Aspects of Renormalization Group Flows

Z. Komargodski
Department of Particle Physics and Astrophysics, Weizmann Institute of Science, Rehovot, Israel

Preface

The aim of the lectures is to review various aspects of renormalization group flows in diverse dimensions.

- Introduction.
- Basic Tools: The Two-Dimensional Case.
- Three Dimensions.
- Four Dimensions.

1
Introductory Comments

The idea that we can try to solve (or usefully approximate) a complicated physical system by trying to ignore slow degrees of freedom is very old. For example, to first approximation, we can treat the nucleus of the atom as static.

Let us present a simple example where the application of this general idea immediately leads to a nontrivial result. Consider the Schrödinger equation with Hamiltonian

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}\lambda^2 x^2 y^2 \ . \tag{1.1}$$

Above λ is a positive coupling constant. The question is whether for nonzero λ the spectrum is continuous or discrete.

Solution: There are flat directions for $xy = 0$, namely $x = 0$ or $y = 0$. If we go very far on these flat directions, say we take $|x| \to \infty$, then the y degree of freedom is very heavy, with frequency $w = \lambda|x|$. So the y degree of freedom needs to be integrated out – it settles in the ground state with energy $E = \frac{1}{2}\lambda|x|$. Hence, the effective Hamiltonian that we get after we integrate out the variable y is

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}\lambda|x| + \cdots \ , \tag{1.2}$$

where the $\cdots$ stand for terms that are suppressed at large x relative to the linear term in $|x|$. We see that the light degree of freedom x is confined (i.e. no flat directions) and therefore has a discrete spectrum.

Exercise 1: Devise a procedure to compute corrections to (1.2) at large $|x|$. How does the first subleading term behave as $|x| \to \infty$?

We see that the ideas of renormalization group are already useful in quantum mechanics. In Quantum Field Theory (QFT) the ideas of the renormalization group are often not only very useful, but also indispensable.

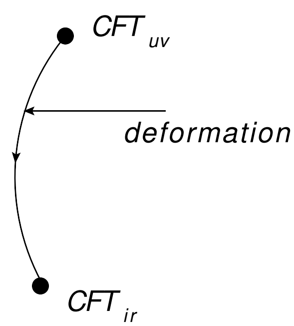
If we think of a very dense lattice of d space-like dimensions of size L and spacing $a << L$ as an approximation to continuum QFT, then we have some complicated system of $(L/a)^d$ quantum-mechanical degrees of freedom. Let us further assume that at each lattice point there are n quantum mechanical degrees of freedom.

The correlation length ξ is measured by correlation functions of local operators on the lattice

$$\langle \mathcal{O}(\vec{r}_1)\mathcal{O}(\vec{r}_2)\rangle \sim e^{-(|r_1-r_2|)/\xi} \ . \tag{1.3}$$

On sufficiently long distance scales the lattice structure always disappears. But usually one would expect $\xi \sim a$ so there is no nontrivial theory to describe at long distances (we say that the theory is gapped). However, if we tune the lattice couplings such that $\xi >> a$, then on sufficiently long distance scales there is a nontrivial continuum QFT. If we tune to the point $\xi \to \infty$ then one has a continuum conformal quantum field theory (CFT): a system without any energy scale.

One can now perturb our lattice theory, say, we can change some coupling constant. This can sometimes lead to a new nontrivial theory which has very large correlation length. This is when we have an interesting RG flow. We can often describe the deformation parameter itself in the language of an operator that we add to our QFT, see figure below.



So a renormalization group flow in the context of QFT is when the continuum CFT_{uv} is perturbed by an operator with scale M that triggers some reorganization of the degrees of freedom, possibly ending with a new CFT_{ir} at very long distances ($\gg M^{-1}$).

This description of QFT is very naturally connected to the physics of quantum spin systems and to high-energy physics. But QFT is so interesting because it also describes entirely different phenomena as well.

Classical Statistical Physics: Consider a classical system with coordinates q_i and momenta p_i, and energy functional $H = \sum_i p_i^2 + V(q_i)$. Thermal physics just means to sum over the entire phase space

$$Z = \int dp_i dq_i e^{-H(p_i,q_i)/T} \ .$$

The integral over momentum is trivial and we remain with

$$Z \sim \int dq_i e^{-V(q_i)/T} \ .$$

Typically, say in some classical lattice system, we would have

$$V(q_i) = K(q_{i+1} - q_i)^2 + v(q_i) + \cdots$$

then if the lattice is dense enough and if we tune T to lie near a phase transition, then the microscopic details would again disappear and we see that we get a d-dimensional Euclidean QFT. Difference such as $q_{i+1} - q_i$ are approximated by continuous derivatives. In this context, the word 'Quantum' is really misleading, because Euclidean QFT can also describe entirely classical systems, such as boiling water (at the second order phase transition point...).

Stochastic Quantization: Suppose we have a particle subject to friction and being stirred by random force

$$m\frac{d\vec{v}}{dt} = -\alpha\vec{v} + \vec{\eta}(t)$$

where $\eta(t)$ is some random stirring force. We can assume, for example, that

$$\langle \vec{\eta}(t_1)\vec{\eta}(t_2)\rangle \sim \delta(|t_1 - t_2|)\ .$$

This is called the Langevin equation. Suppose one reaches some kind of equilibrium at the end (so $\langle\vec{\eta}\rangle = 0$) and one would like to compute $\langle \vec{v}^n \rangle$. Suppose that η was a given function, then we could think of this as follows:

$$\langle \vec{v}^n(t_0)\rangle \sim \int [D\eta(t)] e^{-\frac{1}{2}\vec{\eta}(t)^2}\vec{v}^n_\eta(t_0) \tag{1.4}$$

where v_η is the exact solution corresponding to a given $\vec{\eta}$. If one assumes eventual equilibrium, then $\lim_{t_0\to\infty}\langle \vec{v}^n(t_0)\rangle$ would exist. We could compute it by identifying some effective field theory at zero dimensions. In this simplest Langevin example, these correlation functions are computed (roughly) from

$$\lim_{t_0\to\infty}\langle \vec{v}^n(t_0)\rangle \sim \int d\vec{v} e^{-\alpha\vec{v}^2}\vec{v}^n\ .$$

This procedure is related to the central assumption of stochastic quantization. Regardless of that, one can rewrite many classical stochastic systems in terms of supersymmetric QFT etc. See [1] for a review.

To summarize. we see that QFT describes (among others)

- High-Energy Physics
- Quantum Many body Systems (zero and nonzero temperature)
- Classical Statistical Systems
- Stochastic Systems

Let us now list some general questions that we will be concerned with (not all can be solved at the moment, and not all would be addressed in this mini-course)

- Is any sufficiently nice system without an explicit mass scale always conformal (i.e. enjoys the full conformal group)?
- Starting from CFT_{uv}, which CFT_{ir} can we flow to? are there are interesting constraints?
- When do nontrivial CFTs exist?
- What is the geometry in the space of theories?
- Consequences for experiments?
- Using AdS/CFT, can we learn something about quantum gravity from all of this?
- Is there a useful useful generalization of the renormalization group for systems which are not in equilibrium?

2
Two-Dimensional Models

We consider Euclidean two-dimensional theories that enjoy the isometry group of $\mathbb{R}^2$. Namely, the theories are invariant under translations and rotations in the two space directions. If the theory is local it has an energy momentum tensor operator $T_{\mu\nu}$ which is symmetric and conserved $T_{\mu\nu} = T_{\nu\mu}$, $\partial^\mu T_{\mu\nu} = 0$. These equations should be interpreted as operator equations, namely they must hold in all correlation functions except, perhaps, at coincident points. Here we will study theories where, when possible, these equations hold in fact also at coincident points. In other words, we study theories where there is no local gravitational anomaly. See [2] for some basic facts about theories violating this assumption.

We can study two-point correlation functions of the energy-momentum operator. This correlation function is highly constrained by symmetry and conservation. It takes the following most general form in momentum space:

$$\begin{aligned}\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle &= \frac{1}{2}\left((q_\mu q_\rho - q^2\eta_{\mu\rho})(q_\nu q_\sigma - q^2\eta_{\nu\sigma}) + \rho \leftrightarrow \sigma\right) f(q^2) \\ &+ (q_\mu q_\nu - q^2\eta_{\mu\nu})(q_\rho q_\sigma - q^2\eta_{\rho\sigma})g(q^2)\ .\end{aligned} \tag{2.1}$$

The most general two-point function is therefore fixed by two unknown functions of the momentum squared. (Note that above we have stripped the trivial delta function enforcing momentum conservation.)

