High-dimensional statistics and large-SpeNta expansion OR One can go only so far from home

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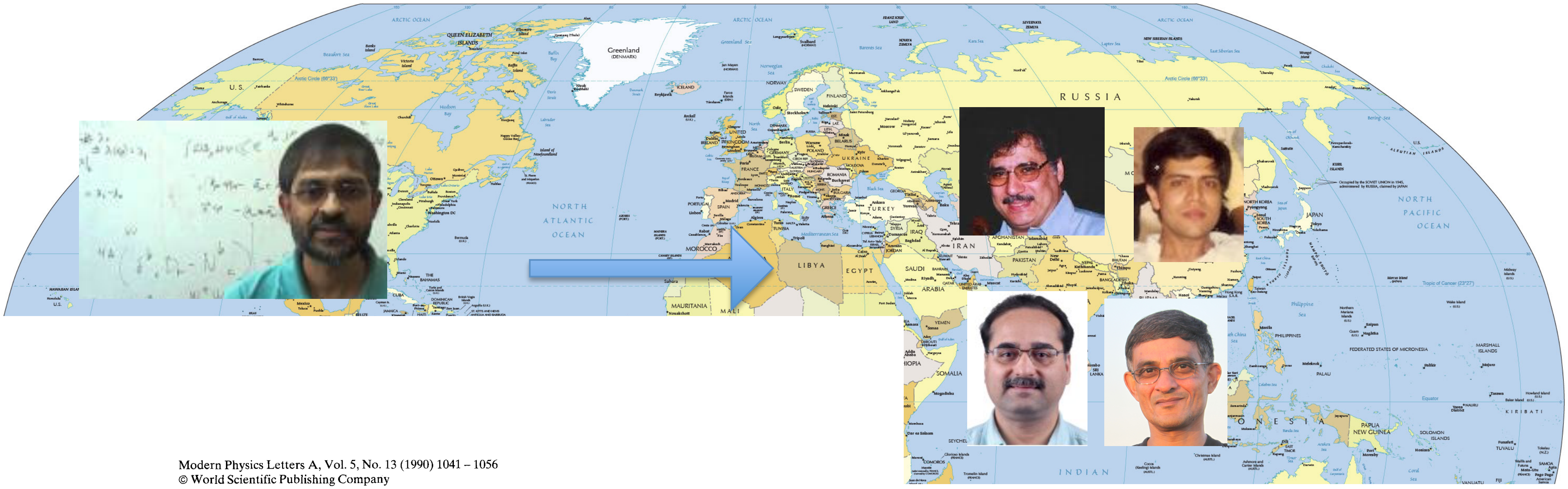
TIFR String Excitations: 1987-88

??
R. Saroja

Fall 1989



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NEW CRITICAL BEHAVIOR IN $d = 0$ LARGE-N MATRIX MODELS

SUMIT R. DAS, AVINASH DHAR, ANIRVAN M. SENGUPTA and SPENTA R. WADIA
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Received 9 January 1990

The non-perturbative formulation of 2-dimensional quantum gravity in terms of the large-N limit of matrix models is studied to include the effects of higher order curvature terms. This leads to matrix models whose potential contains a symmetry breaking term of the form Tr $\phi A \phi A$, where A is a given fixed matrix. This is studied in $d = 0$ dimensions and effectively induces additional terms of the form $(\mathrm{Tr}\,\phi^k)^2$ in the one matrix potential. An exact solution to leading order of the potential $V(\phi) = 1/2\,\mathrm{Tr}\,\phi^2 + g/N\,\mathrm{Tr}\,\phi^4 + g'/N^2\,(\mathrm{Tr}\,\phi^2)^2$ is presented leading to 3 phases: $\gamma = -1/2$ (smooth surfaces), $\gamma = 1/2$ (branched polymer) and $\gamma = 1/3$ (intermediate phase). Including a Tr ϕ^6 term in the potential gives rise to an additional phase with $\gamma = 1/4$. It is conjectured that for the general polynomial potential there are phases with $\gamma = 1/n$, $n = 2, 3, \ldots$. The $\gamma > 0$ phases may correspond to $c > 1$ matter coupled to 2-dimensional gravity.

, *USA*

1. Introduction

It is generally believed that many fundamental issues in string theory are non-perturbative in nature. In this respect the standard first quantized approach seems to be of little use since it is always restricted to two-dimensional surfaces and hence to the perturbation expansion. Nevertheless, it has been realized recently that a deeper understanding of the integration over two-dimensional metrics in the first-

lidean quantum
osed surfaces of

Where was Spenta? (late eighties)

Spring 1991

Modern Physics Letters A, Vol. 6, No. 18 (1991) 1685–1692

CLASSICAL SOLUTIONS OF 2-DIMENSIONAL STRING THEORY

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Received 25 March 1991

We present an exact one-parameter family of solutions to the classical graviton-dilaton system in two dimensions. The solution can be identified as a black hole. We present the solution both in a Schwarzschild-like gauge and in the target space conformal gauge. We discuss possible relations with matrix models.

