Congratulations to Spenta on his many achievements

Story of Two iϵ

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Bangalore, January 2017

Motivation

String theory is supposed to give us a finite quantum theory of gravity

– has no UV divergence

It has IR divergences associated with poles of the propagator.

These can be dealt with using superstring field theory and will not be the topic of discussion today.

Even if we leave aside the expected IR divergences, most of the amplitudes in string theory diverge

– related to the fact that string theory amplitudes are given in the Schwinger parameter representation of the (string) field theory amplitudes

$$(k^2 + m^2)^{-1} = \int_0^\infty ds\, e^{-s(k^2+m^2)}$$

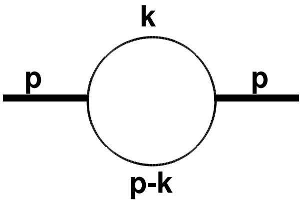
– the lhs is finite for $k^2 + m^2 \neq 0$ but the rhs diverges for $k^2 + m^2 < 0$.

$k^2 = -(k^0)^2 + \vec{k}^2$

An example:

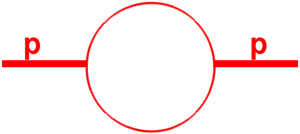
Consider two fields, one of mass M and another of mass m, with M>2m.

Consider one loop mass renormalization of the heavy particle.



Thick line: heavy particle

Thin line: light particle.



$$\delta M^2 = i\,B \int \frac{d^D k}{(2\pi)^D} \exp[-A\{k^2 + m^2\} - A\{(p-k)^2 + m^2\}]$$
$$\{k^2 + m^2\}^{-1}\{(p-k)^2 + m^2\}^{-1}$$

A: a positive constant mimicking UV finiteness of string amplitudes.

B: a constant encoding additional contribution to the vertices and / or propagators.

$$\delta M^2 = i\,B \int \frac{d^Dk}{(2\pi)^D} \exp[-A\{k^2+m^2\} - A\{(p-k)^2+m^2\}]$$
$$\{k^2+m^2\}^{-1}\{(p-k)^2+m^2\}^{-1}$$

We shall work in $\vec{p}=0$ frame.

String motivated approach: Evaluate the original integral using Schwinger parametrization

$$\exp[-A(k^2+m^2)](k^2+m^2)^{-1} = \int_A^\infty dt_1 \exp[-t_1(k^2+m^2)]$$

$$\exp[-A((p-k)^2+m^2)]((p-k)^2+m^2)^{-1} = \int_A^\infty dt_2 \exp[-t_2((p-k)^2+m^2)]$$

Substitute and do momentum integrals (formally)

$$\delta M^2 = -B\,(4\pi)^{-D/2} \int_A^\infty dt_1 \int_A^\infty dt_2\,(t_1+t_2)^{-D/2}$$
$$\exp\left[\frac{t_1 t_2}{t_1+t_2} M^2 - (t_1+t_2)m^2\right]$$

– diverges from the upper end for $M > 2m$.

Sagnotti; Sundborg; Amano,Tsuchiya; $\cdots$

– can be traced to the impossibility of choosing energy integration contour keeping $\mathrm{Re}(k^2+m^2)>0$, $\mathrm{Re}((p-k)^2+m^2)>0$.

A related problem: δM^2 is formally real

Berera

– inconsistent with unitarity.

Proposed Resolutions

1. Analytic continuation

– define the amplitude for M $<$ 2m and then analytically continue to M $>$ 2m.

Problem: In string theory M is not adjustable

– fixed by the requirement of gauge invariance.

Proposed resolution

First compute four point function by analytic continuation and then factorize on poles to compute quantum corrected M^2.

Even this is not straightforward.

For on-shell four point function of massless fields, there is no kinematic region where the amplitude is finite.

Solution

D'Hoker, Phong

Express the formal answer as sum of terms, and analytically continue different terms from different kinematic regions where they are finite.

This works, but $\cdots$

– no systematic generalization to general amplitudes

– it is not clear if the amplitudes so defined satisfy Cutkosky rules needed for unitarity.

2. Moduli space contour deformation

Berera; Witten

Replace $\int_A^\infty dt_i \cdots$ by

$$\int_A^{A+i\infty} dt_i \, e^{-\epsilon t_i} \ldots$$

for each t_i and take $\epsilon \to 0$ limit at the end.

Instead of integrating over the moduli space of Riemann surfaces, we integrate over a subspace of the complexified moduli space.

Example: For one loop mass renormalization

$$\delta M^2 = -B\,(4\pi)^{-D/2}\int_A^{A+i\infty} dt_1 \int_A^{A+i\infty} dt_2\,(t_1+t_2)^{-D/2}$$
$$\exp\left[\frac{t_1 t_2}{t_1+t_2}\,M^2-(t_1+t_2)m^2\right]e^{-\epsilon(t_1+t_2)}$$

– gives finite result

Pros and Cons

– gives a systematic procedure for computing string amplitudes when IR divergences are absent $\surd$

– preserves the notion of integration over the moduli space of Riemann surfaces $\surd$

– not clear if the amplitudes defined this way satisfy Cutkosky rules $\times$

3. Use superstring field theory

– gives string amplitudes as sum over Feynman diagrams with standard momentum integrals.

However the exponential factors in the vertices requires us to treat the energy integrals carefully.

Steps

1. First multiply all external energies by a common complex number u

For imaginary u, we take the loop energy integrals to run along the imaginary axis.

a. All loop energies are imaginary and hence there are no poles in the propagators

$$(k_i^2 + m_i^2)^{-1} = (-(k_i^0)^2 + \vec{k}_i^2 + m_i^2)^{-1}$$

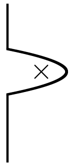
b. The exponential factor

$$\exp[-A \sum_i (k_i^2 + m_i^2)] = \exp[-A \sum_i (-(k_i^0)^2 + \vec{k}_i^2 + m_i^2)]$$

converges at infinity.

