X = \mathsf{O}_X^{-1}$ on $L^2(\mathbb{R})$ are of trace class

discrete spectrum! $\quad \mathrm{e}^{-E_n}, \qquad n = 0, 1, \cdots$

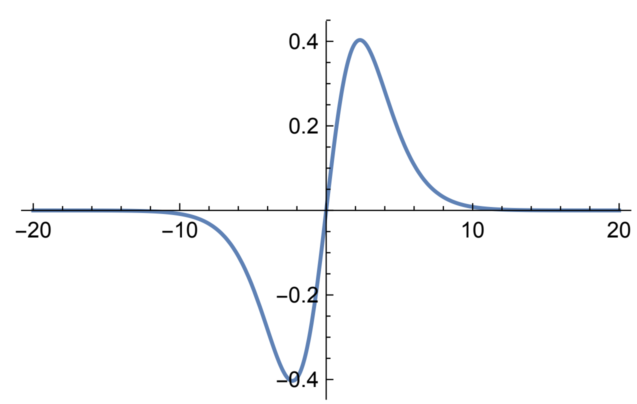
$$\mathrm{tr}\, \rho_X^\ell < \infty, \quad \ell = 1, 2, \cdots$$

similar to confining potentials in Schrödinger theory

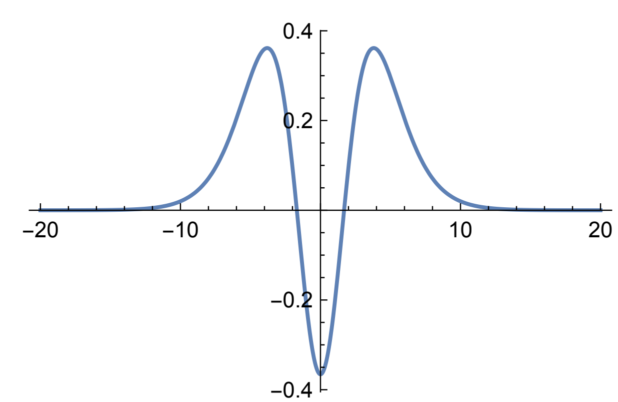
Example: $\quad \mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}} \qquad \hbar = 2\pi$



$E_0 = 2.881815429926296...$



$E_1 = 4.25459152858199...$



$E_2 = 5.28819530714418...$

The Fermi gas

We now consider a canonical ensemble of *N* *non-interacting fermions* with one-particle density matrix

$$\rho_X = \mathsf{O}_X^{-1} = \mathrm{e}^{-\mathsf{H}_X}$$

The canonical partition function is given by

$$Z_X(N,\hbar) = \frac{1}{N!} \int \mathrm{d}x_1 \cdots \mathrm{d}x_N \, \mathrm{det}_{i,j} \rho_X(x_i, x_j)$$

This is a perfectly well-defined quantity for any positive Planck constant and any positive integer *N*, since the density matrix is trace class. It can be explicitly computed in many cases

A conjecture

Let us consider the following *'t Hooft limit* for the canonical partition function $Z_X(N, \hbar)$

$$N \to \infty \qquad \hbar \to \infty \qquad \frac{N}{\hbar} = \lambda \quad \text{fixed}$$

This is a *strongly coupled* quantum mechanical problem. We claim [Grassi-Hatsuda-M.M., M.M.-Zakany] that it is equivalent to the *weakly coupled* topological string, and that the canonical free energy has the following asymptotic expansion

$$\log Z_X(N, \hbar) \sim \sum_{g \geq 0} F_g(\lambda) \hbar^{2-2g}$$

This realizes the genus expansion of topological string theory as the 't Hooft expansion of a manifestly well-defined quantity!

Many, many (successful!) tests of this conjecture in the last two years, by using e.g. matrix model representations for $Z_X(N,\hbar)$. Mathematically it is highly non trivial: the l.h.s. involves operators and their spectral properties, while the r.h.s. involves enumerative geometry and combinatorics.

Similar in spirit to the relation between matrix quantum mechanics and the *c=1* string, and to AdS/CFT. Indeed, in some examples