The form (2.1) holds in any number of dimensions. In two dimensions the two functions $f(q^2)$ and $g(q^2)$ are not independent.

Exercise 2: Show that the two kinematical structures in (2.1) **are not independent in two dimensions.**

Now let us make a further assumption, that the theory is scale invariant (but not necessarily conformal invariant). This allows us to fix the two functions f, g up to a constant[1]

$$f(q^2) = \frac{b}{q^2}\ , \qquad g(q^2) = \frac{d}{q^2}\ . \tag{2.2}$$

We can now calculate the two-point function $\langle T^\mu_\mu(q)T^\mu_\mu(-q)\rangle$

$$\langle T^\mu_\mu(q)T^\mu_\mu(-q)\rangle = (b+d)q^2\ . \tag{2.3}$$

Transforming back to position space this means that $\langle T^\mu_\mu(x)T^\mu_\mu(0)\rangle \sim (b+d)\Box\delta^{(2)}(x)$. In particular, at separated points the correlation function vanishes:

[1] A logarithm is disallowed since it violates scale invariance. (The rescaling of the momentum would not produce a local term.)

$$\langle T^\mu_\mu(x) T^\nu_\nu(0)\rangle_{x\neq 0} = 0 \ . \tag{2.4}$$

In unitary theories, this means that the trace itself is a vanishing operator $T^\mu_\mu = 0$ (since it creates nothing from the vacuum, it must be a trivial operator). This is called the Reeh-Schlieder theorem [3].

The operator equation

$$T^\mu_\mu = 0 \tag{2.5}$$

is precisely the condition for having the full conformal symmetry of $\mathbb{R}^2$, $SO(3,1)$, present.

In fact, in two-dimensions, (2.5) is sufficient to guarantee an infinite symmetry group (the Virasoro algebra). This is because T_{zz} obeys $\bar\partial T_{zz} = 0$ and so it can be multiplied by an arbitrary holomorphic function and still remain conserved. We will elaborate on that a little more later.

The equation (2.5) is satisfied in all correlation functions at separated points, but it may fail at coincident points. We have already seen this phenomenon in (2.3). Such contact terms signal an anomaly of $SO(3,1)$. Other than this potential anomaly, (2.5) means that $SO(3,1)$ is a perfectly good symmetry of the theory. Therefore, we see that scale invariant theories are necessarily conformal invariant in two dimensions [4].[2]

More generally, we find

$$\langle T^\rho_\rho(q) T_{\mu\nu}(-q)\rangle = -(b+d)(q_\mu q_\nu - q^2\eta_{\mu\nu}) \ , \tag{2.6}$$

which is again a polynomial in momentum and hence a contact term in position space, consistently with (2.5). One may wonder at this point whether there exist Quantum Field Theories for which these contact terms are absent because $b+d=0$. Consider the correlation function $\langle T_{11}T_{11}\rangle$. This has support at separated points. It is proportional to $b+d$.

Exercise 3: In light of exercise 2, is this a coincidence?

Therefore, in unitary theories, from reflection positivity (i.e. Reeh-Schlieder theorem) it follows that

$$b+d > 0 \ . \tag{2.7}$$

So far we have seen that scale invariant theories in fact enjoy $SO(3,1)$ symmetry, but the symmetry is afflicted with various contact terms such as (2.6). To see clearly the physical meaning of this anomaly, we can couple the theory to some ambient curved space (there is no dynamics associated to the curved space, it is just a background field). This is done to linear order via $\sim \int d^2x T^{\mu\nu}h_{\mu\nu}$, where $h_{\mu\nu}$ is the linearized metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Hence in the presence of a background metric that deviates only slightly from flat space

$$\begin{aligned}\langle T^\mu_\mu(0)\rangle_{g_{\mu\nu}} &\sim \int d^2x \langle T^\mu_\mu(0) T^{\rho\sigma}(x)\rangle h_{\rho\sigma}(x) \sim (b+d)\int d^2x \left(\partial^\rho\partial^\sigma\delta^2(x) - \eta^{\rho\sigma}\Box\delta^2(x)\right) h_{\rho\sigma} \\ &\sim (b+d)(\partial^\rho\partial^\sigma - \eta^{\rho\sigma}\Box)h_{\rho\sigma} \ .\end{aligned} \tag{2.8}$$

The final object $(\partial^\rho\partial^\sigma - \eta^{\rho\sigma}\Box)h_{\rho\sigma}$ is identified with the linearized Ricci scalar. In principle, if we had analyzed three-point functions of the energy-momentum tensor and so forth,

[2] Some technical assumptions implicit in the argument above are spelled out in [4], where the argument was first developed.

we would have eventually constructed the entire series expansion of the Ricci scalar. Therefore, the expectation value of the trace of the energy-momentum tensor is proportional to the Ricci scalar of the ambient space. This is the famous two-dimensional trace anomaly. It is conventional to denote the anomaly by c (and not by $b + d$ as we have done so far). The usual normalization is

$$T = -\frac{c}{24\pi} R \,. \tag{2.9}$$

c is also referred to as the "central charge" but we will not emphasize this representation-theoretic interpretation here. Our argument (2.7) translates to $c > 0$.

There is a very useful consequence of (2.9). Consider the conservation equation in curved space $\nabla^\mu T_{\mu\nu} = 0$. If we switch to local complex coordinates with a Hermitian metric

$$ds^2 = e^{\varphi} dz d\bar{z} \,,$$

then the conservation equation takes the form

$$\partial_{\bar{z}} T_{zz} + e^{\varphi} \partial_z (e^{-\varphi} T_{z\bar{z}}) = 0 \,.$$

But in two dimensions we have the relation (2.9) so we can substitute this into the second term of the equation above. It is useful to remember that $R = -4e^{-\varphi}\partial\bar{\partial}\varphi$.

This implies that we can definite a holomorphic energy-momentum tensor

$$T'_{zz} = T_{zz} + \alpha c \left(-(\partial\varphi)^2 + 2\partial^2\varphi \right) \,. \tag{2.10}$$

Exercise 4: Fix the numerical coefficient α and verify the appearance of the Schwartzian derivative below.

The advantage of T' is that it is holomorphic, but the disadvantage is that it transforms non-covariantly under holomorphic coordinate transformations $z \to f(z)$ (because φ shift in-homogeneously). T' is what is actually used in most of the literature on 2d CFTs, and the prime is usually omitted. The in-homogenous piece in the transformation rule is the so-called Schwartzian derivative

$$z \to w(z) \,, \qquad \{w, z\} = \frac{w_{zzz}}{w_z} - \frac{3}{2}\left(\frac{w_{zz}}{w_z}\right)^2 \,. \tag{2.11}$$

Using (2.9) we can present another useful interpretation of c. Consider a two-dimensional conformal field theory compactified on a two-sphere $\mathbb{S}^2$ of radius a,

$$ds^2 = \frac{4a^2}{(1+|x|^2)^2} \sum_{i=1}^{2} (dx_i)^2 \,, \qquad |x|^2 = \sum_{i=1}^{2} (x_i)^2 \,. \tag{2.12}$$

The Ricci scalar is $R = 2/a^2$. Because of the anomaly (2.9), the partition function

$$Z_{\mathbb{S}^2} = \int [d\Phi] e^{-\int_{\mathbb{S}^2} \mathcal{L}(\Phi)} \tag{2.13}$$

depends on a. (If the theory had been conformal without any anomalies, we would have expected the partition function to be independent of the radius of the sphere.) We find that

$$\frac{d}{d\log a}\log Z_{\mathbb{S}^2} = -\int_{\mathbb{S}^2}\sqrt{g}\langle T\rangle = \frac{c}{24\pi}\int_{\mathbb{S}^2}\sqrt{g}R = \frac{c}{24\pi}\frac{2}{a^2}Vol(\mathbb{S}^2) = \frac{c}{3}\ . \tag{2.14}$$

Thus the logarithmic derivative of the partition function yields the c anomaly. This particular interpretation of c will turn out to be very useful later.

