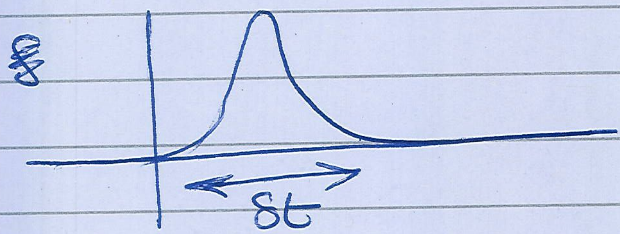
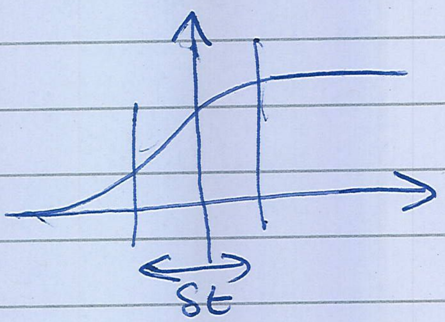
Buchel, Lehner, Myers.
S.R.D, D. Galante, R. Myers.

FAST QUENCHES

· Buchel, Lehner, Myers
— Performed quantum quench by turning on time dependent boundary conditions of a massive scalar field with $\Delta < d$

Numerically computed the response



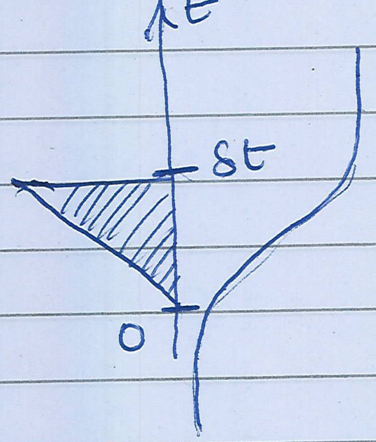


$$\langle \mathcal{O} \rangle \sim (\delta t)^{d-2\Delta}$$

$$\langle \mathcal{E} \rangle \sim (\delta t)^{d-2\Delta}$$

Hints at a universal scaling law

Holographic Explanation



Energy is produced only $0 < t < \delta t$

$\Rightarrow$ We are only interested in the response of the scalar field in this regime.

By causality $\Rightarrow$ need to solve the wave eqns in the causal triangle

As $\delta t \to 0$ this triangle shrinks

$\Rightarrow$ By suitable rescaling may show

- Nonlinearities go away
- Only need to solve the wave eqn in the Δ

— basically in pure AdS

Once this point is realized
— scaling follows.

<u>Puzzle</u> : What happens when
$\delta t \to 0$
and $\Delta > d/2$

Does not seem to reproduce
answers of an abrupt
quench !

QUENCHES IN FREE FIELDS

- $$S = -\int d^dx \left[(\partial\phi)^2 + m^2(t)\phi^2 \right]$$

$$S = \int d^dx \left[\bar{\psi} \partial\!\!\!/ \psi + m(t) \bar{\psi}\psi \right]$$

Find mass profiles for which the theory can be solved for arbitrary quench rates

Consider various limits carefully.

- For scalar field

$$m^2(t) = m^2(A + B \tanh(t/\delta t))$$

$$t \to -\infty \qquad m^2(A-B)$$
$$t \to +\infty \qquad m^2(A+B)$$

- Klein Gordon equation can be solved "in" solutions.

$$u^{in}_{\vec{k}} = \frac{1}{\sqrt{2\omega_{in}}} \exp\left(i\vec{k}\cdot\vec{x} - i\omega_+ t - i\omega_- \delta t \log(2\cosh t\right.$$
$$ {}_2F_1\left(1 + i\omega_- \delta t, i\omega_- \delta t; 1 - i\omega_{in}\delta t; \frac{1}{2}(1 + \tanh t/(\delta t)\right)$$

where

$$\omega_{in}^2 = \vec{k}^2 + m^2(A-B)$$
$$\omega_{out}^2 = \vec{k}^2 + m^2(A+B)$$
$$\omega_\pm = \tfrac{1}{2}(\omega_{out} \pm \omega_{in})$$

