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HOLOGRAPHIC KIBBLE ZUREK

- Aim: To understand Kibble Zurek scaling without the adiabatic-diabatic assumption.

- Strategy:

    - Find a holographic model which has an isolated critical point

    - Turn on a relevant operator with a time dependent coupling which crosses this critical point

    - Calculate response.

- Has been explored in a variety of models — concentrate on one of them.

  — A model of $T=0$ holographic superconductor

Bulk $d+2$ dimensions

One compact direction $\theta \sim \theta +$

$$S = \int d^{d+1}x \sqrt{g} \left\{ \frac{1}{2k^2}\left(R + \frac{d(d+1)}{L^2}\right) - \frac{1}{4q^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{\lambda}\left((\partial_\mu \phi^* + iA_\mu \phi^*)(\partial_\nu \phi - iA_\nu \phi) - m^2 |\Phi|^2 - V(|\Phi|)\right) \right\}$$

Choose $R_{AdS} = 1$ units.

With these boundary conditions and with $\phi = 0$ $\exists$ 2 solutions

(1) AdS_{2+d} SOLITON.

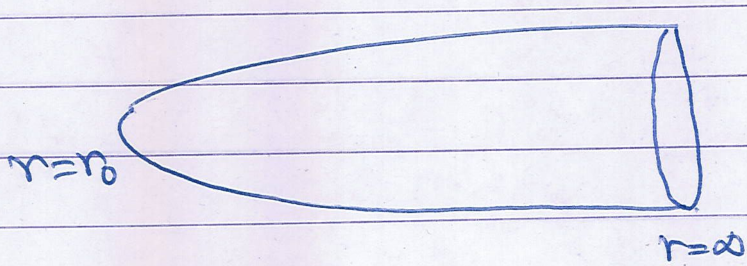
$$ds^2 = \frac{dr^2}{r^2 f_s(r)} + r^2(-dt^2 + d\vec{x}^2_{d-1}) + r^2 f_s(r)\, d\theta^2$$

$A_t = \mu$ (other $A_i = 0$) μ : constant

$$f_s(r) = 1 - \left(\frac{r_0}{r}\right)^{d+1}$$

$$\theta \sim \theta + \frac{4\pi}{(d+1) r_0}$$

- The geometry looks like a cigar ~~at~~ a single time slice



The tip of the cigar is flat space $\mathbb{R}^{d+1} \times$ (time)

Near r_0 : $r = r_0 + \frac{r_0(d+1)}{4} x^2$

$$ds^2 \to dx^2 + \frac{1}{4} r_0^2 (d+1)^2 x^2 \, d\theta^2 + r_0^2(-dt^2 + d\vec{x}$$

- Note $A_t = \mu =$ const. Thus no gauge field strength

- Can extend to a nonzero temperature by imposing a periodicity in imaginary time
  - temp not restricted.

Dual Theory:

$A_\mu \Leftrightarrow J_\mu$ a global $U(1)$ current

Thus

QFT on a circle of radius $\frac{4\pi}{(d+1)r_0}$

a constant chemical potential.

Note: $r \geq r_0$. Thus the modes which are normalizable form a discrete spectrum

$\rightarrow$

Dual theory has a gap.

Chemical potential, but no charge density

RN Black Brane

$$ds^2 = -r^2 f_b(r)\, dt^2 + \frac{dr^2}{r^2 f_b(r)} + r^2 (d\vec{x}^2 + d\theta^2)$$

$$A_t = \mu \left[1 - \left(\frac{r_+}{r} \right)^{d+1} \right]$$

Now θ periodicity unrestricted.

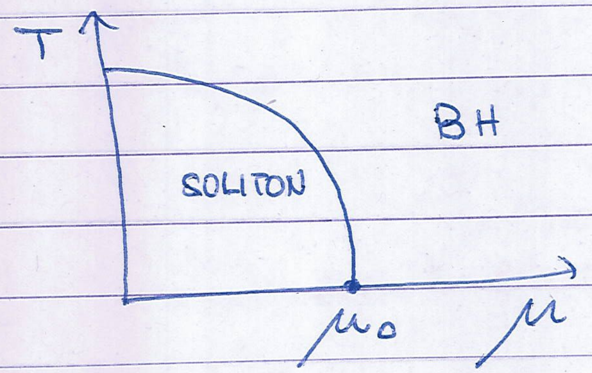
$$T = \frac{r_+}{4\pi} \left\{ (d+1) - \frac{(d-1)^2}{2d} \left(\frac{\mu}{r_+} \right)^2 \right\}$$