The central charge c appears in several other very important places. We would like to explain very briefly why c appears in the entanglement entropy of the vacuum [5]. Suppose we take $\mathbb{R} = A \cup A^c$ and $A = [a, b]$ some interval. The reduced density matrix is defined by

$$\rho_A = Tr_{A^c}(\rho_{vacuum})\ ,$$

with ρ_{vacuum} the density matrix of the normal pure vacuum. Then, the Renyi entropies are defined by

$$S_n = \frac{1}{n-1}Tr_A(\rho_A^n)\ .$$

(The von-Neumann entropy is recovered from the $n \to 1$ limit, if the limit exists.) The S_n can be calculated from the partition function of the theory on an n-sheeted covering of $\mathbb{R}^2$:

$$C_n := \cup_{i=1}^n \mathbb{R}^2_{(i)}$$

such that for any field ϕ, it takes the same values for all $\mathbb{R}^2_{(i)}$ away from A and on A we glue ϕ_i and ϕ_{i+1} in a cyclic fashion. This construction can be regarded technically as some twist field correlation function in a symmetric orbifold theory. Less abstractly, near the points a, b we simply have a conical singularity with the total angle in the range $\theta \in [0, 2\pi n]$. The space C_n constructed above can actually be mapped conformally to the one with $A = [0, \infty]$.[3] Then the C_n space associated to it is manifestly equivalent to the ordinary complex plane after a transformation

$$z = w^{1/n}\ , \qquad w \in C_n\ , z \in \mathbb{R}^2\ .$$

Now, using the fact that the holomorphic energy-momentum tensor does not transform covariantly, using the explicit expressions (2.10),(2.11) we find that the conical defects at a, b behave as primary operators of dimensions

$$\Delta_n = \frac{c}{24}(1 - 1/n^2)\ .$$

The Renyi entropy, being just the partition function on C_n, is therefore given by

$$S_n \sim \frac{1}{n-1}|a-b|^{-c/6(n-1/n)}\ . \tag{2.15}$$

In particular, the $n \to 1$ limit exists and gives

$$S_{von-Neuman} \sim c\log(|a-b|)\ . \tag{2.16}$$

The analysis of the case that A consists of a disjoint union of intervals is far more complicated and we will not discuss it here.

[3] **Exercise 5:** Show that in two dimensions any interval can be conformally mapped to the half line.

We will now consider non-scale invariant theories, i.e. theories where there is some conformal field theory at short distances, CFT_{uv}, and some other conformal field theory (that could be trivial) at long distances, CFT_{ir}. Let us study the correlation functions of the stress tensor in such a case, following [6]. To avoid having to discuss contact terms (which were very important above) we switch to position space. We begin by rewriting (2.1) in position space.

In terms of the complex coordinate $z = x^1 + ix^2$ the conservation equations are $\partial_{\bar{z}} T_{zz} = -\partial_z T$, $\partial_z T_{\bar{z}\bar{z}} = -\partial_{\bar{z}} T$, where T stands for the trace of the energy-momentum tensor. We can parameterize the most general two point functions consistent with the isometries of $\mathbb{R}^2$

$$\begin{aligned} \langle T_{zz}(z) T_{zz}(0) \rangle &= \frac{F(z\bar{z}, M)}{z^4} \ , \\ \langle T(z) T_{zz}(0) \rangle &= \frac{G(z\bar{z}, M)}{z^3 \bar{z}} \ , \\ \langle T(z) T(0) \rangle &= \frac{H(z\bar{z}, M)}{z^2 \bar{z}^2} \ . \end{aligned} \tag{2.17}$$

In the above M stands for some generic mass scale of the theory. As we have seen in our analysis above (2.1), we know that the conservation equation should bring down the number of independent functions to two. Indeed, we find the following relations $\dot{F} = -\dot{G} + 3G$, $\dot{H} - 2H = -\dot{G} + G$, where $\dot{X} \equiv |z^2| \frac{dX}{d|z|^2}$, leaving two real undermined functions (remember that G and F are complex).

Using these relations one finds that the combination $C \equiv F - 2G - 3H$ satisfies the following differential equation

$$\dot{C} = -6H \ . \tag{2.18}$$

However, since H is positive definite, the equation above means that C decreases monotonically as we increase the distance. Let us now identify C at very short and very long distances. At very short and very long distances it is described by the appropriate quantities in the corresponding conformal field theories. As we have explained above, in conformal field theory G, H are contact terms, hence can be neglected as long we do not let the operators collide. On the other hand, $F \sim c$. (It is easy to verify that F is sensitive only to the combination $b + d$ as defined in (3.3). Hence, it is only sensitive to c.)

This shows that C is a monotonic decreasing function that starts from c_{UV} and flows to c_{IR}. Since the anomalies c_{UV}, c_{IR} are defined inherently in the corresponding conformal field theories, this means that the space of 2d CFTs admits a natural foliation and the renormalization group flow can proceed in only one direction in this foliation. No cycles of the renormalization group are allowed.

One can think of c as a measure of degrees of freedom of the theory. In simple renormalization group flows it is easy to understand that c should decrease since we merely integrate out some massive particles. However, there are many highly nontrivial renormalization group flows where there are emergent degrees of freedom, and the result that

$$c_{UV} > c_{IR} \tag{2.19}$$

is a strong constraint on the allowed emergent degrees of freedom.

We can integrate the equation (2.18) to obtain a certain sum rule

$$c_{UV} - c_{IR} \sim \int d\log|z^2|H \sim \int d^2z|z^2|\langle T(z)T(0)\rangle > 0\ . \tag{2.20}$$

Since c can also be understood as the path integral over the two-sphere, the inequality (2.19) can also be interpreted as a statement about the partition function of the massive theory on $\mathbb{S}^2$.

There is also a very interesting reinterpretation using the entanglement-entropy of the interval. when the interval $[a, b]$ is very short compared to the mass scale, then we recover (2.16) with $c \to c_{uv}$. When the interval becomes very long compared to the mass scale, then we have (2.16) with $c \to c_{ir}$. In between there is some interpolating functions. Using information-theoretic properties of entropy, it is possible to show directly [10] from this point of view that $c_{uv} > c_{ir}$. So it would seem that the monotonicity of the renormalization group flow in two dimensions is essentially equivalent to some properties of the von-Neumann entropy.

3
Three-Dimensional Models

Three-dimensional QFT is directly relevant for understanding interesting classical second order phase transitions (boiling water, He_3 etc), as well as quantum critical points that appear condensed matter physics. Finally, it is a useful playground for confinement and other non-perturbative aspects of quantum field theory.

Here we will discuss the symmetries of fixed points, S^3 partition functions, monotonicity of RG flows, connection to entanglement entropy, and supersymmetry.

3.1 Conformal Invariance

There are interesting examples of continuum theories with infinite correlation length but no conformal symmetry. Consider the free gauge field in three dimensions

$$S = \frac{1}{2e^2} \int d^3x F_{\mu\nu}^2$$

One would think that the theory is scale invariant because we can assign the gauge field dimension $1/2$ and thus e would be dimensionless.

However, the only conceivable, conserved, gauge invariant, energy-momentum tensor we could write is

$$T_{\mu\nu} = F_{\mu\rho} F^{\rho}_{\ \nu} - \frac{1}{4}\eta_{\mu\nu} F^2 \ . \tag{3.1}$$

Exercise 6: Show that this energy-momentum tensor is conserved.

It is not traceless in three-dimensions $T^{\mu}_{\mu} = \frac{1}{4}F^2$. It is possible to prove that a local scale current has to be of the form

$$\Delta_\mu = x^\nu T_{\mu\nu} - V_\mu \ , \tag{3.2}$$

where V_μ is called the virial current. This is conserved if an only if $T^{\mu}_{\mu} = \partial^\nu V_\nu$, i.e. the energy-momentum tensor is a gradient of a well-defined operator. In our case, the best we can do is to write a non-gauge invariant scale current

$$\Delta_\mu = x^\nu T_{\mu\nu} - \frac{1}{2} F_{\mu\nu} A^\nu \tag{3.3}$$

which is conserved.

It is not a good scale current because it is not gauge invariant. However, the charge

$$\Delta = \int d^2x \Delta_0 \tag{3.4}$$

is gauge invariant if we assume all the fields decay sufficiently rapidly.

Exercise 7: Show that the scale current (3.3) is conserved and show that the associated charge is gauge invariant.

So we conclude that the theory has no scale current, but a scale charge exists. One can also show that the currents of special conformal transformations don't exist, but in this case, also the associated charges does not exist. So free 3d QED is an example of a theory that is unitary, scale invariant, but not conformal.

It is not really known whether this theory is an exception or there are more theories of this sort (it would be very interesting to find a unitary interacting example – it is fair to say that most people believe such theories do not exist). This free counterexample to the idea that scale invariance+unitarity imply conformal invariance has been discussed in many places, see for example [7],[8] and references therein.

