Fun with K3

Hirosi Ooguri

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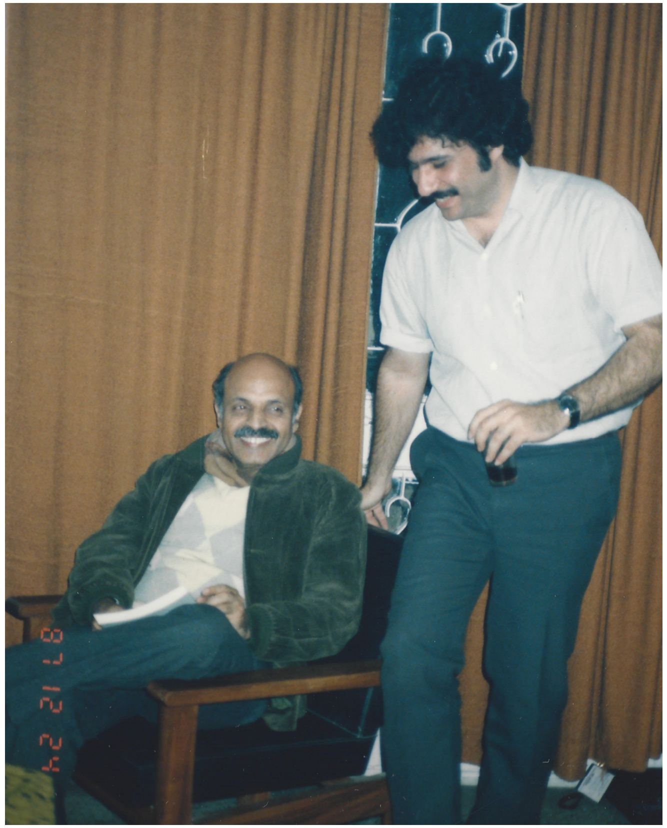
String Theory: Past and Present & SpentaFest
11 - 13 January 2017, ICTS, Bengaluru

Winter School at IIT Kanpur (photos from 24 December 1987)



More stories and photos
at the banquet tonight ...

Together with our friends in China, India, Japan, and Korea, Spenta and I have been organizing the Asian Winter School.

2012 in Kusatsu Hot Springs

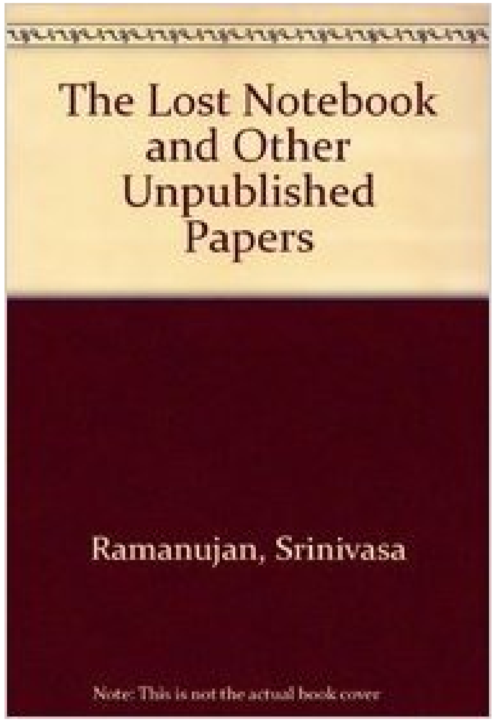
2016 at OIST, Okinawa

2018 in India

Back in India in 1987

Freeman Dyson at the **Ramanujan Centenary Conference** in 1987:

The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to ***enlarge their analytic machinery to include mock theta-functions****.*



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$$\mathrm{ch}_0^R(l=0;\,-1/\tau)=\mathrm{ch}_0^{NS'}(l=1/2;\,\tau)+\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\cosh\pi\alpha}\,\mathrm{ch}^{NS'}(h=\alpha^2/2-1/8;\,\tau)\,,$$

$$\mathrm{ch}_0^{NS}(l=1/2;\,-1/\tau)=-\mathrm{ch}_0^{N}(l=1/2;\,\tau)+\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\cosh\pi\alpha}\,\mathrm{ch}^{NS}(h=\alpha^2/2-1/8;\,\tau)\,.$$

$$\mathrm{ch}_0^{R}(k=1,\,l=0;\,z)=\sum_m q^{m^2/2+m/2+1/4}z^{m+1/2}\frac{1}{1+zq^m}f^{R}(z)\,,$$

$$\mathrm{ch}_0^{R}(k=1,\,l=1/2;\,z)=\sum_m q^{m^2/2+m/2+1/4}z^{m+1/2}\frac{zq^m-1}{1+zq^m}f^{R}(z)\,,$$

$$\mathrm{ch}_0^{NS}(k=1,\,l=0;\,z)=\sum_m q^{m^2/2}z^m\frac{zq^{m-1/2}-1}{1+zq^{m-1/2}}f^{NS}(z)\,,$$