The black brane is extremal when

$$r_+ = \mu \frac{(d-1)}{\sqrt{2d(d+1)}}$$

- Need to minimize free energy to determine which of these are preferred

- Ryu, Takayanagi and Nishioka:



The line of (1st order) Hawking page type of transition

$$r_0 = r_+ \left[1 + \frac{d-1}{2d} \left(\frac{\mu}{r_+} \right) \right]^{\frac{1}{d+1}}$$

At $T=0$ this happens at

$$\mu_0 = \frac{r_0 (d+1) (2d)^{\frac{d-1}{2(d+1)}}}{(d-1)^{\frac{d}{d+1}} (d+1)^{1/2}}$$

- Question: What happens when we turn on a scalar field

FROM NOW ON $T=0$

- Probe approximation 1

Set $V(\Phi) = 0$ Then $\lambda = 1$

and $q \gg k$.

Then

Scalar + Maxwell

does not backreact on the background metric. — work in SOLITON phase

Ryu, Nishioka & Takayanagi:

$$-\frac{(d+1)^2}{4} < m^2 < -\frac{(d+3)(d-1)}{4}$$

Eqn. of motion for scalar

$$\left[-\frac{1}{r^2}(\partial_t - i\mu)^2 + \frac{1}{r^d}\partial_r\left(r^{d+2} f_s(r)\partial_r\right)\right]\Phi - m^2\Phi = 0$$

Asymptotic solution

$$\Phi(r,t) \to J(t)\, r^{-\Delta_-}\left(1 + O(1/r^2)\right) + A(t)\, r^{-\Delta_+}\left(1 + O(1/r^2)\right)$$

- In this mass window 2 possible quantizations

  Standard $J(t)$ source
  Alternative $A(t)$ source

  Look for solutions without source and regular at the tip.

- ~~May rescale~~ $\cancel{r \to r/r_0}$
  $\cancel{\mu \to \mu/r_0}$
  ~~To set~~ $\cancel{r_0 = 1}$

- RNT solved the coupled Scalar+Maxwell equations numerically and showed that $\exists\ \mu > \mu_0$ such that there is such a solution.

- When we have nonlinearity in the scalar field there is an additional coupling λ.

Choose $\lambda \gg q^2$ $\lambda \gg k^2$.

Then dynamics of scalar has ~~~~ parametrically small backreaction on both gravity and gauge background.

- Good enough to study the nonlinear scalar equation alone

The static eqn is

$$\frac{\mu^2}{r^2} + \frac{1}{r^d}\partial_r\left(r^{d+2} f_s(r)\partial_r\right)\Phi - m^2\Phi - V'(\Phi) = 0$$

Rescale

$$r_{new} = \frac{r}{r_0} \qquad t_{new} = t r_0 \qquad \mu_{new} = \frac{\mu}{r_0}$$

To set $r_0 = 1$

~~Fix~~

- Near $r=1$ $f_s(r) \simeq (d+1)(r-1)$
  Thus a regular soln. near $r=1$ behaves as

$$\Phi = \Phi_t + \Phi_t'(r-1) + O\left((r-1)^2\right)$$

Substituting in the eqn. get

$$\Phi_t' = \frac{\Phi_t(\Phi_t^2+m^2)}{d+1} - \frac{\mu^2 \Phi_t}{d+1}$$

- Choosen gauge s.t Φ_{static} = real

- Normally if you shoot with some Φ_t $\rightarrow$ end up with a soln which has a source piece at $r=\infty$

- However for $\mu > \mu_0$ a new solution appears which has zero source

- Since this solution is nontrivial
  $\Rightarrow \langle O \rangle \neq 0.$

- Energy density

$$\mathcal{E} = -\frac{1}{2}\int dr \sqrt{g}\, \phi^4 < 0$$

· Zero Mode:

Exactly at $\mu = \mu_0$ this soln. is in fact a ZERO MODE
— a soln. of the linearized equ. of motion

· Define

$$\rho(r) \equiv \int_r^\infty \frac{ds}{s^2 f_s^{1/2}(s)}$$

$$\rho(r) = \begin{cases} 1/r & \text{as } r \to \infty \\ \rho_* + \dfrac{2\sqrt{r-1}}{\sqrt{d+1}} & r \to 1 \end{cases}$$

ρ_* is a finite number
for AdS_5 $\rho_* = 1.311$

· Also redefine the field

$$\Phi(r,t) = \frac{1}{[r(\rho)]^{\frac{d-2}{2}}} \left(\frac{d\rho}{dr}\right)^{1/2} 2\Phi(\rho,t)$$