3.2 A More Careful Look into Free 3d QED (for advanced students)

The discussion above is essentially correct, but there is a more precise way to phrase it. In three dimensions a free gauge field and a free scalar field are completely equivalent

$$\partial_\mu \phi = \epsilon_{\mu\nu\rho} F^{\nu\rho} \ . \tag{3.5}$$

The Bianchi identity corresponds to the Klein-Gordon equation.

One has to distinguish the case that the gauge symmetry acting on A_μ is compact from the non-compact case. In the former, one can have a nontrivial flux of the magnetic field through two-cycles. That means that ϕ is a periodic scalar, so that it can wind through the dual one-cycle. If we decompactify the gauge symmetry, the period of the scalar goes to zero. So the scalar dual to a gauge field with noncompact gauge symmetry is of zero radius (this is a scalar without a zero mode).

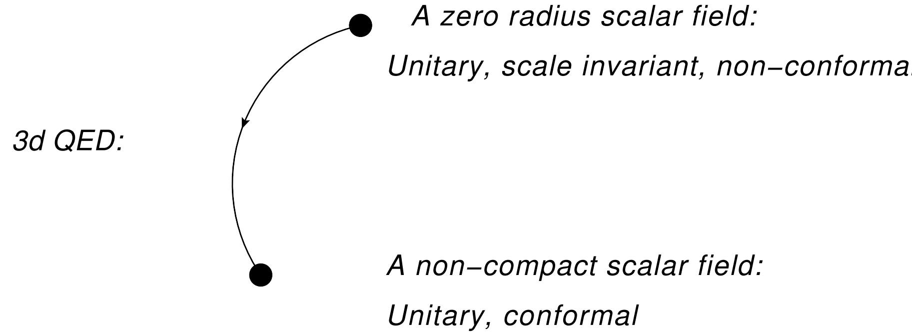
The clash between having a conserved energy momentum tensor and conformal invariance is clearly visible in the language of ϕ. A traceless energy-momentum tensor for ϕ would take the form

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 + \frac{1}{8}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\phi^2 \tag{3.6}$$

Exercise 8: Show that this is conserved and traceless.

However, this energy-momentum tensor is inconsistent with the periodicity of ϕ. One can naturally interpret the period of ϕ as some dimensionful parameter (of dimension 1/2). In the far UV, the periodicity goes to zero (since it has mass dimension) and therefore one finds that the UV theory is a scalar without a zero mode. This is a unitary scale but non-conformally invariant theory. In the infrared the scalar effectively de-compactifies. Then, the energy momentum tensor (3.6) becomes admissible. Hence, the infrared theory is just an ordinary, conformal, non-compact free scalar field.

One can therefore summarize the free 3d gauge field model in the following diagram:



One can test this interpretation of the 3d QED model by, for example, computing the entanglement entropy on a disk. One indeed finds that this approaches infinity logarithmically in the UV (we will discuss why this has to be the case below) and a finite value, identical to that of a free non-compact scalar field, in the infrared [9].

3.3 S^3 Partition Functions

An interesting question is whether there exists a function in three dimensions satisfying something similar to (2.19). The problem consists of identifying a candidate quantity that could satisfy such an inequality and then proving that it indeed does so.

There are various ways to define quantities in higher-dimensional field theories that share some common features with c. For example, in conformal theories in two dimensions c is equivalent to the free energy density of the system, divided by the appropriate power of the temperature. One could define a similar object in higher-dimensional field theories. However, one quickly finds that it is not monotonic [11]. This already shows that inequalities such as (2.19) are quite delicate, and they fail if one chooses to measure the number of effective degrees of freedom in the wrong way (albeit a very intuitive and seemingly natural way).

Progress on the problem of identifying a candidate quantity generalizing (2.19) happened quite recently [12],[13]. The conjecture arose independently from studies in AdS/CFT and from studies of $\mathcal{N} = 2$ supersymmetric 3d theories.

Any conformal field theory on $\mathbb{R}^3$ can be canonically mapped to a theory on the curved space $\mathbb{S}^3$. This is because $\mathbb{S}^3$ is stereographically equivalent to flat space (thus the metric on $\mathbb{S}^3$ is conformal to $\mathbb{R}^3$). In three dimensions there are no trace anomalies, and hence the partition function over $\mathbb{S}^3$ has no logarithms of the radius. (This should be contrasted with the situation in two dimensions, (2.14).)

Consider

$$Z_{\mathbb{S}^3} = \int [d\Phi] e^{-\int_{\mathbb{S}^3} \mathcal{L}(\Phi)} \ . \tag{3.7}$$

This is generally divergent and takes the form (for a three-sphere of radius a)

$$\log Z_{\mathbb{S}^3} = c_1(\Lambda a)^3 + c_2(\Lambda a) + F \ . \tag{3.8}$$

Terms with inverse powers of Λ are dropped since they are not part of the continuum theory. They can be tuned away by adding the cosmological constant counter-term and the Einstein-Hilbert counter-term. However, no counter term can remove the finite part F.[1]

Imagine a three-dimensional flow from some CFT_{uv} to some CFT_{ir}. Then we can (in principle) compute F_{uv} and F_{ir} via the procedure above. The conjecture is

$$F_{uv} > F_{ir} \ . \tag{3.9}$$

Let us outline the computation of F is simple examples. Take a free massless scalar $\mathcal{L} = \frac{1}{2}(\partial\Phi)^2$. To put it in a curved background while preserving conformal invariance (more precisely, Weyl invariance) we write in d dimensions

$$S = \frac{1}{2}\int d^3x\sqrt{g}\left((\nabla\Phi)^2 + \frac{d-2}{4(d-1)}R[g]\Phi^2\right) \ . \tag{3.10}$$

This coupling to the Ricci scalar is necessary to preserve Weyl invariance. Weyl invariance means that the action is invariant under rescaling the metric by any function. We achieve this by accompanying the action on the metric with some action on the fields. For the action above, Weyl invariance means that the action is invariant under

$$g \to e^{2\sigma}g \ , \qquad \phi \to e^{-\frac{1}{2}\sigma}\phi \ . \tag{3.11}$$

We can now compute the partition function on the three-sphere by diagonalizing the corresponding differential operator $-\log Z_{S^3} = \frac{1}{2}\log\det\left(-\nabla^2 + \frac{1}{8}R\right)$. The Ricci scalar is related to the radius in three dimensions via $R = \frac{6}{a^2}$. The eigenfunctions are of course well known. The eigenvalues are

$$\lambda_n = \frac{1}{a^2}\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right) \ ,$$

and their respective multiplicities are

$$m_n = (n+1)^2 \ .$$

The free energy on the three-sphere due to a single conformally coupled scalar is therefore

$$-\log Z_{\mathbb{S}^3} = \frac{1}{2}\sum_{n=0}^{\infty} m_n\left(-2\log(\mu_0 a) + \log\left(n+\frac{3}{2}\right) + \log\left(n-\frac{1}{2}\right)\right) \ . \tag{3.12}$$

We have inserted an arbitrary scale μ_0 to soak up the dependence on the radius of the sphere. Since there are no anomalies in three dimensions, we expect that there would be no dependence on μ_0 eventually.

[1] More precisely, one can have the gravitational Chern-Simons term, but this cannot affect the real part of F. We disregard the imaginary part of F in our discussion.

This sum clearly diverges and needs to be regulated. We choose to regulate it using the zeta function. One finds that with this regulator $\sum_{n=0} m_n = \zeta(-2) = 0$ and therefore a logarithmic dependence on the radius is absent, as anticipated. We remain with

$$F_{scalar} = -\frac{1}{2}\frac{d}{ds}\left[2\zeta(s-2,\frac{1}{2}) + \frac{1}{2}\zeta(s,\frac{1}{2})\right] = \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx 0.0638$$

One can perform a similar computation for a free massless Dirac fermion field and one finds

$$F_{fermion} = \frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2} \approx 0.219$$

The absolute value of the partition function of a massless Majorana fermion is just a half of the result above. We see that the counting of degrees of freedom is quite nontrivial.

An interesting fact is that a nonzero contribution to F arises also from *topological* degrees of freedom. This has to be contrasted with the situation in two dimensions, where c was defined through a local correlation function and hence was oblivious to topological matter. For example, let us take Chern-Simons theory associated to some gauge group G,

$$S = \frac{k}{4\pi}\int_M Tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) , \tag{3.13}$$

where k is called the level. This theory has no propagating degrees of freedom. Indeed, the equation of motion is

$$0 = F = dA + A\wedge A ,$$

which means that the curvature of the gauge field vanishes everywhere. Such gauge fields are called flat connections. The space of flat connection on the manifold M is fixed completely by topological properties of the manifold.