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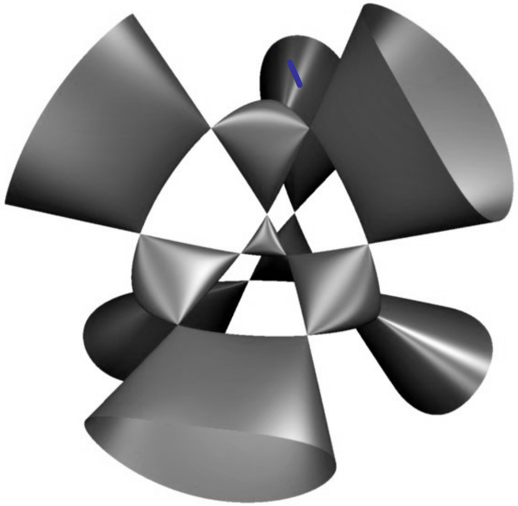
Eguchi + Taormina (1988)



We studied various examples of the N=4 superconformal sigma model with K3 target space and found that there is <u>a part of the spectrum that is identical in all examples</u>.

We then realized that <u>this part</u> is counting 1/4 BPS states.

$$Z_{\frac{1}{4}BPS} = tr\left[(-1)^{F+\bar{F}} q^{L_0-\frac{1}{4}} e^{2\pi i z J_0^3}\right]$$

$$= 8\left[\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\right)^2 + \left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2 + \left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2\right]$$

Eguchi, Taormina, Yang + H.O. (1989)

Elliptic genus of K3 expanded in N=4 superconformal characters

$$F(\tau) \equiv \sum_h (N_{h,1} - 2N_{h,0}) q^h = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots$$

$$F(-1/\tau) = \sum_h (N_{h,1} - 2N_{h,0}) \cdot \tilde{q}^h \longrightarrow 2\sqrt{-i\tau}q^{-1/8} - 12 + \cdots .$$

This observation seems to imply that the q-expansion coefficients of $F(\tau)$ are all positive and the symmetry of the generic non-linear σ-model is just the $N = 4$ superconformal symmetry, though I have no rigorous proof for it.

My Ph.D thesis (1989)

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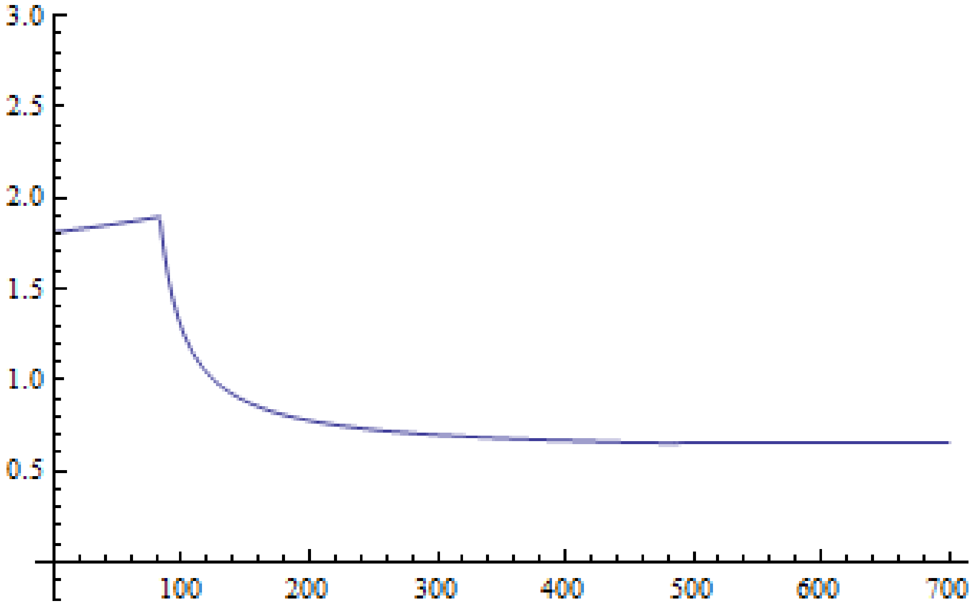
Upper bound on scaling dimension
of the lowest non-BPS state



Total Hodge number

We can also combine this with the modular bootstrap to find constraints on non-BPS spectrum of the Calabi-Yau sigma-model.