1. Introduction

Recently there has been considerable interest in the problem of quantized 2-dimensional gravity coupled to a scalar field, which in the absence of gravity would be a $c = 1$ system. There are presently two approaches to this problem — the matrix model[1, 3–9] and the continuum Liouville theory.[10–12] There are encouraging results which indicate that the weak coupling tree level S-matrices computed in the matrix model can be obtained by Liouville theory calculation.[a] One of the interesting results of the continuum Liouville theory is that this system has a 2-dimensional target space, in that the general couplings are functions of two variables one of them being the conformal mode of the 2-dimensional metric.[2] The reasoning that led to this proposal was similar to that which leads to the σ-model approach to critical

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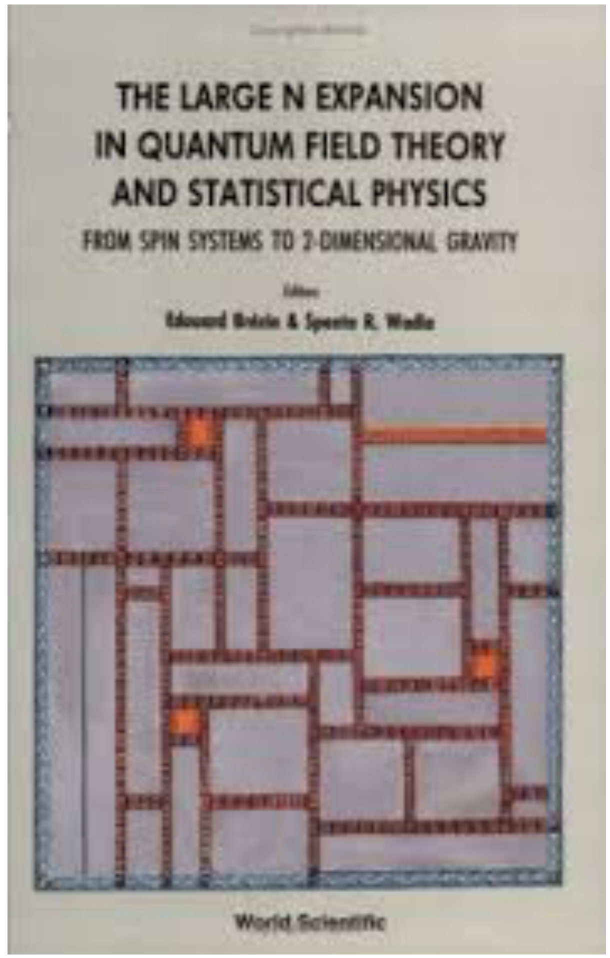
[a]By a Liouville theory here we mean a string in 2-dimensional target space with the following backgrounds: $G_{\mu,\nu} = \eta_{\mu\nu}$ (metric), $\Phi = Q\eta/2$ (dilaton) and $T = a\exp(Q\eta/2)$ (tachyon), where η is the Liouville mode. However, the 2-D gravity theory coupled to matter can admit of more general backgrounds having more complicated dependences on η. The reasoning that led to this proposal[2] is similar to the one that led to the σ-model approach in critical string theory.

ICTS_1

Large N Expansion



Recurring theme we heard today

- Emphasis on physical intuition
- Emphasis on unity of physics

Large N expansion as a fixed point

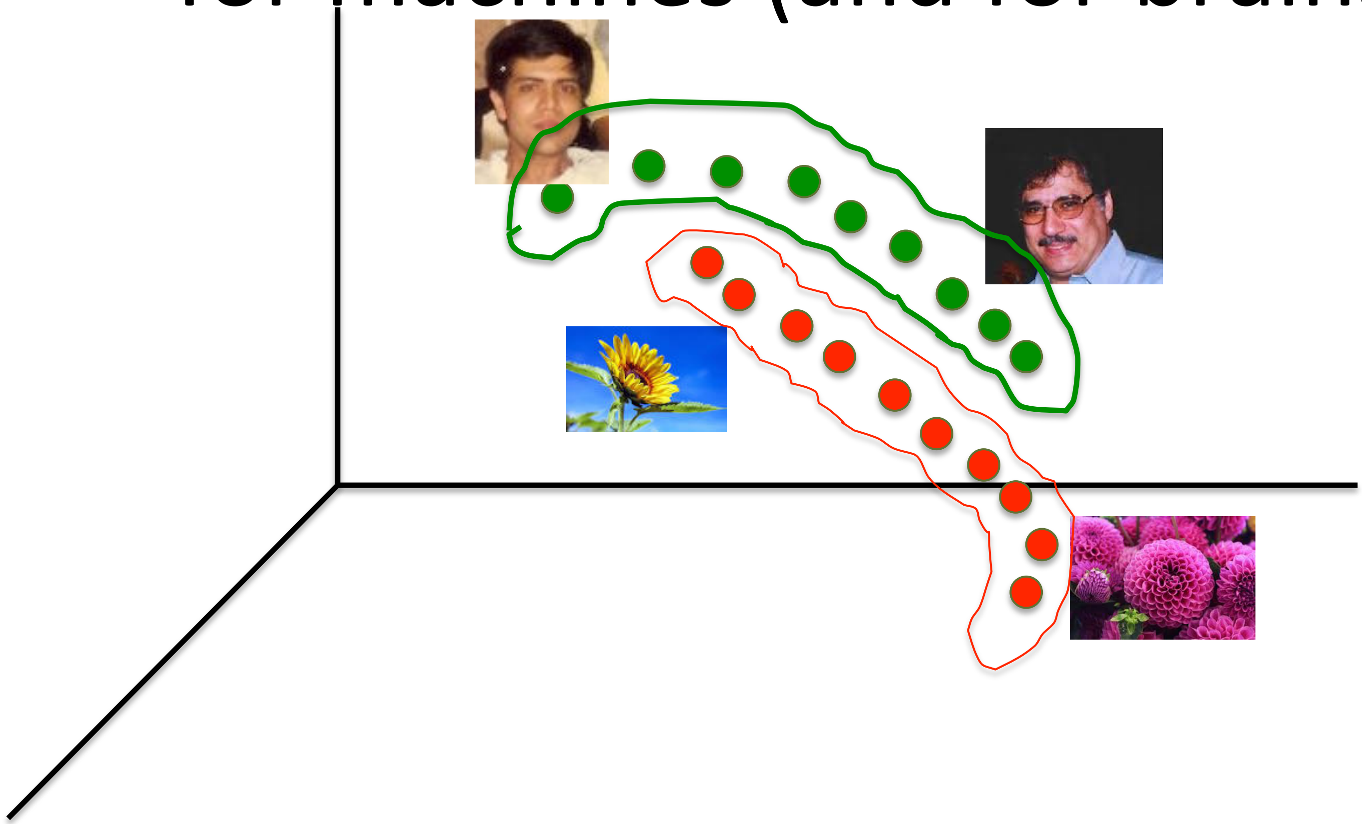
Kondo Physics:
Sengupta, Georges (1995)
Parcollet, Georges, Kotliar, Sengupta (1998)