Similarly there are "out" solutions

$$v_{\vec{k}} = \frac{1}{\sqrt{2\omega_{out}}} \exp\left(i\vec{k}\cdot\vec{x} - i\omega_+ t - i\omega_- \delta t \log\left(2\cosh\frac{t}{\delta t}\right)\right)$$
$$ {}_2F_1\Big(1 + i\omega_-\delta t, i\omega_-\delta t; 1 + i\omega_{out}\delta t$$
$$; \tfrac{1}{2}(1 - \tanh t/(\delta t)\Big)$$

As $t \to -\infty$

$$\log\left(2\cosh\frac{t}{\delta t}\right) \to \log\left(e^{-t/\delta t}\right) = -\frac{t}{\delta t}$$

Thus exponent in $u_{\vec{k}}^{in}$ becomes

$$i\vec{k}\cdot\vec{x} - i\omega_+ t + i\omega_- t = i\vec{k}\cdot\vec{x} - i\omega_{in} t$$

~~Thus~~

$$ {}_2F_1(\ldots) \to 1$$

$$u_{\vec{k}} \to \frac{1}{\sqrt{2\omega_{in}}} \exp\left(i\vec{k}\vec{x} - i\omega_{in} t\right)$$

- There is a Bogoliubov transformation which relates these

$$\alpha_{\vec{k}} = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma(1-i\omega_{in}\delta t)\,\Gamma(-i\omega_{out}\delta t)}{\Gamma(-i\omega_{+}\delta t)\,\Gamma(1-i\omega_{+}\delta t)}$$

$$\beta_{\vec{k}} = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma(1-i\omega_{in}\delta t)\,\Gamma(i\omega_{out}\delta t)}{\Gamma(i\omega_{-}\delta t)\,\Gamma(1+i\omega_{-}\delta t)}$$

$$u_{\vec{k}} = \alpha_{\vec{k}} v_{\vec{k}} + \beta_{\vec{k}} v^{*}_{-\vec{k}}$$

- Consider the initial state vacuum. This is of course the "in" vacuum

$$\phi(x,t) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \left[a_k u_k + h.c. \right]$$

$$[a_{\vec{k}}, a^{+}_{\vec{k}'}] = \delta^{d-1}(\vec{k}-\vec{k}')$$

- Suppose we calculate

$$\langle 0 | \phi^2 | 0 \rangle$$

$$\langle \phi^2 \rangle = \frac{1}{(2\pi)^{d-1}} \int \frac{d^{d-1}k}{2\omega_{in}} |{}_2F_1|^2$$

- This is of course UV divergent. A priori nothing to do with time dependence. Even at $t=-\infty$

$$\langle \phi^2 \rangle = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{2\sqrt{k^2+m^2}}$$

— the standard coincident point divergence.

- One might have thought that the subtracted (normal ordered) answer should be finite.

- Consider, however, possible counterterms one may add to this

$$\langle \phi^2 \rangle_{ren} = \frac{\Omega_{d-2}}{2(2\pi)^{d-1}} \int dk \left(\frac{k^{d-2}}{\omega_{in}} |F|^2 - f_{ct} \right)$$

where $\Omega_{d-2} = \dfrac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$

To see what counterterms can appear consider the fixed mass answer

$$\int dk \cdot \frac{k^{d-2}}{\sqrt{k^2+m^2}}$$

$$= \int dk\, k^{d-3}\left(1 - \frac{m^2}{2k^2} + \frac{3m^4}{8k^4} - \frac{5m^6}{16k^6} + \dots\right)$$

Might have thought replace $m^2 \to m^2(t)$ and we are done.

However, by dimension counting we can have more terms

e.g in $d=6$ m^4 term is log div.
But there can be also terms like

$$\partial_t^2 m^2$$

at the same order of divergences

- Indeed if one performs a large k expansion of $|{}_2F_1|^2$ — such divergences do show up.