Keller + H.O., arXiv: 1209.4649

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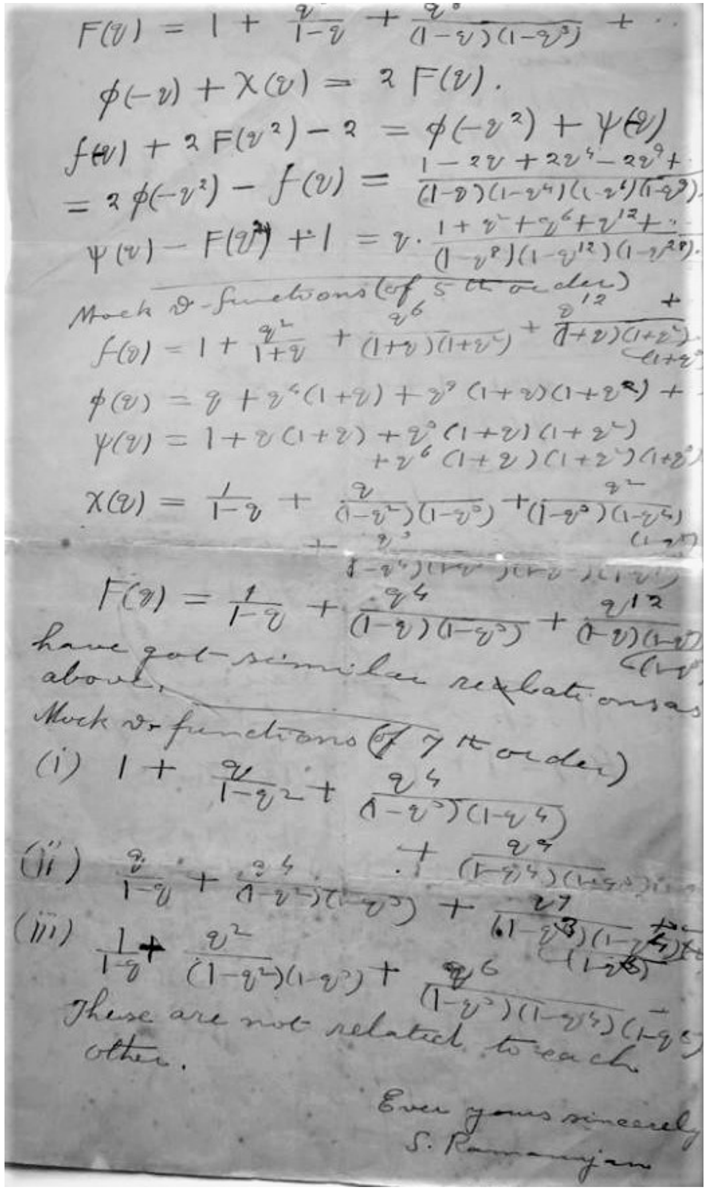
21 years later ...

Representations of Mathieu group M24

n	1	2	3	4	5
A_n	45	231	770	2277	5796

Eguchi, Tachikawa + H.O., arXiv: 1004.0956

Examining Ramanujan's last letter to G. H. Hardy at the Wren Library of Trinity College in Cambridge, March 2015 (*thanks to David Tong*).



$F(q) = 1 + \frac{q}{1-q} + \frac{q^3}{(1-q)(1-q^3)} + \cdots$

$\phi(-q) + \chi(q) = 2F(q).$

$f(q) + 2F(q^2) - 2 = \phi(-q^2) + \psi(q)$

$= 2\phi(-q^2) - f(q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$

$\psi(q) - F(q^{1/2}) + 1 = q \cdot \frac{1 + q^2 + q^6 + q^{12} + \cdots}{(1-q^8)(1-q^{12})(1-q^{28})\cdots}$

Mock ϑ-functions (of 5th order)

$f(q) = 1 + \frac{q^2}{1+q} + \frac{q^6}{(1+q)(1+q^2)} + \frac{q^{12}}{(1+q)(1+q^2)(1+q^3)} + \cdots$

$\phi(q) = q + q^4(1+q) + q^9(1+q)(1+q^3) + \cdots$

$\psi(q) = 1 + q(1+q) + q^3(1+q)(1+q^2) + q^6(1+q)(1+q^2)(1+q^3) + \cdots$

$\chi(q) = \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \cdots$

$F(q) = \frac{1}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{12}}{(1-q)(1-q^3)(1-q^5)} + \cdots$

have got similar relations as above.