The new field has asymptotics

$$2\Phi[\rho,t] \sim J(t)\rho^{\alpha} + A(t)\rho^{1-\alpha}$$

$$\alpha = \Delta_- - d/2$$

- The full wave equation becomes

$$\left(-\partial_t^2 + 2i\mu\partial_t\right)\Psi = \mathcal{P}\Psi - \mu^2\Psi + \frac{r^{2-d}}{\sqrt{f_s(r)}}|\Psi|^2\Psi$$

$\mathcal{P}$ is a Schrödinger operator

$$\mathcal{P} = -\partial_\rho^2 + V(\rho)$$

$$V(\rho) = m^2 r^2 + \frac{4d(d+2)r^{2d+2} - 4d(d+3)r^{d+1} - (d-1)^2}{16\, r^{d-1}(r^{d+1}-1)}$$

- As a function of ρ

$$V(\rho) = \begin{cases} \dfrac{m^2 + \frac{d(d+2)}{4}}{\rho^2} + O(\rho^2) & \rho \to 0 \\[2ex] -\dfrac{1}{4(\rho_* - \rho)^2} + O(1) & \rho \to \rho_* \end{cases}$$

- The singularity at $\rho = \rho_*$ is in fact a coordinate artifact
Compare with Laplacian with $\ell = 0$ in 2d $\quad ds^2 = dy^2 + y^2 d\theta^2$

$$\nabla^2\psi = \frac{1}{y}\partial_y(y\,\partial_y)\psi$$

With $\chi = \sqrt{y}\,\psi \qquad \nabla^2\psi = \frac{1}{\sqrt{y}}\left(\partial_y^2 + \frac{1}{4y^2}\right)\chi$

For our problem define

$$y = \rho_* - \rho$$

$$\tilde{\psi} \equiv \frac{\psi}{\sqrt{y}}$$

The operator in equation 18 can be now written in terms of the 2d laplacian

$$\partial_y^2 \psi = \sqrt{y}\left(\nabla^2 \tilde{\psi} - \frac{1}{4y^2}\tilde{\psi}\right)$$

Thus eigenvalue problem for θ is

$$\left[-\nabla^2 + \frac{1}{4y^2} + V_0\right]\tilde{\psi} = E\tilde{\psi}$$

The effective potential is

$$V_1(y) = V_0 + \frac{1}{4y^2}$$

- For $d=3$

$$V_1(y) = \left(m^2+\frac{15}{4}\right) r^2 + \underbrace{\frac{1}{4}\left[\frac{1+3r^4}{r^2-r^6} + \frac{1}{y^2}\right]}_{V_2(y)}$$

May show that

$$V_2(y) \geq 0.$$

$\Rightarrow$ for $m^2 > -\frac{15}{4} = m^2_{BF}$

All eigenvalues of $\mathcal{P}$ are positive.

- The linearized equ. of motion is

$$(\mathcal{P} - \mu^2)\psi = 0$$

$\Rightarrow$ There is a $\mu = \mu_0$ when a zero mode appears.

For $\mu > \mu_0$ negative eigenvalues appear — but $\psi \sim e^{i\omega t}$ ω still real

But new soln has lower energy

Exponents:

$\mu = \mu_0$ is a critical point.
Exponents can be computed analytically.

For $J = 0$ $\quad \mu = \mu_c + \varepsilon$
assume

$$\Psi = \varepsilon^{\beta} (\Psi_0 + \epsilon \psi_1 + \cdots)$$

Eqn is $\qquad \mathcal{D} \psi + G(\rho) \psi^n = 0$

Eigenvalue of $\mathcal{D} \sim O(\varepsilon)$.
Thus

$$\varepsilon . \varepsilon^{\beta} \psi_0 + G \varepsilon^{n\beta} \psi_0 = 0$$

$\psi_0 \sim O(1)$ iff $\quad \beta = \frac{1}{n-1}$

$\Rightarrow$

$$\psi \sim (\mu - \mu_c)^{\frac{1}{n-1}}$$

· Similarly at $\mu = \mu_c$ write the field

$$\psi(\rho) = J\,\chi(\rho) + J^{\gamma}\,\psi_s(\rho).$$

The effect of this is that J now acts as a source for the normalizable part.

$$J(\mathcal{D}\chi) + J^{\gamma}\mathcal{D}\psi_s + G(\rho)\left[J\chi + J^{\gamma}\psi_s\right]^n = 0$$

- When $\mathcal{D}\psi_s \neq 0$ possible to get a $O(1)$ soln for ψ_s with $\gamma = 1$.