Singular Value Distribution of Random Matrices:
Sengupta, Mitra (1999)

Information Theory of MIMO Systems:
Moustakas, Baranger, Balents, Sengupta, Simon (2000)
Moustakas, Simon, Sengupta (2003)

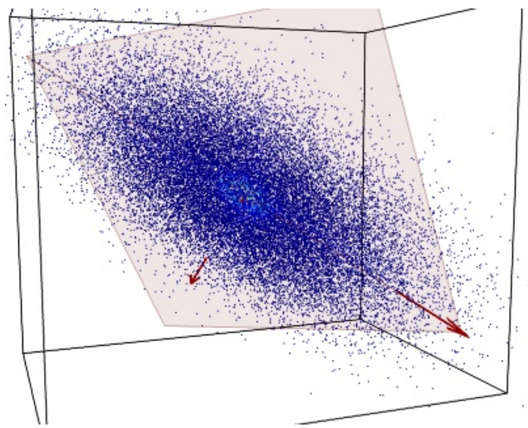
Algorithmic Phase Transitions in Compressed Sensing:
Ramezanali, Mitra, Sengupta (2015)

High dimensional pattern recognition: for machines (and for brains)



Curse of dimensionality

Dimension Reduction 101: Principal Components Analysis (PCA)



Diagonalize covariance matrix with Element given by:

$$C_{ij} = \sum_{a} (X_{ia} - \bar{X}_i)(X_{ja} - \bar{X}_j)$$

http://www.hgu.mrc.ac.uk/Research/Hill/Fig_4.jpg

PCA-> Singular Value Decomposition (SVD)

If we define $\hat{X}_{ia} = X_{ia} - \bar{X}_i$ then $C = \hat{X}\hat{X}^T$

Diagonalizing covariance matrix *C* is the same thing as doing a singular value decomposition of deviations *X* hat, and ordering singular vectors according to the size of singular values.

$$\underset{m \times n}{\overset{\hat{X}}{\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ x_{m1} & & & x_{mn} \end{pmatrix}}} \approx \underset{m \times r}{\overset{U}{\begin{pmatrix} u_{11} & \cdots & u_{1r} \\ \vdots & \ddots & \\ u_{m1} & & u_{mr} \end{pmatrix}}} \underset{r \times r}{\overset{S}{\begin{pmatrix} s_{11} & 0 & \cdots \\ 0 & \ddots & \\ \vdots & & s_{rr} \end{pmatrix}}} \underset{r \times n}{\overset{V^{\mathsf{T}}}{\begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \\ v_{r1} & & v_{rn} \end{pmatrix}}}$$

SVD for Denoising/Compressing Images

Figure 2: 1^{st}, 50^{th}, and 120^{th} image approximations of the image for $\sigma = 0\%$

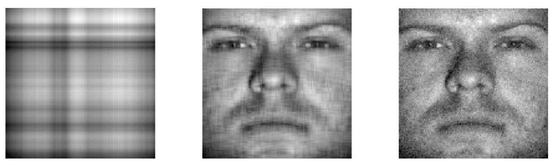


Figure 3: 1^{st}, 14^{th}, and 120^{th} image approximations of the image for $\sigma = 5\%$

Figure 4: 1^{st}, 9^{th}, and 120^{th} image approximations of the image for $\sigma = 10\%$

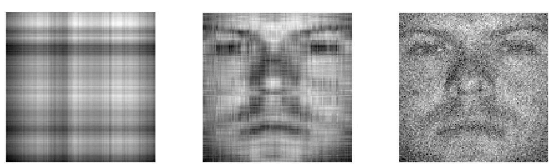


Figure 5: 1^{st}, 5^{th}, and 120^{th} image approximations of the image for $\sigma = 15\%$

From Poon, Ng and Sridharan, 2009

$$X_{\mathrm{recon}} = \bar{X} + U S_K V^{\dagger}$$

$$S_K = \begin{pmatrix} s_{11} & \cdots & \cdots & 0\cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & s_{KK} & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

Q: What *K* is optimal?

A: Departure from singular values of pure noise.