Mock ϑ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)} + \cdots$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)} + \cdots$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)} + \cdots$

These are not related to each other.

Ever yours sincerely
S. Ramanujan

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However, M24 cannot be full symmetry of a single K3 sigma model.

Gaberdiel, Hohenegger + Volpato, arXiv:1106.4315

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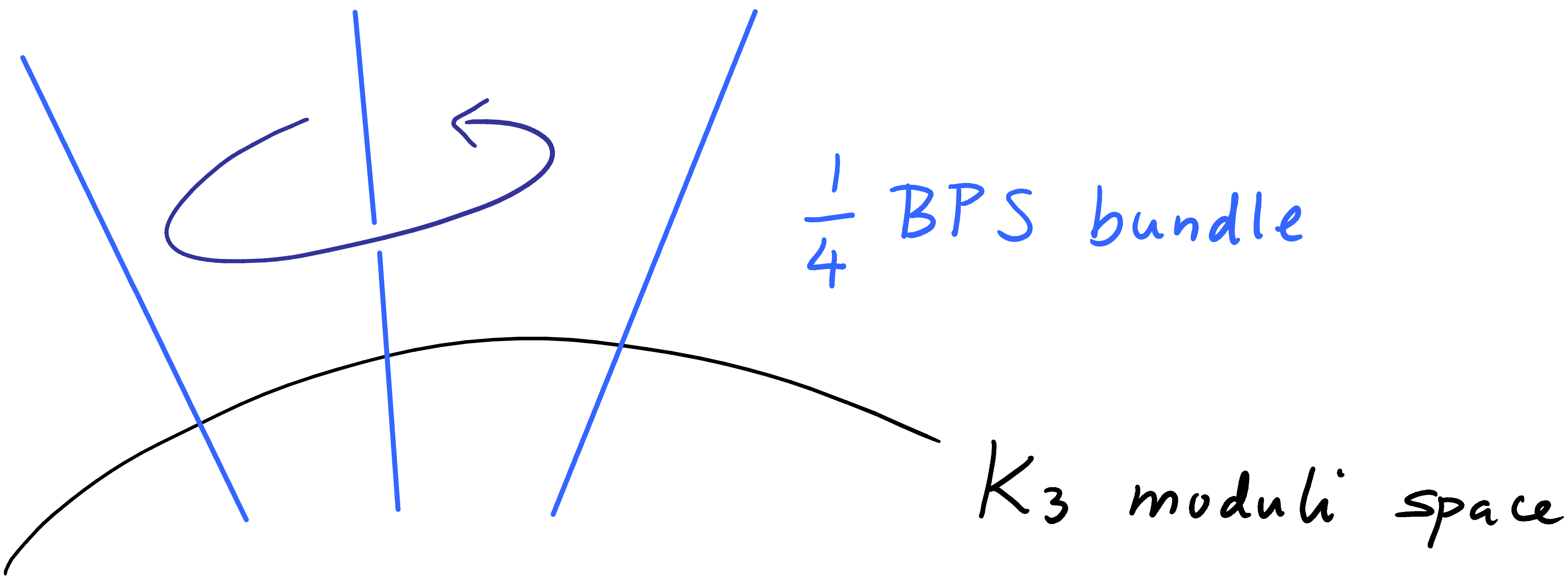
Can it be symmetry of 1/4 BPS states counted by the elliptic genus?

How does the 1/4 BPS bundle fiber over the K3 moduli space?



How does the 1/4 BPS bundle fiber over the K3 moduli space?

Today, I will discuss more modest issues on the base K3 moduli space.

There has been a paradox on the K3 moduli space for 27 years ...

Tension between Two Fundamental Papers in 1988 - 1989

The moduli space of N=4 SCFT with c = 6 is O(4,20)/O(4) x O(20).