$Z_X(N,\hbar) =$ partition function of a 3d SCFT on a 3-manifold

$F_g(\lambda) =$ free energy of its superstring theory dual

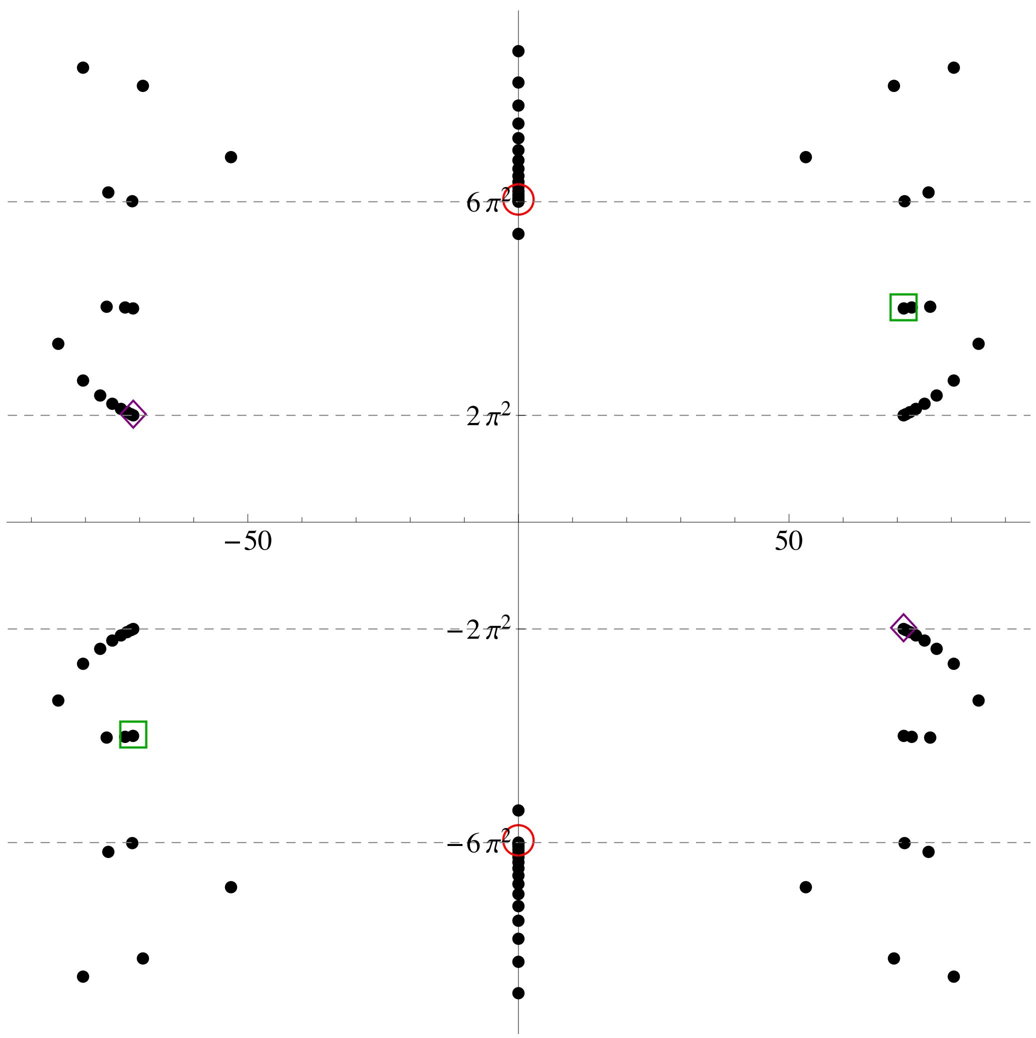
Resumming the string perturbation series

"It is desirable to evaluate the relevance of the large *N* limit. The question is very simple: to what extent and in which domains of coupling is a thermodynamic computation in which $N \to \infty$ relevant for the case of finite *N*"

S. Wadia, "A study of U(N) lattice gauge theory in two dimensions", 1979

Can we reconstruct the canonical free energy of the Fermi gas from its asymptotic, $1/N$ expansion?

It turns out that, in many examples, the genus expansion of these topological strings is factorially divergent but *Borel summable* [Grassi-M.M.-Zakany, Couso-M.M.-Schiappa]



Borel plane for the genus expansion in the geometry

$$e^{x} + e^{p} + e^{-x-p} = \text{constant}$$

and $\lambda = \frac{3}{4\pi}$

Borel summability is not enough

One could then think that the Borel resummation of the genus expansion will reproduce the exact free energy of the Fermi gas. Indeed, in many examples in Quantum Mechanics like the quartic oscillator, Borel summable series can be resummed to the "good" answer.

However, in the case of topological strings, the Borel resummation of the perturbative series is *different* from the exact answer, and they only agree up to *exponentially small effects*. The reason of the mismatch is the presence of *complex instantons*, which do not obstruct Borel summability but still lead to non-trivial non-perturbative effects.

Conclusions

Models of strings based on non-interacting fermions were until now restricted to low dimensional targets. Our construction extends them to the rich (and infinite) family of topological strings on non-compact CYs.

This construction leads to *exact quantization conditions* for the spectrum of the relevant operators, in terms of enumerative invariants. It also leads to the exact spectrum for a new family of quantum integrable systems related to non-compact CYs (Goncharov-Kenyon).

Our construction provides a non-perturbative definition of topological string theory on non-compact CYs, and a very concrete model of stringy/quantum geometry which is worth exploring further.