Describe the distribution in a different coordinate system

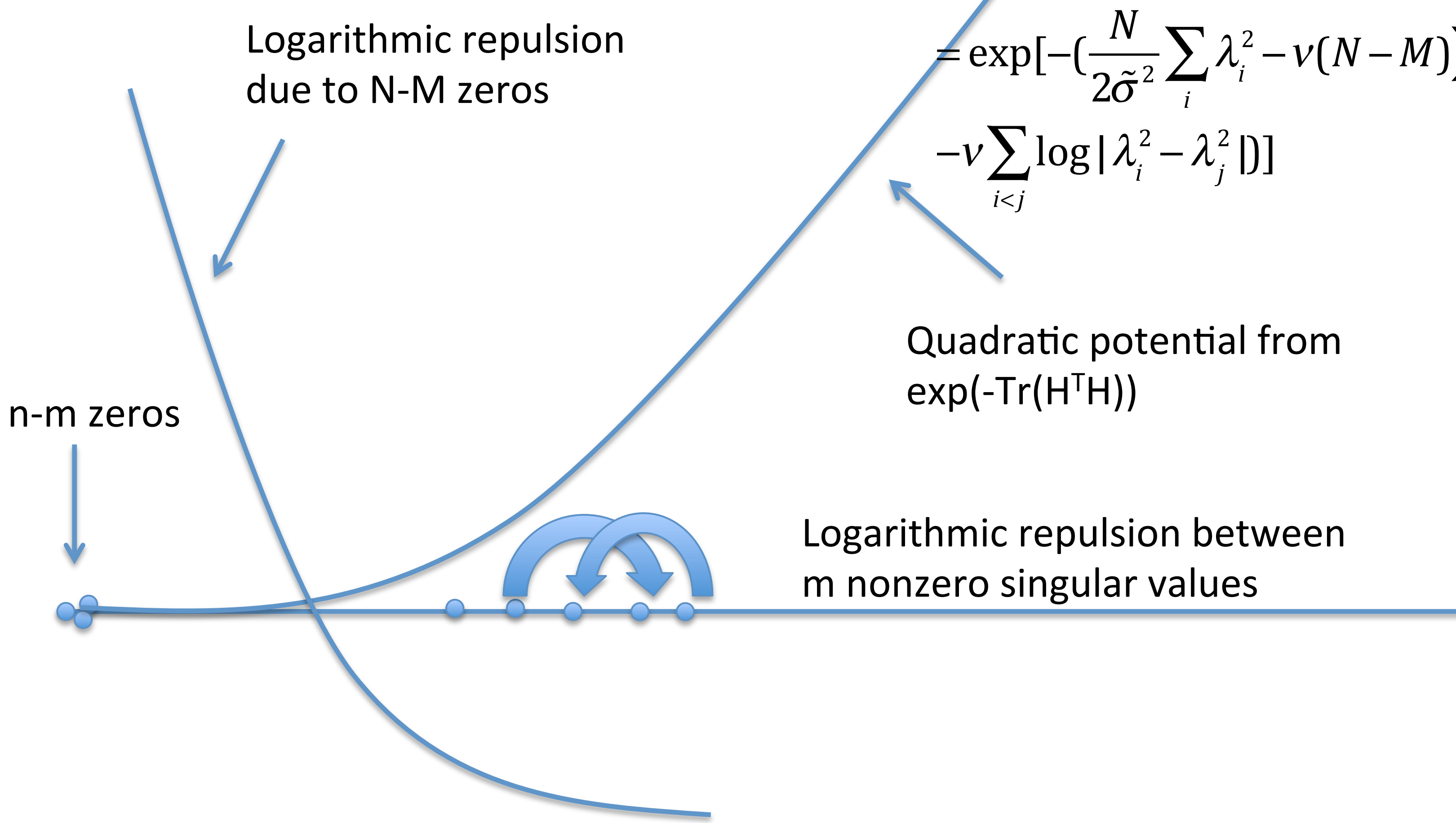
$$H = U\Lambda V^{\dagger} = U\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_M & 0 & \cdots & 0 \end{pmatrix} V^{\dagger}$$

$$P(H)\prod_{ia} dH_{ia} = C\exp[-\frac{N}{2\tilde{\sigma}^2} tr(H^2)]\prod_{ia} dH_{ia}$$

$$\rightarrow Ce^{-\frac{N}{2\tilde{\sigma}^2}\sum_i \lambda_i^2} \prod_{i<j} |\lambda_i^2 - \lambda_j^2|^{\nu} \prod_i \lambda_i^{\nu(N-M+1)-1} (\prod_i d\lambda_i) d\mu(U) d\mu(V)$$

Coulomb of Singular Values

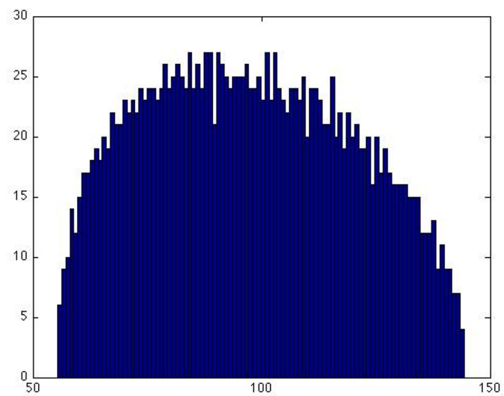
$$
\begin{aligned}
p(\lambda_1,\ldots,\lambda_N) &\propto \exp[-E(\lambda_1,\ldots,\lambda_N)] \\
&= \exp[-(\frac{N}{2\tilde{\sigma}^2}\sum_i \lambda_i^2 - \nu(N-M)\sum_i \log\lambda_i \\
&\quad -\nu\sum_{i<j}\log|\lambda_i^2-\lambda_j^2|)]
\end{aligned}
$$



SVD of `Just Noise'

$$\rho(\lambda)=\frac{a}{\lambda}\sqrt{(\lambda_{\max}^2-\lambda^2)(\lambda^2-\lambda_{\min}^2)}$$
$$where$$
$$\lambda_{\min,\max}=\sigma\sqrt{(M+N)\mp 2\sqrt{MN}}$$