It is <u>not Kaehler</u> and <u>does not factorizes</u>.

I will resolve the tension from the worldsheet perspective.

It turns out to be due to a new type of anomalies.

OBSERVATIONS ON THE MODULI SPACE OF SUPERCONFORMAL FIELD THEORIES

Nathan SEIBERG*

Institute for Advanced Study, Princeton, NJ 08540, USA

Received 18 January 1988

Some aspects of the moduli space of superconformal field theories are discussed. It is helpful to consider the conformal field theory as a background for propagation of strings and to exploit the space-time interpretation. Using this point of view we show that the metric on the moduli space of $N = 4$ superconformal field theory with $c = 6$ is locally that of $O(20,4)/O(20) \times O(4)$. We

ON EFFECTIVE FIELD THEORIES DESCRIBING (2,2) VACUA OF THE HETEROTIC STRING*

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Vadim S. KAPLUNOVSKY***

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Received 29 May 1989

Classical vacua of the heterotic string corresponding to $c = 9$, $N = (2,2)$ superconformal theories on the world sheet yield low-energy effective field theories with $N = 1$ space-time supersymmetry in four dimensions, gauge group $E_6 \otimes E_8$, several families of **27** and $\overline{\mathbf{27}}$ matter fields, and moduli fields. String theory relates matter fields to moduli; in this article we relate the kinetic terms in the effective lagrangian for both moduli and matter fields to the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ Yukawa couplings. Geometrically, we recover the result (obtained previously via the type II superstring and $N = 2$ supergravity) that moduli space is a direct product of two Kähler manifolds of restricted type, spanned by the moduli related respectively to the **27** and $\overline{\mathbf{27}}$ matter fields. The

Based on the work with J. Gomis, Z. Komargodski, N. Seiberg, and Y. Wang:

Shortening Anomalies in Supersymmetric Theories

Jaume Gomis,[1] Zohar Komargodski,[2] Hirosi Ooguri,[3,4] Nathan Seiberg,[5] and Yifan Wang[6]

[1] Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5, Canada

[2] Weizmann Institute of Science, Rehovot 76100, Israel

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[4] Kavli IPMU, WPI, University of Tokyo, Kashiwa, 77-8583, Japan

[5] School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA

[6] Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Dedicated to John Schwarz on his 75th birthday

In 2d N =(2,2) SCFT, the conformal manifold factorizes locally:

$$\mathcal{M}_{chiral} \times \mathcal{M}_{twisted\ chiral}$$

both Kähler

Dixon, Kaplunovsky, Louis (1990)

...

Gomis, Hsin, Komargodski, Schwimmer, Seiberg,Theisen, arXiv:1509.08511

However,

O(2*n*, 2*n*)/O(2*n*) x O(2*n*) for the sigma-model on T^2*n*. Narain (1986)

O(4, 20)/O(4) x O(20) for the sigma-model on K3. Seiberg (1988)

They do not factorize, not even Kähler.

?

R - symmetry is enhanced:

$\mathcal{N}=2$ superconformal : $U(1)$ R-symmetry

$\Rightarrow$ T^{2n} : $SO(2n)$

K_3 : $SU(2)$

Normally, when a particular global symmetry implies special properties, enhancing that symmetry does not ruin those properties.

We have counter-examples to this expectation.

When R - symmetry is enhanced:

$$^{\exists} J_{++} \ : \ R \text{ charge } (2,0)$$

ϕ_{+-}: chiral primary of R charge $(1,1)$

$\tilde{\phi}_{+-}$: twisted chiral primary of R charge $(1,-1)$

$$\Rightarrow \quad \phi^{i}_{+-}(x_1)\,\tilde{\phi}^{\tilde{j}}_{+-}(x_2) \sim \frac{c^{i\tilde{j}}}{x_1^{--} - x_2^{--}}\, J_{++}(x_2)$$

(e.g. K3 sigma-model : $\phi^{i}_{+-} = k^{i}_{a\bar{b}}\, \psi_+^{a} \psi_-^{\bar{b}}$

$\tilde{\phi}^{j}_{+-} = k^{j}_{a\bar{c}}\, g^{\bar{c}d}\, \Omega_{db}\, \psi_+^{a} \psi_-^{b}$

$$\phi^{i}(x_1)\,\tilde{\phi}^{j}(x_2) \sim \frac{\eta^{ij}}{x_1^{--} - x_2^{--}}\, \Omega_{ab}\, \psi_+^{a} \psi_+^{b}$$)