The partition function of CS theory on the three sphere has been discussed in [14]. In particular, for $U(1)$ CS theory the answer is $\frac{1}{2}\log k$ while for $U(N)$ it is

$$F_{CS}(k,N) = \frac{N}{2}\log(k+N) - \sum_{j=1}^{N-1}(N-j)\log\left(2\sin\frac{\pi j}{k+N}\right) . \tag{3.14}$$

We see that the contribution from a topological sector can be in fact arbitrarily large as we take the level k to be large.

Let us now check the inequality (3.9) in a simple renormalization group flow. One can start from the conformal field theory described by $U(1)_k$ CS theory coupled to N_f Dirac fermions of charge 1. This is a conformal field theory because the Lagrangian has no coupling constant that can run. (The CS coefficient is discrete because it is topological in nature.) This conformal field theory is weakly coupled when $k >> 1$. Hence, the F coefficient is

$$F_{UV} \approx \frac{1}{2}\log k + N_f\left(\frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2}\right)$$

Let us now deform this by a mass term. The fermions disappear, but there is a pure CS term in the infrared with a shifted level $k \pm N_f/2$, where the sign depends on the sign of the mass term. Hence,

$$F_{IR} \approx \frac{1}{2} \log \left(k \pm N_f/2 \right) \ ,$$

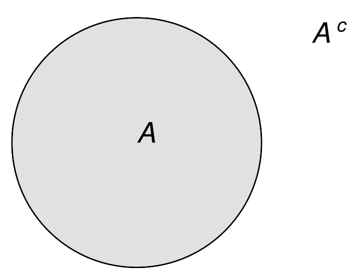
and one can convince oneself that in the regime where our analysis is valid,

$$F_{uv} > F_{ir} \tag{3.15}$$

holds true.

We would now like to discuss the physical interpretation of F, which is far from obvious. The partition function over S^3 does not have an obvious interpretation in terms of a Hilbert space so it is not clear what does it count.

Consider the entanglement entropy across a disk, so in the notation of the previous section, $A = D^2$, a two-dimensional disk.



In three dimensions a logarithm is not allowed in the von-Neumann entropy of a disk. (Roughly speaking, this is because there are no conformal anomalies.) In general, we expect the von-Neumann entropy of a disk of radius r to take the form

$$S_{von-Neumann} = \Lambda r + S \ , \tag{3.16}$$

where Λ is some UV cutoff. This linear divergence can be removed by adding a local counter-term on the boundary of the disk $\int_{S^1} d\gamma$. However, S is physical, it cannot be removed by adding any admissible local counterterm.

Exercise 9: (Challenging) suppose we add on the boundary of the disk the counter-term $\int_{S^1} d\gamma \ \kappa$ where κ is the extrinsic curvature. This would seem to make S ambiguous. Explain why the counter term $\int_{S^1} d\gamma \ \kappa$ is disallowed.

In the paper [15] it was shown that

$$F = S \ . \tag{3.17}$$

Therefore, one can interpret the S^3 partition function as the entanglement entropy across a disk. Additionally, one can also think of this entanglement entropy as the thermodynamic entropy in hyperbolic space [15]. We see again that monotonicity of the renormalization group flow (3.15) is again intimately related to the entanglement entropy.

There is not yet a conventional field theoretic proof of this inequality (3.9), but using the relation with the Entanglement Entropy, the inequality follows from some inequalities satisfied by the density matrix. Various issues with this construction are discussed in [17]. Of course, in all of these discussions one assumes that the entanglement entropy exists in the continuum theory.

3.4 Back to Free QED$_3$

Above we have computed the F associated to the theory of a free fermion and to a free conformally-coupled boson. Here we would like to go back to the subtle theory of a free gauge field in three dimensions and compute the F associated to it.

Let us start with some general comments. First, since the gauge field is dual to a scalar, one would be tempted to use (3.10). But remember that the gauge field is dual to a periodic scalar, so the second term in (3.10) is disallowed. This is of course closely related to the fact that the final term in the energy-momentum tensor (3.6) does not respect the gauged discrete shift symmetry. We can ask under what conditions Z_{S^3} is going to come out independent of a, the radius of S^3. Z_{S^3} would be independence of a if there exists a local, well-defined, V_μ such that

$$T^\mu_\mu = \nabla^\rho V_\rho \ . \tag{3.18}$$

This equation guarantees the existence of a conserved scale invariance current,

$$\Delta_\mu = x^\nu T_{\nu\mu} - V_\mu \ . \tag{3.19}$$

Since such a local scale current does not exist for the free gauge field, there might be an interesting dependence on a. This means that we might not be able to assign a finite F to the theory of a free gauge field.

The partition function of the free scalar field on S^3 is obtained by starting from the action

$$S = \frac{1}{2}\int d^3x\sqrt{g}\left((\nabla\phi)^2 + \alpha R\phi^2\right) \tag{3.20}$$

On the sphere of radius a, $R = d(d-1)/a^2$. α is a general real coefficient such that if $\alpha = 1/8$ it is conformally coupled. Only $\alpha = 0$ is consistent with the discrete shift symmetry.

The partition function as a function of α is $F = \frac{1}{2}\log\det\left(-\nabla^2 + \alpha R\right)$. To compute it one again recalls that the eigenvalues of the Laplacian on the sphere of radius a:

$$\lambda_n = \frac{1}{a^2}n(n+2) \ , \tag{3.21}$$

and they come with multiplicity $m_n = (n+1)^2$. Hence,

$$F = \frac{1}{2}\sum_n m_n \log\left(\lambda_n + \frac{6\alpha}{a^2}\right) \ . \tag{3.22}$$

Let us investigate the dependence on the a that appears inside the logarithm first. It takes the form $-\log(a)\sum_n m_n$. Employing zeta-function regularization $\sum_n n^2 = \zeta(-2) = 0$ and hence the coefficient of $\log(a)$ is zero.

We remain with

$$F = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)^2\log\left(n(n+2) + 6\alpha\right) \tag{3.23}$$

This is divergent as N^3 where N is some effective cutoff on the modes. This corresponds to a cosmological constant counter-term on S^3. There is also a subleading linear divergence corresponding to an Einstein-Hilbert counter-term. So the above sum needs a regulator.

We see a certain sickness for α corresponding to the periodic free scalar, $\alpha = 0$. This is easily interpreted as coming from the fact we have a non-compact scalar zero mode once we remove the curvature coupling. So this is a divergence from the infinite volume of target space. It is an infrared divergence.

We can compute the sum (3.23) with a zeta-function regulator. A helpful formula that we will use is

$$\sum_{n=0}^{\infty}(n+1)^2\log(n+c) = -\frac{d}{ds}\left[\zeta(s-2,c)+2(1-c)\zeta(s-1,c)+(c-1)^2\zeta(s,c)\right]\Bigg|_{s=0} . \tag{3.24}$$

Now let us consider $\alpha = 0$, where the existence of a divergence due to the non-compact target space has already been noted. We take $\alpha = \frac{1}{6}\epsilon$ to regulate things. We find

$$F[\alpha = \frac{1}{6}\epsilon] = \frac{1}{2}\log(\epsilon) + \frac{\log(\pi)}{2} + \frac{\zeta(3)}{4\pi^2} + \mathcal{O}(\epsilon) . \tag{3.25}$$

Exercise 10: Derive (3.24) and (3.25).

The coefficient $\frac{1}{2}$ in front of the first log would be generally replaced by the dimension of moduli space of the theory divided by two. Note that this partition function is now tending to minus infinity. When ϵ is small, the scalar is effectively allowed to probe distances or order $\phi \sim \epsilon^{-1/2}R^{-1/2}$. Therefore, if we have a scalar with period $sqrtf$ (where f has dimension 1), we expect that the leading logarithmic piece in the F function would be

$$F = \frac{1}{2}\log(f^{-1}R^{-1}) = -\frac{1}{2}\log(fR) \tag{3.26}$$

We see that when the scalar has zero radius $F \to \infty$ which is completely consistent with the qualitative picture of the flow in the figure on page 12.