- This is very much like QFT in curved space-time
  — Renormalization requires counterterms which involve derivatives of metric
  — Curvature invariants.

- In this case it turns out that the counter-terms can be obtained — with precise coefficients — from an adiabatic expansion.

Solve the wave eqn in WKB.

$$\frac{d^2 u_k}{dt^2} + \left[k^2 + m^2(t)\right] u_k = 0$$

$$u_k = \frac{1}{\sqrt{2\Omega_k(t)}} \exp\left[i\vec{k}\cdot\vec{x} - i\int^t \Omega_k(t')\,dt'\right]$$

$$\Rightarrow \quad \Omega_k^2 = \omega_k^2 - \frac{1}{2}\frac{\partial_t^2 \Omega_k}{\Omega_k} + \frac{3}{4}\left(\frac{\partial_t \Omega_k}{\Omega_k}\right)^2$$

$$\omega_k^2 = k^2 + m^2(t).$$

Solve this in a derivative expansion

- The adiabatic answer is

$$\langle \phi^2 \rangle_{\text{adiabatic}} = \frac{\Omega_{d-2}}{2(2\pi)^{d-1}} \int dk \, \frac{k^{d-2}}{\Omega_k(t)}$$

Now expand for large k

$$\frac{1}{\Omega_k} \simeq \frac{1}{k}\Big[1 - \frac{m^2(t)}{2k^2} + \frac{3m^4(t)}{8k^4} - \frac{5m^6(t)}{16k^6} + \frac{\partial_t^2 m^2}{8k^4} - \frac{5}{32k^6}\Big\{(\partial_t m^2)^2 + 2m^2 \partial_t^2 m^2\Big\}\Big] + \cdots$$

This gives counterterms

$$f_{ct} = k^{d-3} - \frac{1}{2} m^2 k^{d-5} + \frac{1}{8} k^{d-7}(3m^4 + \partial_t^2 m^2) + \cdots$$

- Pause: Why does this work? Adiabatic expansions are supposed to work for SLOW variations.

  However to read off UV we need contributions for large k near the cutoff.

By definition, the mass scale m

$$m \ll \Lambda \quad , \quad \delta t \gg \Lambda^{-1}$$

So what we need is just this – the physics at the cutoff scale does not care about the value of $m\delta t$

- Explicit expansion of $|F|^2$ verifies this.

- These counterterms can be thought of as counterterms which should be added to the action

  - Put the theory on curved space-time
  - Regard $m(t)$ as a scalar field

Since we are considering free field the counterterms should be all field independent

$$S_{ct} = -\int d^dx \sqrt{g}\, \Big[s_{00} \Lambda^d + s_{10}\, m^2 \Lambda^{d-2} + s_{20}\, m^4 \Lambda^{d-4}$$
$$+ \left(s_{30} m^6 + s_{31}\, m^2 \nabla^2 m^2 \right) \Lambda^{d-6}$$
$$+ \left(s_{40} m^8 + s_{41}\, m^4 \nabla^2 m^2 + s_{42}\, m^2 \nabla^4 m^2 \right) \Lambda^{d-8}$$
$$+ \cdots \Big]$$

We may then read out the renorm. quantities by

$$Z = \int [\mathcal{D}\phi]_\Lambda \exp\left[-i S_0 - i S_{ct} \right]$$

$$\langle \phi^2 \rangle_{ren} = -2i \left[\frac{1}{\sqrt{-g}} \frac{\delta}{\delta m^2} \log Z \right]_{\substack{g=\eta \\ \Lambda\to\infty}}$$

$$\langle T_{\mu\nu} \rangle_{ren} = -2i \left[\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \log Z \right]_{\substack{g=\eta \\ \Lambda\to\infty}}$$

- Note that the same counterterms should renormalize both, and the Ward identity

$$\frac{dE}{dt} = \frac{1}{2} \frac{\partial m^2}{\partial t} \langle \phi^2 \rangle$$