In $\mathcal{N}=(2,2)$ superspace : $Z=(x,\theta,\bar{\theta})$

$$\mathcal{O}(Z)=\phi(x)+\cdots \quad (\bar{D}_+\mathcal{O}=0,\ \bar{D}_-\mathcal{O}=0)$$

$$\widetilde{\mathcal{O}}(Z)=\tilde{\phi}(x)+\cdots \quad (\bar{D}_+\widetilde{\mathcal{O}}=0,\ D_-\widetilde{\mathcal{O}}=0)$$

$$\mathcal{J}_{++}(Z)=J_{++}(x)+\cdots \quad : \text{very short multiplet}$$

$(C^{i\tilde{j}}\to 1$ for now $)$

$$\Rightarrow \quad \mathcal{O}(Z_1)\,\widetilde{\mathcal{O}}(Z_2)\sim\frac{1}{z_{1\bar{2}}^{--}}\,\mathcal{J}_{++}(Z_2)+\cdots$$

$$z_{1\bar{2}}=x_1-i\,\theta_1\bar{\theta}_1-(\bar{x}_2+i\,\theta_2\bar{\theta}_2)+2i\,\theta_1\bar{\theta}_2$$

$$\mathcal{O}(Z_1)\,\tilde{\mathcal{O}}(Z_2) \sim \Big(\frac{1}{z_{1\bar{2}}^{--}} + \underbrace{c\,\delta^{(2)}(z_{1\bar{2}})\,\theta_{12}^{+}\bar{\theta}_{12}^{+}}_{\text{Contact term allowed by supersymmetry}} \Big)\,\mathcal{J}_{++}(Z_2)$$

By adjusting c,

$$\big(\bar{D}_{+}\mathcal{O}(Z_1)\big)\,\tilde{\mathcal{O}}(Z_2) \sim \delta^{(2)}(z_{1\bar{2}})\,\theta_{12}^{+}\,\mathcal{J}_{++}(Z_2)$$

or

$$\mathcal{O}(Z_1)\big(\bar{D}_{+}\tilde{\mathcal{O}}(Z_2)\big) \sim \delta^{(2)}(z_{1\bar{2}})\,\theta_{12}^{+}\,\mathcal{J}_{++}(Z_2)$$

We cannot set both $\bar{D}_{+}\mathcal{O} = 0$ and $\bar{D}_{+}\tilde{\mathcal{O}} = 0$

What can go wrong with spurion analysis:

$$\mathcal{L} + \int d\theta^+ d\theta^- \, \lambda \, \mathcal{O} + \int d\theta^+ d\bar{\theta}^- \, \tilde{\lambda} \, \tilde{\mathcal{O}}$$

$\lambda, \tilde{\lambda}$: background chiral and twisted chiral superfields

OPE singularities in $\mathcal{O}$ and $\tilde{\mathcal{O}}$ give rise to :

$\bar{D}_+ \mathcal{O} \sim (\bar{D}_- \tilde{\lambda}) \mathcal{J}_{++}$ or $\bar{D}_+ \tilde{\mathcal{O}} \sim (D_- \lambda) \mathcal{J}_{++}$

depending on counter-terms

"Shortening Anomalies"

$$\mathcal{L} + \int d\theta^+ d\theta^- \, \lambda \, \mathcal{O} + \int d\theta^+ d\bar{\theta}^- \, \tilde{\lambda} \, \tilde{\mathcal{O}}$$

$$\bar{D}_+ \mathcal{O} \sim (\bar{D}_- \tilde{\lambda}) \, \mathcal{J}_{++} \quad \text{or} \quad \bar{D}_+ \tilde{\mathcal{O}} \sim (D_- \lambda) \, \mathcal{J}_{++}$$

Marginal couplings λ and $\tilde{\lambda}$ cannot be promoted to background superfields in short representations simultaneously.

$$\Downarrow$$

Results of $\mathcal{N}=(2,2)$ spurion analysis are invalidated.

e.g. $\mathcal{M}_{\text{chiral}} \times \mathcal{M}_{\text{twisted chiral}}$

$$\mathcal{L} + \int d\theta^+ d\theta^- \lambda \mathcal{O} + \int d\theta^+ d\bar{\theta}^- \tilde{\lambda} \tilde{\mathcal{O}}$$

$$\bar{D}_+ \mathcal{O} \sim (\bar{D}_- \tilde{\lambda}) \mathcal{J}_{++} \quad \text{or} \quad \bar{D}_+ \tilde{\mathcal{O}} \sim (D_- \lambda) \mathcal{J}_{++}$$

Marginal couplings λ and $\tilde{\lambda}$ cannot be promoted to background superfields in short representations simultaneously.