Since the periodic scalar and the gauge field are equivalent, we see that the free gauge field needs to be assigned infinite positive F (we again quote only the leading logarithmic term):

$$F_{S^3}^{Maxwell} = -\frac{\log(e^2 a)}{2} . \tag{3.27}$$

This allows us an interesting reinterpretation of the result $\frac{1}{2}\log k$ of the $U(1)_k$ CS theory, quoted above (3.14). If we add a CS term of level k to a free Maxwell field, it picks up a mass $m \sim e^2 k$ and hence the logarithm needs to be "cut-off" at spheres of radius $a^{-1} \sim e^2 k$. Therefore we expect to find in the infrared $\frac{1}{2}\log k$ with a positive sign in front of the logarithm. This is precisely the result in CS theory, including the pre-factor.

We summarize that the "number of degrees of freedom" associated to a free gauge field is formally infinite. In fact, this is necessary for the F-theorem to hold: we can start from the free gauge field and flow to the topological CS term with any level k. Since the latter has $F = \frac{1}{2}\log(k)$, the only consistent F that we must assign to the free gauge field is infinite. This is somewhat unfortunate, because many interesting models whose dynamics we would like to understand (e.g. QED+flavors in 3d) start their life in the ultraviolet from free gauge fields and so the F-theorem does not easily lead to interesting bounds. However, one can still place interesting bounds by applying tricks such as in [21].

Note a conceptual difference from the c-theorem in two dimensions (2.19). In the two-dimensional case, topological degrees of freedom do not contribute. In three dimensions, one must count the topological degrees of freedom as well.

3.5 The S^3 Partition Function can be Computed in $\mathcal{N} = 2$ Theories

If one wants to gain some information about F beyond free (or weakly coupled) models, non-perturbative methods are needed.

It turns out that if one starts from $\mathcal{N} = 2$ (i.e. four supercharges) QFT in flat space ($\mathbb{R}^3$), then one can put the theory on S^3 while preserving four supercharges, which are not the same ones that are preserved in flat space. A technical requirement is that the flat space theory has at least one R-symmetry.

This statement is obvious for superconformal field theories in flat space, since they can be placed on S^3 using a stereographic transformation as above. The surprising fact is that this can be done even if the theory is non-conformal.

One can then use supersymmetric localization to compute F in many interesting examples. In particular, the inequality (3.9) is always satisfied. To learn about these developments see [18],[19],[20] and references therein.

4
Four-Dimensional Models

We saw that in two dimensions the natural monotonic property of the RG evolution was tightly related to the trace anomaly in two dimensions. In three dimensions the main role was played by the three-sphere partition function (there are no trace anomalies in three dimensions).

In four dimensions there are two trace anomalies and the monotonic property of renormalization group flows concerns again with these anomalies. The anomalous correlation function is now

$$\langle T_{\mu\nu}(q) T_{\rho\sigma}(p) T_{\gamma\delta}(-q-p)\rangle$$

And again, like in our analysis in two dimensions, there are contact-terms which are necessarily inconsistent with $T^\mu_\mu = 0$. In four dimensions it turns out that there are two independent trace anomalies. Introducing a background metric field we have

$$T^\mu_\mu = aE_4 - cW^2 \ , \tag{4.1}$$

where $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the Euler density and $W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ is the Weyl tensor squared. These are called the a- and c-anomalies, respectively.

It was conjectured in [22] (and shortly after studied extensively in perturbation theory in [23],[24]) that if the conformal field theory in the ultraviolet, CFT$_{UV}$, is deformed and flows to some CFT$_{IR}$ then

$$a_{UV} > a_{IR} \ . \tag{4.2}$$

The four-dimensional c-anomaly does not satisfy such an inequality (this can be seen by investigating simple examples) and also the free energy density divided by the appropriate power of the temperature does not satisfy such an inequality.

In two and three dimensions we have seen that the quantities satisfying inequalities like (2.19), (3.9) are computable from the partition functions on $\mathbb{S}^2$ and $\mathbb{S}^3$, respectively. Similarly, in four dimensions, since the four-sphere is conformally flat, the partition function on $\mathbb{S}^4$ selects only the a-anomaly. Indeed,

$$\partial_{\log r} \log Z_{\mathbb{S}^4} = -\int_{\mathbb{S}^4} \sqrt{g}\langle T^\mu_\mu\rangle = -a\int_{\mathbb{S}^4} \sqrt{g} E_4 = -64\pi^2 a \ .$$

In the formula above r stands for the radius of $\mathbb{S}^4$. Real scalars contribute to the anomalies $(a,c) = \frac{1}{90(8\pi)^2}(1,3)$, Weyl fermion: $(a,c) = \frac{1}{90(8\pi)^2}(11/2,9)$, and a $U(1)$ gauge field: $(a,c) = \frac{1}{90(8\pi)^2}(62,36)$.

In four dimensions, a free gauge field is a conformal field theory because (3.1) is traceless. So the problems with the free gauge field that we have discussed at great length in three dimensions do not exist in four dimensions. (Similar issues arise for the free two-form gauge theory, but this does not appear naturally in the ultraviolet of interesting models in four dimensions.)

We will now present an argument for (4.2), but before that, it is useful to repeat the two-dimensional story from a new point of view. The main idea is to promote various coupling constants to background fields [25],[26]. Our discussion below is more intuitive than rigorous, but it can be made entirely rigorous using the formalism of [27],[28]. Following the more rigorous and complete approach would force us to introduce a lot of notation and much more algebra, but the essential idea and results would not be changed.

4.1 Two-Dimensional Models Revisited

Imagine any renormalizable QFT (in any number of dimensions) and set all the mass parameters to zero. The extended symmetry includes the full conformal group. If the number of space-time dimensions is even then the conformal group has trace anomalies. If the number of space-time dimensions is of the form $4k + 2$, there may also be gravitational anomalies. We will keep ignoring gravitational anomalies here.

Upon introducing the mass terms, one violates conformal symmetry *explicitly*. Thus, in general, the conformal symmetry is violated both by trace anomalies and by an operatorial violation of the equation $T^\mu_\mu = 0$ in flat space-time. The latter violation can always be removed by letting the coupling constants transform. Indeed, replace every mass scale M (either in the Lagrangian or associated to some cutoff) by $Me^{-\tau(x)}$, where $\tau(x)$ is some background field (i.e. a function of space-time). Then the conformal symmetry of the Lagrangian is restored if we accompany the ordinary conformal transformation of the fields by a transformation of τ. To linear order, $\tau(x)$ always appears in the Lagrangian as $\sim \int d^dx\, \tau T^\mu_\mu$. Setting $\tau = 0$ one is back to the original theory, but we can also let τ be some general function of space-time. The variation of the path integral under such a conformal transformation that also acts on $\tau(x)$ is thus fixed by the anomaly of the conformal theory in the ultraviolet. This idea allows us to study some questions about general RG flows using the constraints of conformal symmetry. We will sometimes refer to τ as the dilaton.

Consider integrating out all the high energy modes and flow to the deep infrared. Since we do not integrate out the massless particles, the dependence on τ is regular and local. As we have explained, the dependence on τ is tightly constrained by the conformal symmetry. Since in even dimensions the conformal group has trace anomalies, these must be reproduced by the low energy theory. The conformal field theory at long distances, CFT$_{IR}$, contributes to the trace anomalies, but to match to the defining UV theory, the τ functional has to compensate precisely for the difference between the anomalies of the conformal field theory at short distances, CFT$_{UV}$, and the conformal field theory at long distances, CFT$_{IR}$.

Let us see how these ideas are borne out in two-dimensional renormalization group flows. Let us study the constraints imposed by conformal symmetry on action functionals of τ (which is a background field). An easy way to analyze these constraints is to introduce a fiducial metric $g_{\mu\nu}$ into the system. Weyl transformations act on the dilaton and metric according to $\tau \to \tau + \sigma$, $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$. If the Lagrangian for the dilaton and metric is Weyl invariant, upon setting the metric to be flat, one finds a conformal invariant theory for the dilaton. Hence, the task is to classify local diff$\times$ Weyl invariant Lagrangians for the dilaton and metric background fields.

It is convenient to define $\hat g_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$, which is Weyl invariant. At the level of two derivatives, there is only one diff$\times$Weyl invariant term: $\int \sqrt{\hat g}\hat R$. However, this is a topological term, and so it is insensitive to local changes of $\tau(x)$. Therefore, if one starts from a diff$\times$Weyl invariant theory, upon setting $g_{\mu\nu} = \eta_{\mu\nu}$, the term $\int d^2x(\partial\tau)^2$ is absent because there is no appropriate local term that could generate it.