SHOW NUMERICAL RESULTS

- Fast quench regime

$$m\,\delta t \ll 1$$

But $\qquad \Lambda\,\delta t \gg 1.$

In this regime, to lowest order in the dimensionless coupling

$$k = m\,\delta t$$

we can get exact analytic expressions

$$\boxed{-A = B = -1/2}$$

- Odd $d \geq 5$

$$\langle \phi^2 \rangle_{ren} = (-1)^{\frac{d-1}{2}} \frac{\pi}{2^{d-2}} \partial_t^{d-4} m^2 + O(\delta t^{6-d})$$

- Even $d \geq 4$

$$\langle \phi^2 \rangle_{ren} = (-1)^{d/2} \log(\mu\,\delta t) \frac{\partial_t^{d-4} m^2}{2^{d-3}}$$

$$\Rightarrow \boxed{\begin{aligned} \langle \phi^2 \rangle &\sim m^2 (\delta t)^{4-d}. \\ \langle \phi^2 \rangle &\sim m^2 (\log \delta t)\,(\delta t)^{4-d}. \end{aligned}}$$

- For $d=3$

$$\langle \phi^2 \rangle_{ren} = -\frac{m}{4\pi} - \frac{m^2 \delta t}{16} \log \frac{1}{2}\left(1 - \tanh \frac{t}{\delta t}\right) + \cdots$$

- Similar scaling for
  - × CFT → m quench
  - × CFT → CFT quench.

- Similar results for
  - × $T_{\mu\nu}$
  - × higher spin currents.

SHOW FIGURES

In fact this scaling holds for arbitrary initial and final mass provided

$$m_i \delta t, \; m_f \delta t \ll 1.$$

Consider the general expression.

$$\langle \phi^2 \rangle = \int \frac{k^{d-2} dk}{\omega_{in}} \left\{ |{}_2F_1|^2 - f_{ct} \right\}$$

The hypergeometric which appears is

$${}_2F_1 \Big[1 + i\omega_- \delta t \,;\; i\omega_- \delta t \,;\; 1 - i\omega_{in} \delta t \,;\; \tfrac{1}{2}\left(1 + \tanh t/\delta t\right) \Big]$$

$$\omega_{in}^2 = k^2 + m_i^2$$
$$\omega_{out}^2 = k^2 + m_f^2 .$$

Now redefine $\quad \tilde{k}^2 \equiv k^2 + m_f^2$

$$\omega_{in}^2 = \tilde{k}^2 + \delta m^2$$
$$\omega_{out}^2 = \tilde{k}^2 .$$

$$\langle \phi^2 \rangle = \int_{m_f}^{\infty} \frac{\tilde{k}\, d\tilde{k}}{\omega_{in}} \left(\tilde{k}^2 - m_f^2\right)^{\frac{d-3}{2}} \left[|F|^2 - f_{ct} \right]$$

This looks like quench from $\delta m^2 \to 0$

Rewrite using $\tilde{q} = \tilde{k}\,\delta t$

$$\langle \phi^2 \rangle = (\delta t)^{4-d} \int_{m_f \delta t}^{\infty} \frac{\tilde{q}^{d-2}\, d\tilde{q}}{\omega_{in}} \left(1 - \frac{m_f^2\,\delta t^2}{\tilde{q}^2}\right)^{\frac{d-3}{2}} \left[|F|^2 - f_{ct} \right]$$

When $m_f\,\delta t \ll 1$

and $\delta m\,\delta t \ll 1 \Rightarrow m_i\,\delta t \ll 1$

This becomes

$$(\delta t)^{4-d} \int_0^{\infty} \frac{\tilde{q}^{d-2}\, d\tilde{q}}{\omega_{in}} \left[|F|^2 - f_{ct} \right]$$

— exactly as before.

- Comparison with linear response.

  Note that the leading answer is proportional to m^2.

  → It must be that the result is exactly the same as linear response

  → Detailed comparison shows that this is indeed the case.