Suppose $\gamma > 1$ $\quad J(\mathcal{D}\chi) + G J^n \chi^n$ Not possibl

- When $\mathcal{D}\psi_s = 0$, however must have
$$\gamma = 1/n$$
$$\mathcal{D}\chi + G(\rho)\,\psi_s = 0$$

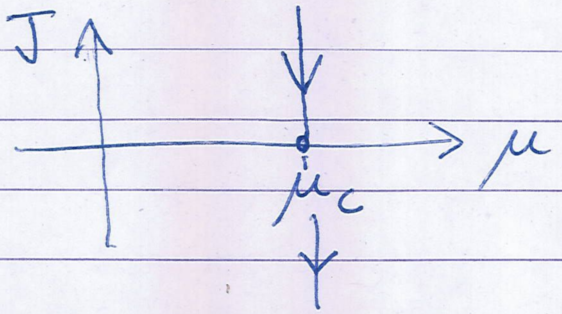
- At $\mu = \mu_c$ there is a zero mode $\Rightarrow$ there is a soln $\mathcal{D}\psi_s = 0$
$$\Rightarrow \quad \psi_s \sim |J|^{1/n} \Rightarrow \langle \mathcal{O} \rangle \sim |J|^{1/n}$$

Performing Quantum Quench

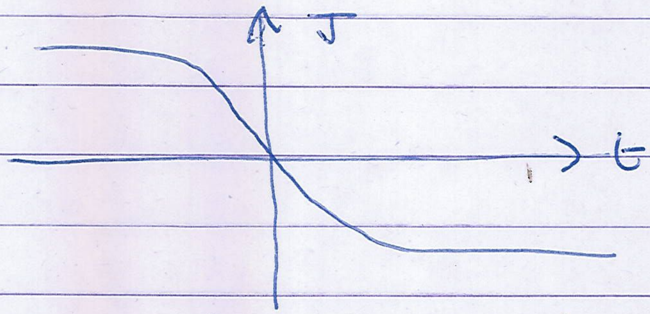
- We will study the system with

$$\mu = \mu_c$$

and $J(t)$



$J(t)$ is chosen of the form



e.g $\quad J(t) = J_0 \tanh(vt)$

- Presence of a nonzero J at $\mu = \mu_c$
$\Rightarrow$ there is a gap. at $t \to -\infty$
estimate

$$E_{gap} \sim J_0^{2/3}$$

Instantaneous gap $\quad [J(t)]^{2/3}$

- Thus as $t \to -\infty$ the soln ~~of~~ to the eqn. of motion

$$[-\partial_t^2 + 2i\mu\partial_t]\Psi = \mathcal{D}\Psi + G(\rho)\Psi|\Psi|^{n-1}$$

is well approximated by the Adiabatic soln. ie

$$\Psi = \Psi_0[\rho, J(t)] .$$

where Ψ_0 satisfies

$$\mathcal{D}\Psi_0 + G(\rho)\Psi_0|\Psi_0|^2 = 0$$

with boundary condition given by J.

- Adiabatic expansion:

$$\Psi[\rho, t] = \Psi_0(\rho, J(t)) + \epsilon\Psi_1(\rho, t) + O(\epsilon^2)$$

ϵ : Rescale $\partial_t \to \epsilon\partial_t$

- $$(\mathcal{D} + 3G\Psi_0^2)\,\mathrm{Re}\,\Psi_1 = 0$$

$$(\mathcal{D} + 6\Psi_0^2)\,\mathrm{Im}\,\Psi_1 = 2\mu\partial_t\Psi_0$$

- Clearly the soln with specified boundary conditions

$$\mathrm{Re}\,\Psi_1 = 0$$

$$\mathrm{Im}\,\Psi_1 = 2\mu\,\dot{J}(t)\int_0^{\rho_*} d\rho'\, \mathcal{G}(\rho,\rho')\,\frac{\partial \Psi_0(\rho')}{\partial J}$$

$\mathcal{G}$ is the Green's fn. of operator

$$\tilde{\mathcal{D}} = \left(\mathcal{D} + G(\rho)\,\Psi_0^2\right)$$

Ψ_1 : soln of $\tilde{\mathcal{D}}\Psi = 0$ regular at $\rho=\rho_*$

Ψ_2 : " " " regular at $\rho=\infty$

$$\mathcal{G}(\rho,\rho') = \begin{cases} \dfrac{1}{W(\Psi_1,\Psi_2)}\,\Psi_1(\rho')\Psi_2(\rho) & \rho<\rho' \\[2ex] \dfrac{1}{W(\Psi_1,\Psi_2)}\,\Psi_2(\rho')\,\Psi_1(\rho) & \rho>\rho' \end{cases}$$

- The Wronskian is a constant $\Rightarrow$ Evaluate at $\rho=\rho_*$

$$\Psi_1 = C\sqrt{\rho_*-\rho} \qquad \Psi_2 = A\sqrt{\rho_*-\rho} + B\sqrt{\rho_*-\rho}\,\log(\rho_*-\rho)$$

$$W = -BC$$

- Exactly at $\mu = \mu_c$ $\mathcal{D}$ has a zero mode, ie. a soln which satisfies the correct boundary conditions at both ends

$\Rightarrow$ for $\mu = \mu_c$ $B = 0$

$\Rightarrow$ ξ diverges.

in absence of nonlinear term.