The key is to recall that unitary two-dimensional theories have a trace anomaly

$$T^{\mu}_{\mu} = -\frac{c}{24\pi} R\,. \tag{4.3}$$

One must therefore allow the Lagrangian to break Weyl invariance, such that the Weyl variation of the action is consistent with (4.3). The action functional which reproduces the two-dimensional trace anomaly is

$$S_{WZ}[\tau, g_{\mu\nu}] = \frac{c}{24\pi} \int \sqrt{g} \left(\tau R + (\partial\tau)^2\right)\ . \tag{4.4}$$

We see that even though the anomaly itself disappears in flat space (4.3), there is a two-derivative term for τ that survives even after the metric is taken to be flat. This is of course the familiar Wess-Zumino term for the two-dimensional conformal group. (It also appears as the Liouville or linear dilaton action in the context of conformal field theory.)

Consider now some general two-dimensional RG flow from a CFT in the UV (with central charge c_{UV} and a CFT in the IR (with central change c_{IR}). Replace every mass scale according to $M \to Me^{-\tau(x)}$. We also couple the theory to some background metric. Performing a simultaneous Weyl transformation of the dynamical fields and the background field $\tau(x)$, the theory is non-invariant only because of the anomaly $\delta_\sigma S = \frac{c_{UV}}{24} \int d^2x\sqrt{g}\sigma R$. Since this is a property of the full quantum theory, it must be reproduced at all scales. An immediate consequence of this idea is that also in the deep infrared the effective action should reproduce the transformation $\delta_\sigma S = \frac{c_{UV}}{24} \int d^2x\sqrt{g}\sigma R$. At long distances, one obtains a contribution c_{IR} to the anomaly from CFT$_{IR}$, hence, the rest of the anomaly must come from an explicit Wess-Zumino functional (4.4) with coefficient $c_{UV} - c_{IR}$. In particular, setting the background metric to be flat, we conclude that the low energy theory must contain a term

$$\frac{c_{UV} - c_{IR}}{24\pi} \int d^2x(\partial\tau)^2\ . \tag{4.5}$$

Note that the coefficient of this term is universally proportional to the difference between the anomalies and it does not depend on the details of the flow. Higher-derivative terms for the dilaton can be generated from local diff×Weyl invariant terms, and there is no a priori reason for them to be universal (that is, they may depend on the details of the flow, and not just on the conformal field theories at short and long distances).

Zamolodchikov's theorem that we reviewed in the first chapter follows directly from (4.5). Indeed, from reflection positivity we must have that the coefficient of the term (4.5) is positive, and thus, the inequality is established.

We can be more explicit. The coupling of τ to matter must take the form $\tau T^{\mu}_{\mu} + \cdots$, where the corrections have more τs. To extract the two-point function of τ with two derivatives we must use the insertion τT^{μ}_{μ} twice. (Terms containing τ^2 can be lowered once, but they do not contribute to the two-derivative term in the effective action of τ.) As a consequence, we find that

$$\left\langle e^{\int \tau T^\mu_\mu d^2x} \right\rangle = \cdots + \frac{1}{2}\int\int \tau(x)\tau(y)\langle T^\mu_\mu(x)T^\mu_\mu(y)\rangle d^2x d^2y + \cdots$$
$$= \cdots + \frac{1}{4}\int \tau(x)\partial_\rho\partial_\sigma\tau(x)\left(\int (y-x)^\rho (y-x)^\sigma \langle T^\mu_\mu(x)T^\mu_\mu(y)\rangle d^2y\right) d^2x + \cdots . \quad (4.6)$$

In the final line of the equation above, we have concentrated entirely on the two-derivative term. It follows from translation invariance that the y integral is x-independent

$$\int (y-x)^\rho (y-x)^\sigma \langle T^\mu_\mu(x)T^\mu_\mu(y)\rangle d^2y = \frac{1}{2}\eta^{\rho\sigma}\int y^2\langle T^\mu_\mu(0)T^\mu_\mu(y)\rangle d^2y . \quad (4.7)$$

To summarize, one finds the following contribution to the dilaton effective action at two derivatives

$$\frac{1}{8}\int d^2x\tau\Box\tau \int d^2y y^2\langle T(y)T(0)\rangle . \quad (4.8)$$

According to (4.5), the expected coefficient of $\tau\Box\tau$ is $(c_{UV} - c_{IR})/24\pi$, and so by comparing we obtain

$$\Delta c = 3\pi \int d^2y y^2 \langle T(y)T(0)\rangle . \quad (4.9)$$

As we have already mentioned, $\Delta c > 0$ follows from reflection positivity (which is a property of unitary theories). Equation (4.9) precisely agrees with the classic results about two-dimensional flows (2.20).

4.2 Back to Four Dimensions

One starts by classifying local diff×Weyl invariant functionals of τ and a background metric $g_{\mu\nu}$. Again, we demand invariance under $g_{\mu\nu} \longrightarrow e^{2\sigma}g_{\mu\nu}$, $\tau \longrightarrow \tau + \sigma$. We will often denote $\hat{g} = e^{-2\tau}g_{\mu\nu}$. The combination $\hat{g}$ transforms as a metric under diffeomorphisms and is Weyl invariant.

The most general theory up to (and including) two derivatives is:

$$f^2 \int d^4x \sqrt{-\det \hat{g}}\left(\Lambda + \frac{1}{6}\hat{R}\right) , \quad (4.10)$$

where we have defined $\hat{R} = \hat{g}^{\mu\nu}R_{\mu\nu}[\hat{g}]$. Since we are ultimately interested in the flat-space theory, let us evaluate the kinetic term with $g_{\mu\nu} = \eta_{\mu\nu}$. Using integration by parts we get

$$S = f^2 \int d^4x e^{-2\tau}(\partial\tau)^2 . \quad (4.11)$$

One can use the field redefinition $\Psi = 1 - e^{-\tau}$ to rewrite this as

$$S = f^2 \int d^4x \Psi\Box\Psi . \quad (4.12)$$

One can also study terms in the effective action with more derivatives. With four derivatives, one has three independent (dimensionless) coefficients

$$\int d^4x\sqrt{-\hat{g}}\left(\kappa_1\hat{R}^2 + \kappa_2\hat{R}^2_{\mu\nu} + \kappa_3\hat{R}^2_{\mu\nu\rho\sigma}\right) . \quad (4.13)$$

It is implicit that indices are raised and lowered with $\hat{g}$. Recall the expressions for the Euler density $\sqrt{-g}E_4$ and the Weyl tensor squared $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$,$W^2_{\mu\nu\rho\sigma} =$

$R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ We can thus choose instead of the basis of local terms (4.13) a different parameterization

$$\int d^4x\sqrt{-\hat{g}}\left(\kappa_1'\hat{R}^2 + \kappa_2'\hat{E}_4 + \kappa_3'\hat{W}^2_{\mu\nu\rho\sigma}\right) \ . \tag{4.14}$$

We immediately see that the κ_2' term is a total derivative. If we set $g_{\mu\nu} = \eta_{\mu\nu}$, then $\hat{g}_{\mu\nu} = e^{-2\tau}\eta_{\mu\nu}$ is conformal to the flat metric and hence also the κ_3' term does not play any role as far as the dilaton interactions in flat space are concerned. Consequently, terms in the flat space limit arise solely from $\hat{R}^2$. A straightforward calculation yields

$$\int d^4x\sqrt{-\hat{g}}\hat{R}^2\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = 36\int d^4x\left(\Box\,\tau - (\partial\tau)^2\right)^2 \sim \int d^4x\frac{1}{(1-\Psi)^2}\left(\Box\Psi\right)^2 . \tag{4.15}$$

So far we have only discussed diff×Weyl invariant terms in four-dimensions, but from the two-dimensional re-derivation of the c-theorem we have shown above, we anticipate that the anomalous functional will play a key role.

The most general anomalous variation one needs to consider takes the form $\delta_\sigma S_{anomaly} = \int d^4x\sqrt{-g}\sigma\left(cW^2_{\mu\nu\rho\sigma} - aE_4\right)$. The question is then how to write a functional $S_{anomaly}$ that reproduces this anomaly. (Note that $S_{anomaly}$ is only defined modulo diff×Weyl invariant terms.) Without the field τ one must resort to non-local expressions, but in the presence of the dilaton one has a local action.

It is a little tedious to compute this local action, but the procedure is straightforward in principle. We first replace σ on the right-hand side of the anomalous variation with τ

$$S_{anomaly} = \int d^4x\sqrt{-g}\tau\left(cW^2_{\mu\nu\rho\sigma} - aE_4\right) + \cdots \ . \tag{4.16}$$

While the variation of this includes the sought-after terms, as the $\cdots$ indicate, this cannot be the whole answer because the object in parenthesis is not Weyl invariant. Hence, we need to keep fixing this expression with more factors of τ until the procedure terminates. Note that $\sqrt{-g}W^2_{\mu\nu\rho\sigma}$, being the square of the Weyl tensor, is Weyl invariant, and hence we do not need to add any fixes proportional to the c-anomaly This makes the c-anomaly "Abelian" in some sense.