Marcenko
And
Pastur,
1967



Histogram of singular values
For a 2000X10000 random matrix

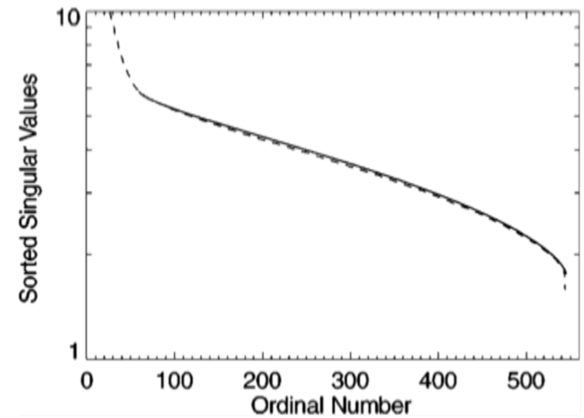
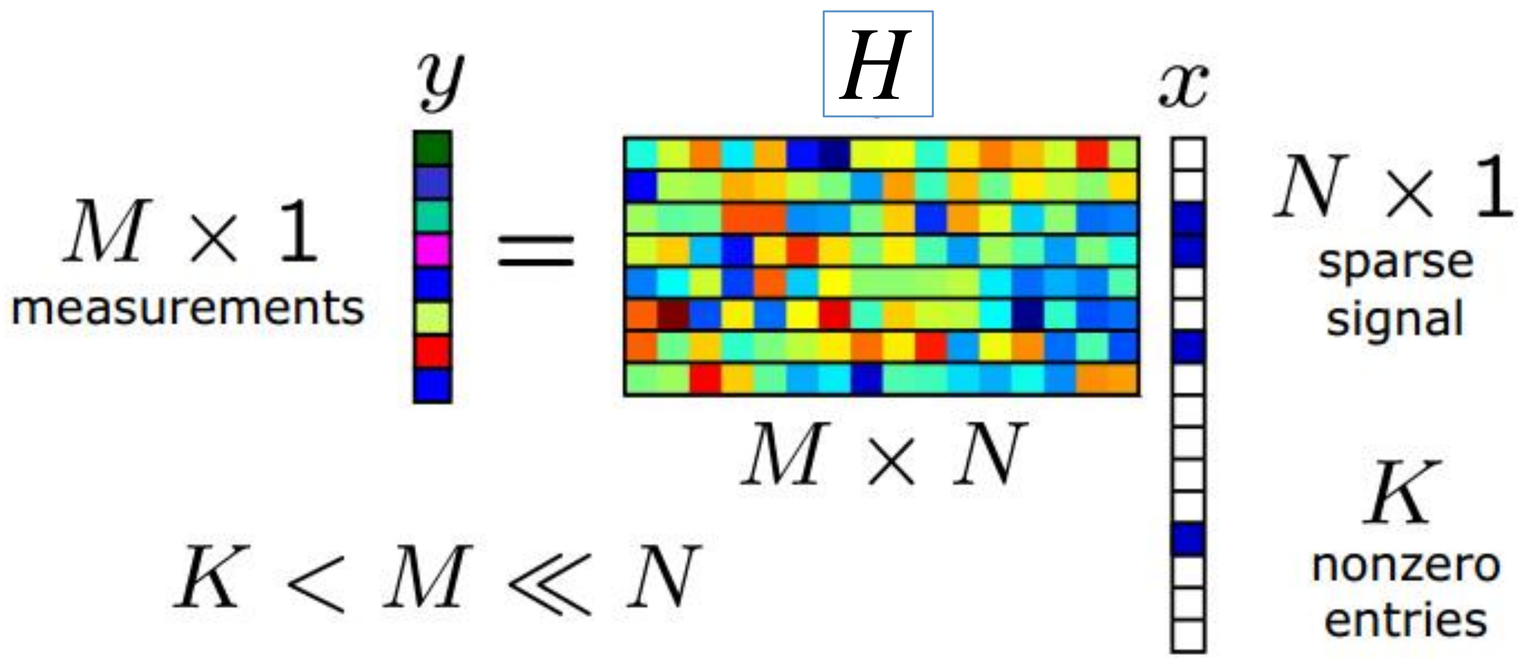


FIG. 3. Comparison of singular values from a SVD of an fMRI data set with the theoretical formula for a noise only matrix.