2. Next we deform u to 1 along the first quadrant of the complex u-plane

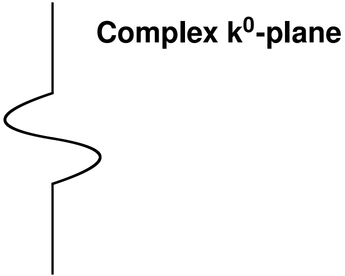
During this deformation some poles of the propagators may approach the integration contour.



Deform the integration contour away from the poles so that no pole crosses the contour.

Keep the ends of the integration contour fixed at $\pm i\infty$.

A typical contour shape



Result: This deformation is always possible

i.e. we do not encounter a situation where a pair of poles approach each other from opposite sides of the contour, making it impossible to deform the contour avoiding poles.

Pius, A.S.

3. Define the physical amplitude as the u $\rightarrow$ 1 limit from the first quadrant

– gives a well defined expression for the amplitude

$$
\begin{aligned}
\delta M^2 \;=\; & i \int \frac{d^D k}{(2\pi)^D} \exp[-A\{k^2 + m^2\} - A\{(p-k)^2 + m^2\}] \\
& \{k^2 + m^2\}^{-1}\{(p-k)^2 + m^2\}^{-1} B(k)
\end{aligned}
$$

Poles in the k^0 plane (for $\vec{p} = 0$):

$$
Q_1 \equiv \sqrt{\vec{k}^2 + m^2}, \quad Q_2 \equiv -\sqrt{\vec{k}^2 + m^2},
$$

$$
Q_3 \equiv p^0 + \sqrt{\vec{k}^2 + m^2}, \quad Q_4 \equiv p^0 - \sqrt{\vec{k}^2 + m^2}
$$

For p^0 imaginary, take k^0 contour along imaginary axis.

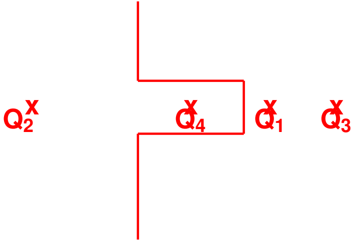
Q_1, Q_3 to the right and Q_2, Q_4 to the left of the imaginary axis.

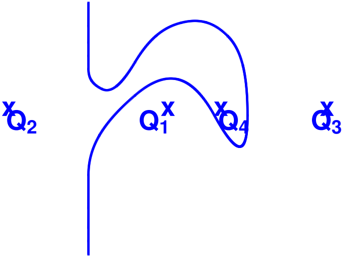
$$Q_1 \equiv \sqrt{\vec{k}^2 + m^2}, \quad Q_2 \equiv -\sqrt{\vec{k}^2 + m^2},$$

$$Q_3 \equiv p^0 + \sqrt{\vec{k}^2 + m^2}, \quad Q_4 \equiv p^0 - \sqrt{\vec{k}^2 + m^2}$$

As p^0 approaches real axis, the poles approach the real axis.

Two situations depending on the value of $\vec{k}$.





Note: Q_1, Q_3 to the right and Q_2, Q_4 to the left of the contour in both diagrams.

This gives finite result for the integral.

Pros and Cons

1. We lose the interpretation of the amplitude as integrals over (complexified) moduli spaces of Riemann surfaces $\times$

2. This prescription gives results satisfying Cutkosky rules

Pius, A.S. $\surd$

3. For dealing with IR divergent amplitudes we anyway have to use superstring field theory $\surd$

Question: Are these prescriptions equivalent?

– ill defined for the first prescription since it has not be prescribed for general amplitudes

– focus on the second and third prescription

Result: These two prescriptions give identical result

– allows us to preserve the pros of both

1. expressed as integral over complexified moduli space of Riemann surfaces

2. satisfies Cutkosky rules

Sketch of the proof

1. Choose $u = re^{i\theta}$ with $\pi/4 < \theta < \pi/2$, $0 < r < \infty$

2. For this value of u, define $F_e(u)$ as the result obtained in the third prescription, taking all the loop energy integrals along the $k^0 = |k^0| e^{i\theta}$ contour

3. Use Schwinger parameter representation for the propagators

$$(k_i^2 + m_i^2)^{-1} = \int_0^{i\infty} dt_i \exp[-t_i(k_i^2 + m_i^2)]$$

t_i integral converges and so it is an actual equality, not a formal equality.

4. Now carry out the momentum integrals and express the result as an integral over Schwinger parameters

– call this $F_m(u)$.

$$F_m(u) = \int_0^{i\infty} dt_1 \int_0^{i\infty} dt_2 \cdots f(\{t_j\})$$

5. $F_e(u)$ and $F_m(u)$ are manifestly equal for $u = re^{i\theta}$ with $\pi/4 < \theta < \pi/2$, $0 < r < \infty$

6. Show that the integrals defining $F_m(u)$ and $F_e(u)$ are analytic in the entire first quadrant of the u-plane

Therefore they must give the same result as $u \to 1$.

Summary

1. Superstring loop amplitudes, expressed as integrals over the moduli spaces of Riemann surfaces, are most often divergent.

2. There are two systematic procedures to define such amplitudes

– integrate over complexified moduli spaces of Riemann surfaces

– or use superstring field theory and express the loop amplitudes as integrals over momentum space.

Our analysis shows that these two procedures give us the same answer, and hence are equivalent.