- For the operator $\hat{\mathcal{D}}$ the lowest eigenvalue can be estimated perturbatively.

We know the soln is of form

$$\Psi_0 \sim Jx + J^{1/3}\Psi_s$$

$$\Rightarrow \quad \Psi_0^2 \sim J^{2/3}$$

$\Rightarrow$ Smallest eigenvalue $\sim J^{2/3}$

$$\Rightarrow \quad \xi \sim J^{-2/3}.$$

$\Rightarrow$

$$\mathrm{Im}\, \Psi^{(1)} = \frac{\mu \dot{J}}{J^{2/3}} \frac{\partial \Psi^{(0)}}{\partial J} = \frac{\mu \dot{J}}{J^{4/3}}$$

This is same order as $2\Psi^{(0)} \sim J^{1/3}$ when

$$\mu \dot{J} \sim J^{5/3}.$$

- Consider this for

$$J(t) = J_0 \tanh(vt)$$

Clearly adiabaticity condition is valid as $t \to -\infty$.

To probe breakdown of adiabaticity assume self consistently $vt_* \ll 1$

$$t_* \sim \mu^{3/5} (J_0 v)^{-2/5}$$

Consistency requires

$$(\mu v) \ll J_0^{2/3}$$

Critical dynamics

Adiabatic expansion $\equiv$ Taylor series expansion in v
For $t > t_*$ this breaks down.

Will show : there is now a new small v expansion in fractional powers.

First separate out the source part

$$\Phi = \rho^{\alpha} J + \Phi_s .$$

In critical region $J = J_0 v t$
Rescale

$$t = (J_0 v)^{-2/5}\, \eta \qquad \Phi_s = (J_0 v)^{1/5}\, \chi$$

Then various terms in the equation of motion organize themselves in powers of $(J_0 v)^{2/5}$

To lowest order

$$\mathcal{D}\chi = (J_0 v)^{2/5} \left\{ 2i\mu\, \partial_\eta \chi - G(\rho)\, |\chi|^2 \chi - \eta\, (\mathcal{D}\rho^\alpha) \right\}$$

$$+ O\left[(J_0 v)^{4/5} \right]$$

- Consider now an expansion

$$\chi(\rho, t) = \sum_n \chi_n(\eta)\, \varphi_n(\rho).$$

where

$$\mathcal{D}\varphi_n(\rho) = \lambda_n \varphi_n(\rho)$$

$\Rightarrow$

$$\lambda_n \chi_n = (J_0 v)^{2/5} \left[2i\mu\, (\partial_\eta \chi_n) - \sum_{n_1 n_2 n_3} \ell^n_{n_1 n_2 n_3} \chi^*_{n_3} \chi_{n_2} \chi_{n_1} + \mathcal{T}_n\, \eta \right] + O\left((J_0 v)^{4/5} \right)$$

$$\mathcal{T}_n = \int d\rho\, \varphi_n^*(\rho)\, (\mathcal{D}\rho^\alpha)$$

$$\ell^n_{n_1 n_2 n_3} = \int d\rho\, G(\rho)\, \varphi_n^*(\rho)\, \varphi^*_{n_3}(\rho)\, \varphi_{n_2}(\rho)\, \varphi_{n_1}(\rho).$$

Clearly the ZERO MODE $x_0(\eta)$ has $O(1)$ solution. — higher modes start with $O[(J_0 v)^{2/5}]$

Thus a soln is of the form

$$x_n(\eta) = \delta_{n0}\, \xi_0(\eta) + J_0 v^{2/5}\, \xi_n(\eta) + \cdots$$

where

$$2 i \mu \partial_\eta \xi_0 = C^0_{000} |\xi_0|^2 \xi_0 + J_0 \eta$$

Going back to original variables

$$\Psi_s(\rho, t, v) = (J_0 v)^{1/5}\, \Psi_s(\rho, t (J_0 v)^{2/5}, 1)$$

$\Rightarrow$

$$\langle \Theta(t, v) \rangle = (J_0 v)^{1/5}\, F\left(t (J_0 v)^{2/5}\right)$$