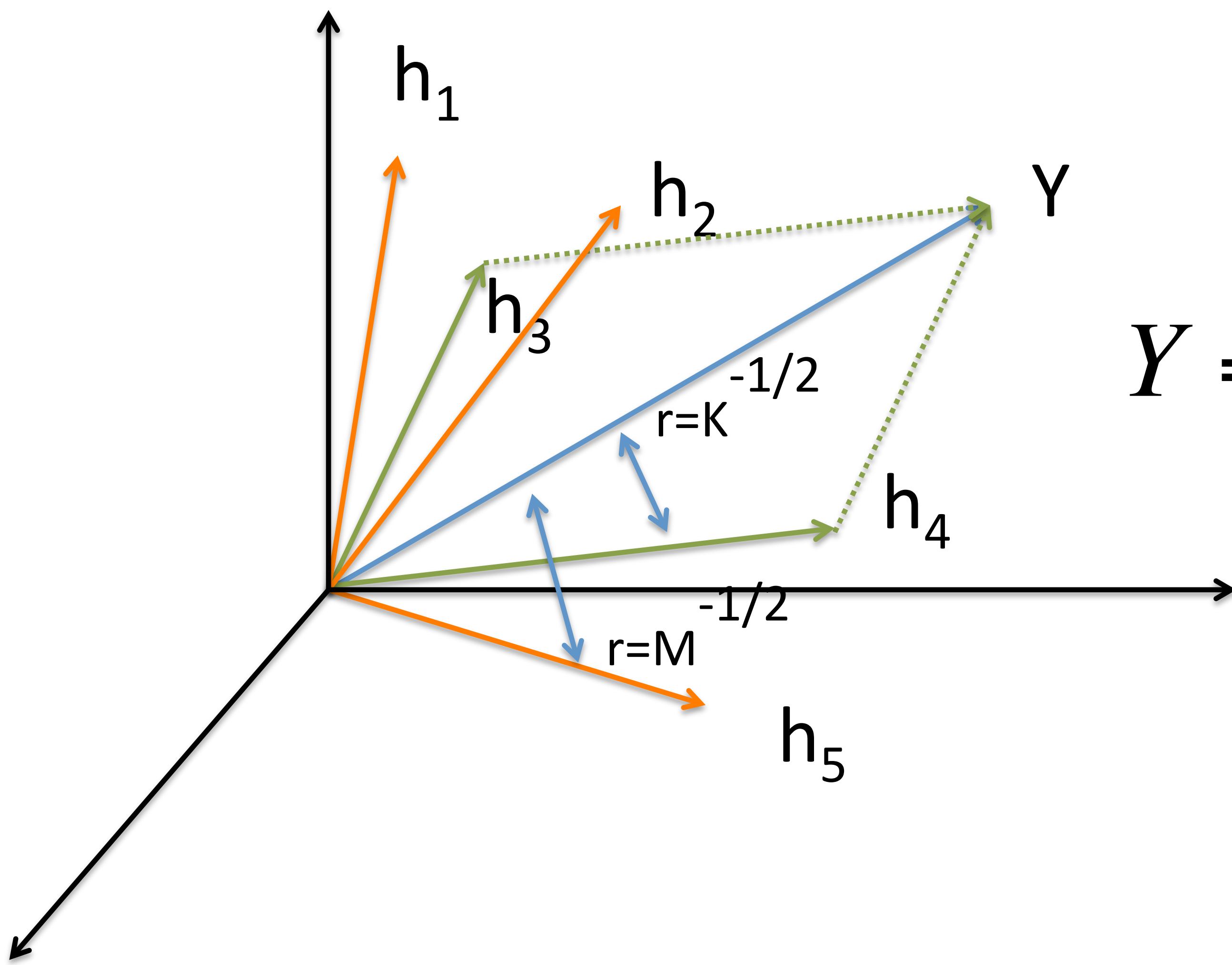
Sengupta and Mitra,
PRE 1999

Gavish, Donoho 2013

Compressed Sensing



Explaining Data



$$Y = \sum_a h_a X_a$$

Convex relaxation

Compressed sensing:

$$y \approx Wx_0, \text{ with } y \in R^M, x_0 \in R^N, M < N,$$

$$\text{but } ||x_0||_0 \le K < M.$$

$$\text{Find } \hat{x} = \arg\min_x ||x||_1 \ \text{ subject to } y = Wx.$$

$$\text{Is } \hat{x} \approx x_0 \text{ when } W \text{ is a random matrix?}$$

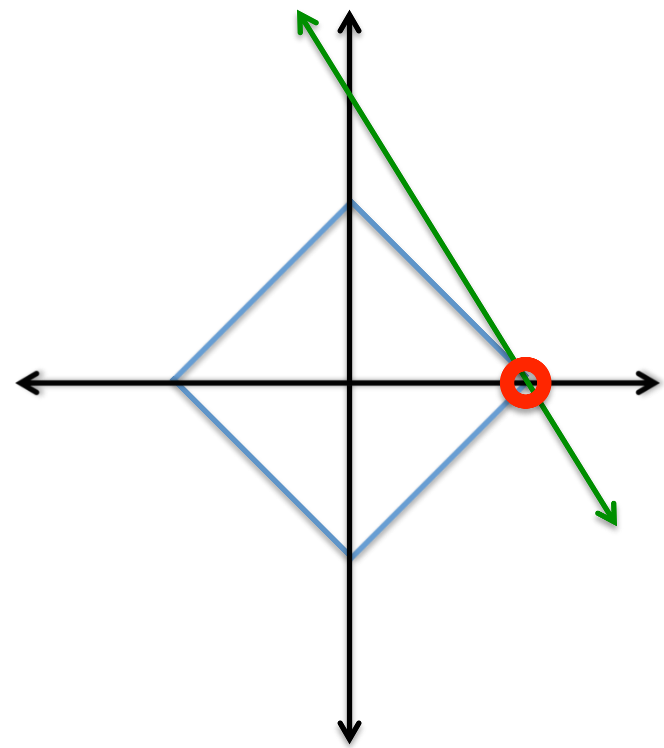
Donoho and Tanner 2009

Sparsity enforced by l_1 norm

L0-L1 relaxation:

Minimize $\| X \|_1$

Subject to $Y = HX$



Works of Candes, Tao
Donoho, from around 2004

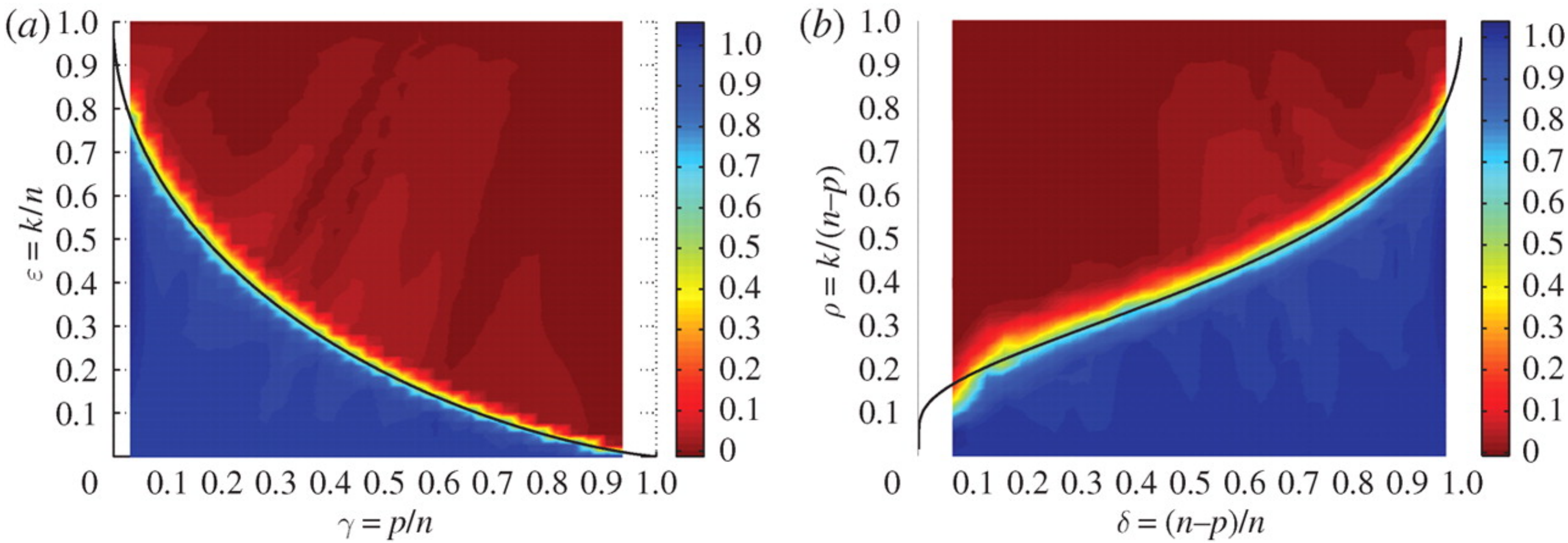
Large Dimensions

The transition happens with

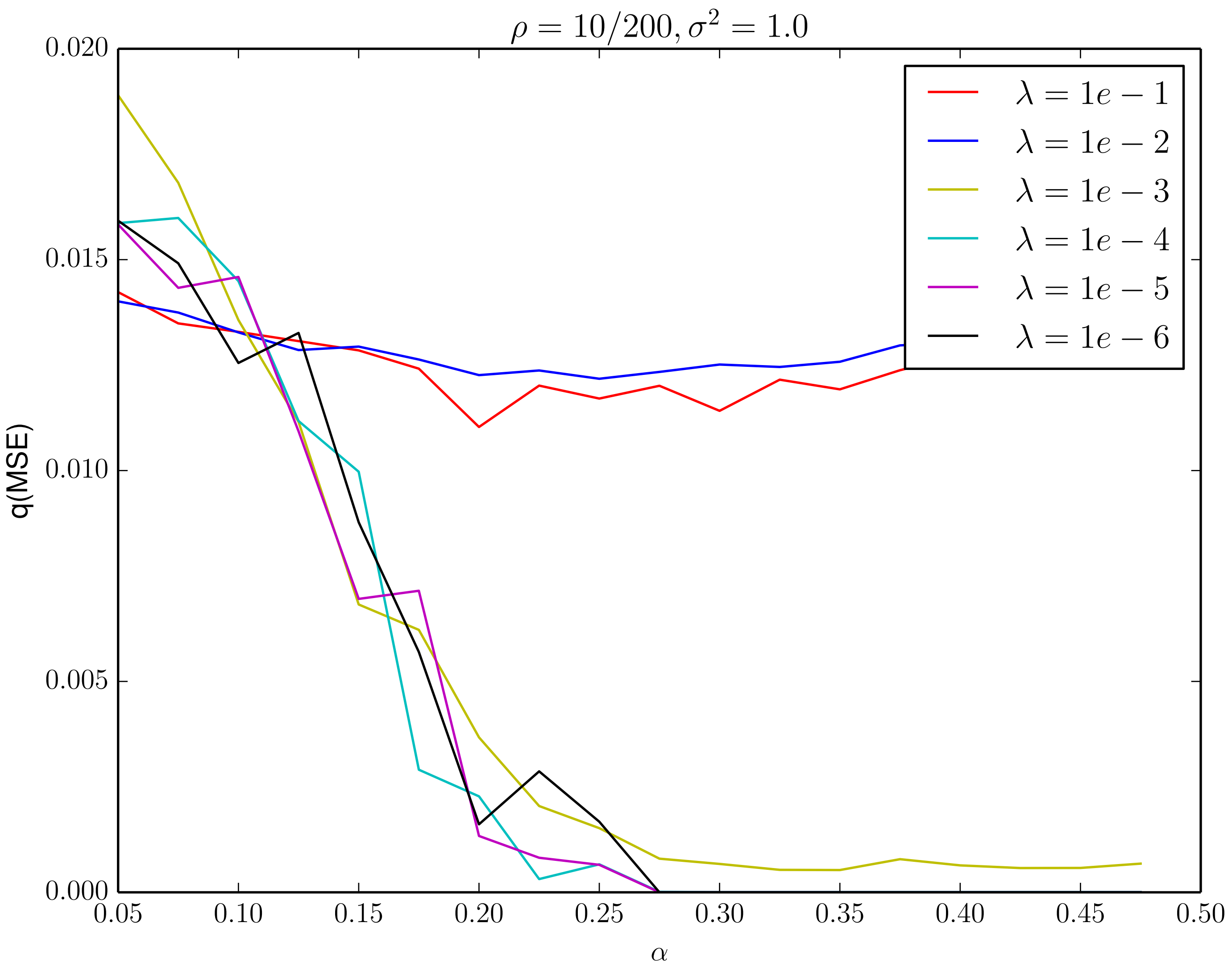
$$N, M, K \to \infty, \text{ holding}$$

$$M / N = \alpha, K / N = \rho \text{ fixed.}$$

Phase Transition in Compressed Sensing by L0-L1 Relaxation



Donoho and Tanner, Phil. Trans. Roy. Soc 2009
Replica Method Analysis: Kabashima
Wadayama and Tanaka, 2009
Ganguly and Sompolinsky, 2010