To my knowledge, this is the first instance where the spurion argument fails to work.

Wess-Zumino Perspective:

$$\mathcal{L} + \int d^4\theta \, (\mathcal{A} \, \mathcal{O} + \mathcal{B} \, \tilde{\mathcal{O}})$$

Kähler potential rather than superpotential.

Assuming $Z(\mathcal{A}, \mathcal{B})$ is invariant under $\delta \mathcal{B} = \bar{D}_+ \psi_- + D_- \psi_+$,

$$\delta \log Z(\mathcal{A}, \mathcal{B}) \,|\, \delta \mathcal{A} = \bar{D}_+ \chi$$

$$\sim \int d^2x \, d^4\theta \, \bar{D}_+ D_- \mathcal{B} \cdot \bar{D}_- \chi \cdot \mathcal{J}_{++}$$

cannot be removed by adding local terms to $\log Z$

i.e. cohomologically non-trivial.

$$\Longleftrightarrow \quad \bar{D}_+ \mathcal{O} \sim (\bar{D}_- \tilde{\lambda}) \, \mathcal{J}_{++} \quad \text{where} \quad \tilde{\lambda} = \bar{D}_+ D_- \mathcal{B} .$$

Curvature Perspective:

- $$\phi^i(x_1)\,\tilde{\phi}^{\tilde{j}}(x_2) \sim \frac{C^{i\tilde{j}}}{x_1^{--} - x_2^{--}}\, J_{++}(x_2)$$

  $\Rightarrow$ mixed curvature $R_{i\tilde{j}\bar{k}\bar{\tilde{l}}} \sim C_{i\tilde{j}}\, \bar{C}_{\bar{k}\bar{\tilde{l}}}$

  obstructs $\mathcal{M}_{\text{chiral}} \times \mathcal{M}_{\text{twisted chiral}}$

- $\mathcal{N} = (4,4)$ SCFT :

  non-trivial holonomies in outer automorphism of $\mathcal{N}=4$ algebra $\Rightarrow$ $\mathcal{N}=2$ subalgebra cannot be chosen consistently.

Constraints on RG flows:

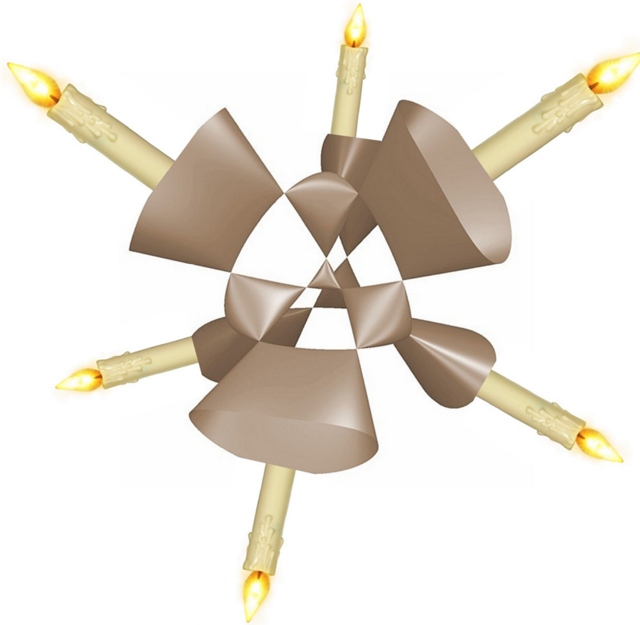
Similarly to 't Hooft anomalies

$\exists\, \mathcal{N}=(2,2)$ RG flow with all marginal couplings

$\Rightarrow$ no shortening anomalies

There cannot be $\mathcal{N}=(2,2)$ gauged linear sigma-models that UV complete K_3 or T^4 SCFT's and cover their conformal manifolds.

(c.f. T^2 SCFT $\Leftarrow$ GLSM with $W = \mathcal{P}G_3(X_1, X_2, X_3)$)



Thank you for your friendship
for the last 30 years.
Many happy returns!