The "non-Abelian" structure coming from the Weyl variation of E_4 is the key to our construction. The a-anomaly is therefore quite distinct algebraically from the c-anomaly.

The final expression for $S_{anomaly}$ is (see [29], where the anomaly functional was presented in a form identical to what we use in this note)

$$\begin{aligned} S_{anomaly} = &-a\int d^4x\sqrt{-g}\Big(\tau E_4 + 4\big(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\big)\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\tau + 2(\partial\tau)^4\Big) \\ &+ c\int d^4x\sqrt{-g}\tau W^2_{\mu\nu\rho\sigma} \ . \end{aligned} \tag{4.17}$$

Note that even when the metric is flat, self-interactions of the dilaton survive. This is analogous to what happens with the Wess-Zumino term in pion physics when the background gauge fields are set to zero and this is also what we saw in two dimensions.

Setting the background metric to be flat we thus find that the non-anomalous terms in the dilaton generating functional are

$$\int d^4x \left(\alpha_1 e^{-4\tau} + \alpha_2(\partial e^{-\tau})^2 + \alpha_3 \left(\Box\tau - (\partial\tau)^2\right)^2\right) \ , \qquad (4.18)$$

where α_i are some real coefficients.

The a-anomaly has a Wess-Zumino term, leading to the additional contribution

$$S_{WZ} = 2(a_{UV} - a_{IR}) \int d^4x \left(2(\partial\tau)^2\Box\tau - (\partial\tau)^4\right) \ . \qquad (4.19)$$

The coefficient is universal because the total anomaly has to match (as we have explained in detail in two dimensions).

We see that if one knew the four-derivative terms for the dilaton, one could extract $a_{UV} - a_{IR}$. A clean way of separating this anomaly term from the rest is achieved by rewriting it with the variable $\Psi = 1 - e^{-\tau}$. Then the terms in (4.18) become

$$\int d^4x \left(\alpha_1\Psi^4 + \alpha_2(\partial\Psi)^2 + \frac{\alpha_3}{(1-\Psi)^2}(\Box\Psi)^2\right) \ , \qquad (4.20)$$

while the WZ term (4.19) is

$$S_{WZ} = 2(a_{UV} - a_{IR}) \int d^4x \left(\frac{2(\partial\Psi)^2\Box\Psi}{(1-\Psi)^3} + \frac{(\partial\Psi)^4}{(1-\Psi)^4}\right) \ . \qquad (4.21)$$

We see that if we consider background fields Ψ which are null ($\Box\Psi = 0$), α_3 disappears and only the last term in (4.21) remains. Therefore, by computing the partition function of the QFT in the presence of four null insertions of Ψ one can extract directly $a_{UV} - a_{IR}$.

Indeed, consider all the diagrams with four insertions of a background Ψ with momenta k_i, such that $\sum_i k_i = 0$ and $k_i^2 = 0$. Expanding this amplitude, $\mathcal{A}$, to fourth order in the momenta k_i, one finds that the momentum dependence takes the form $s^2 + t^2 + u^2$ with $s = 2k_1 \cdot k_2$, $t = 2k_1 \cdot k_3$, $u = 2k_1 \cdot k_4$. Our effective action analysis shows that the coefficient of $s^2 + t^2 + u^2$ is directly proportional to $a_{UV} - a_{IR}$.

In fact, one can even specialize to the so-called forward kinematics, choosing $k_1 = -k_3$ and $k_2 = -k_4$. Then the amplitude is only a function of $s = 2k_1 \cdot k_2$. $a_{UV} - a_{IR}$ can be extracted from the s^2 term in the expansion of the amplitude around $s = 0$. Continuing s to the complex plane, there is a branch cut for positive s (corresponding to physical states in the s-channel) and negative s (corresponding to physical states in the u-channel). There is a crossing symmetry $s \leftrightarrow -s$ so these branch cuts are identical.

To calculate the imaginary part associated to the branch cut we utilize the optical theorem. The imaginary part is manifestly positive definite. Using Cauchy's theorem we can relate the low energy coefficient of s^2, $a_{UV} - a_{IR}$, to an integral over the branch cut. Fixing all the coefficients one finds

$$a_{UV} - a_{IR} = \frac{1}{4\pi} \int_{s>0} \frac{Im\mathcal{A}(s)}{s^3} \ . \qquad (4.22)$$

As explained, the imaginary part $Im\mathcal{A}(s)$ can be evaluated by means of the optical theorem, and it is manifestly positive. Since the integral converges by power counting (and thus no subtractions are needed), we conclude

$$a_{UV} > a_{IR} \,. \tag{4.23}$$

Note the difference between the ways positivity is established in two and four dimensions. In two dimensions, one invokes reflection positivity of a two-point function (reflection positivity is best understood in *Euclidean* space). In four dimensions, the Wess-Zumino term involves four dilatons, so the natural positivity constraint comes from the forward kinematics (and hence, it is inherently *Minkowskian*).

Let us say a few words about the physical relevance of $a_{UV} > a_{IR}$. Such an inequality constrains severely the dynamics of quantum field theory, and the allowed renormalization group flows. In favorable cases can be used to establish that some symmetries must be broken or that some symmetries must be unbroken. In a similar fashion, if a system naively admits several possible dynamical scenarios one can use $a_{UV} > a_{IR}$ as an additional handle.

The work of [15] shows that a can be also obtained from the entanglement entropy across an S^2, i.e. $A = D^3$ in our previous notation. But so far it has not been shown that (4.23) can be derived by manipulating the entanglement entropy and the inequalities it obeys.

4.3 Scale vs Conformal Invariance

As we review in the section about two-dimensional theories, the problem in two dimensions was essentially solved long ago. In three dimensions the problem is open. In four dimensions, there has been a lot of recent work on the subject. The problem is almost solved. Let us briefly explain what we mean by "almost." The tools used to arrive at the results below are very similar to what we have discussed in the previous two sections.

Suppose one is given a unitary scale invariant theory that is not conformal. That means that there exists some V_μ such that $T^\mu_\mu = \partial^\rho V_\rho$. The theory is non-conformal if there does not exist a scalar $\mathcal{O}$ such that $V_\mu = \partial_\mu \mathcal{O}$. In fact, for the results below to hold true, one does not quite need to assume the existence of V_μ – having a well-defined scale charge Δ is sufficient. (So the results also hold for the two-form gauge theory.)

In [30] it was shown that under these assumptions, for any state vector $|X\rangle$ in the scale invariant theory, we have

$$\langle VAC|T^\mu_\mu(p)T^\mu_\mu(q)|X\rangle = 0 \,, \qquad p^2 = q^2 = 0 \,. \tag{4.24}$$

(The momentum of the state $|X\rangle$ is $-p-q$.) If one could prove (4.25) for any p, q then it would follow that $T^\mu_\mu = 0$ and thus the theory is conformal. However, one cannot hope to be able to prove that $T^\mu_\mu = 0$ because in many four-dimensional models one can improve the energy-momentum tensor.[1] The best one can hope to prove is that $T^\mu_\mu = \Box\mathcal{O}$.

In [8] the argument of [30] was generalized to prove the following

$$\forall n \,., \forall |X\rangle . \,, \qquad \langle VAC|T^\mu_\mu(p_1)T^\mu_\mu(p_2)\ldots T^\mu_\mu(p_n)|X\rangle = 0 \,, \qquad p_i^2 = 0 \,. \tag{4.25}$$

It was then argued that this allows to conclude that $T = \Box\mathcal{O}$. The argument is not a proof, but it relies on some very simple and intuitive analogy with S-matrix theory. A more precise statement of what the argument shows is that, at least as far as the theory is flat space is concerned, all its local properties must be consistent with conformal invariance (if one assumes unitarity and scale invariance).

[1] For example, in scalar field theory, we can add the term $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$ to the energy-momentum tensor.

We would like to finish with a short remark on the situation in higher dimensions: the analysis was repeated in six dimensions in [31]. Despite many encouraging facts, a proof that the Euler anomaly is monotonic does not exist yet. In five dimensions, the S^5 partition function has a physical finite part, which one should hope is monotonic in RG flows. But currently there is no argument to that effect.

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