$\rho = 10/200, \sigma^2 = 1.0$
$\lambda = 1e-1$
$\lambda = 1e-2$
$\lambda = 1e-3$
$\lambda = 1e-4$
$\lambda = 1e-5$
$\lambda = 1e-6$
q(MSE)
0.020
0.015
0.010
0.005
0.000
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
α

Random-like deep network

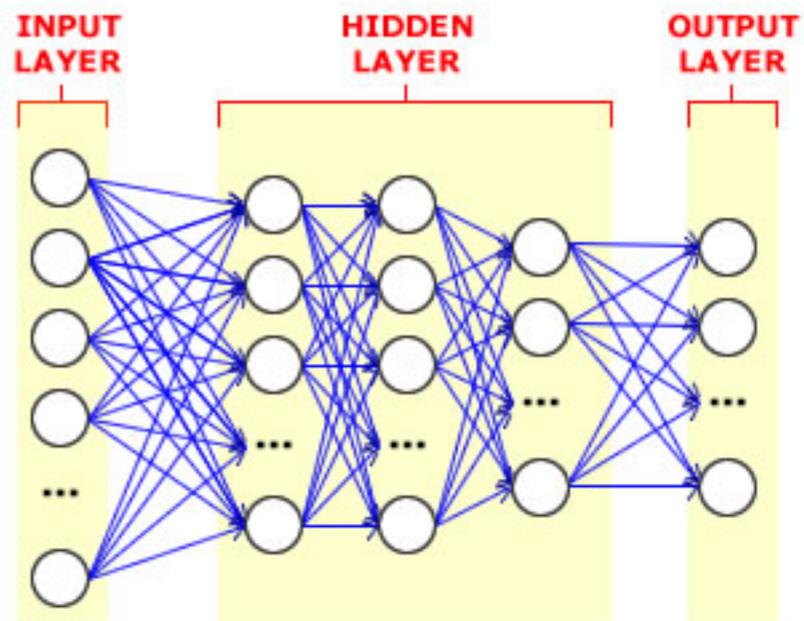
Deep nets

$$h^{(1)}_{M_1\times 1} \approx \sigma(W^{(1)\dagger}_{M_1\times N} x_{N\times 1} + b_1),$$

$$h^{(2)}_{M_2\times 1} \approx \sigma(W^{(2)\dagger}_{M_2\times M_1} h^{(1)}_{M_1\times 1} + b_2)$$

$$\ldots$$

$$h^{(d)}_{M_d\times 1} \approx \sigma(W^{(d)\dagger}_{M_2\times M_1} h^{(d-1)}_{M_1\times 1} + b_d)$$



From http://www.andrewsnoke.com

‘Pre-image’ of the final layer activation

- Fully-connected layers of deep nets

$$h^{(1)} \approx \sigma(W^{(1)\dagger}x \;\; + b_1),$$
$$h^{(2)} \approx \sigma(W^{(2)\dagger}h^{(1)} + b_2)$$
$$\cdots$$
$$h^{(d)} \approx \sigma(W^{(d)\dagger}h^{(d-1)} + b_d)$$

Call this map Φ_f :

$$h^{(d)} = \Phi_f(x)$$

→

What is the preimage of Φ_f ?

Reconstruction :

$$\tilde{x} = \arg\min_x l(\Phi_f(x), h^{(d)}) + R(x)$$

Mahendran and Vedaldi 2015

Alternative ‘generative’ model

Alternative: run the net backwards randomly zeroing some coordinates

Arora, Liang and Ma 2016

$$\tilde{h}^{(d-1)} \approx \tau(W^{(d)}h^{(d)}) \odot n_{d-1}^{drop}$$

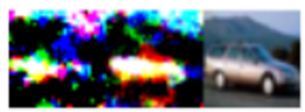
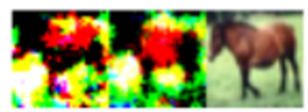
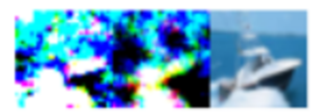
$$\tilde{h}^{(d-2)} \approx \tau(W^{(d-1)}h^{(d-1)}) \odot n_{d-2}^{drop}$$

$$\ldots$$

$$\tilde{x} \approx \tau(W^{(1)}\tilde{h}^{(1)}) \odot n_{0}^{drop},$$

Define a distribution:

$P(\tilde{x} \,|\, h^{(d)})$

Are deep nets reversible when W's are random?

Namely, is $h^{(d)} \approx \Phi_f(\tilde{x})$?

Reversibility in One-layer 'Deep' Net

Run the net backwards

randomly zeroing some coordinates

$\tilde{x} = \gamma \, \mathrm{ReLU}(Wh) \odot n^{drop}$

$\tilde{h} = \mathrm{ReLU}(W^{\dagger}\tilde{x} + b)$

Compute joint distribution $P(\tilde{h}_i, h_i)$

Turns out that $W^{\dagger}\tilde{x} + b$ is nearly Gaussian.

Use that to show $\tilde{h}_i \approx h_i$.

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Cengiz Pehlevan (Simons)

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