A GENERAL RESULT IN QFT

Consider

$$S = S_{CFT} + \int d^dx \, \lambda(t) \mathcal{O}_\Delta(x, t).$$

where

$$\lambda(t) = (\delta\lambda)\, h\left(\frac{t}{\delta t}\right).$$

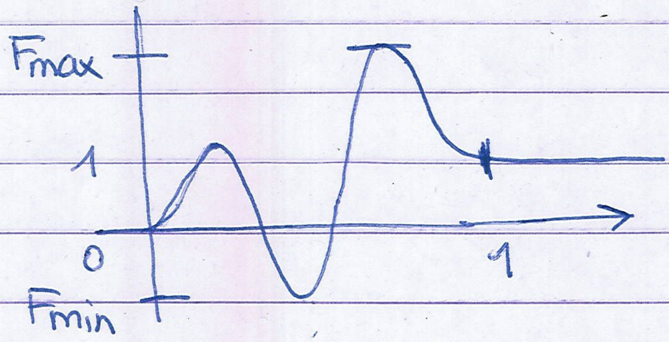
Take the fn. as follows

$$\lambda(t) = \begin{cases} \lambda_0 & t < 0 \\ \lambda_0 + \delta\lambda\, F(t/\delta t) & 0 \leq t \leq \delta t \\ \lambda_0 + \delta\lambda \equiv \lambda_1 & \end{cases}$$

The function

$$F(y) : \quad F(0) = 0, \quad F(1) = 1$$

and always $O(1)$.



Regime

$$\lambda_0 (\delta t)^{d-\Delta}, \quad \lambda_1 (\delta t)^{d-\Delta} \ll 1$$

$$(\lambda_0 + F_{max}\delta\lambda)\, \delta t^{d-\Delta} \ll 1 \qquad (\lambda_0 + F_{min}\delta\lambda)\, \delta t^{d-\Delta} \ll 1$$

- Now start computing in perturbation theory. The expansion is

$$\langle \mathcal{O}(t)\rangle = \langle \mathcal{O}(t)\rangle_{\lambda_0} - \cancel{i}\int_0^t dt' \int d^{d-1}x'\, G_R(x-x';t-t')\, \cancel{\lambda(t')}\, \delta\lambda\, F(t'/\delta t)$$

where G_R is the retarded Green's function in the $\lambda=\lambda_0$ theory

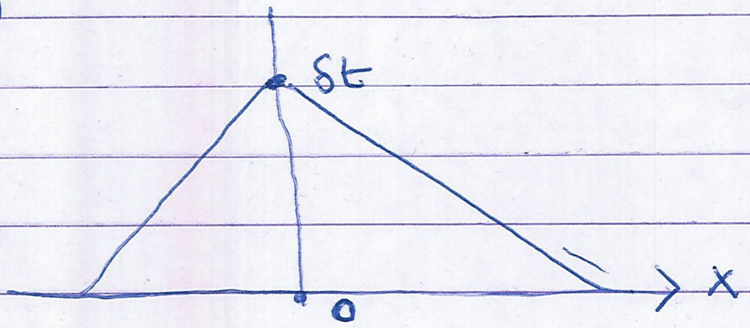
$$G_R(x-x';t-t') = i\,\theta(t-t')\,\langle 0|[\mathcal{O}(x,t),\mathcal{O}(x',t')]|0\rangle_{\lambda_0}$$

We assume that this can be also renormalized ~~by~~
– Possibly by adiabatic expansion

- Now consider

$$t \leq \delta t$$

G_R is causal $\Rightarrow$ (x',t') must lie in the past light cone of $(x,\delta t)$



- This means
$$x - x' \sim O(\delta t)$$
as well

- In our limit
$$\lambda_0 [\delta t]^{d-\Delta} \ll 1$$
Thus
$$\lambda_0^{\frac{1}{d-\Delta}} \delta t \ll 1$$
ie both Δx and Δt — support of the Green's fn. are much smaller than the mass scale of the λ_0 theory

$$\Rightarrow \quad (G_R)_{\lambda_0} = (G_R)_{CFT} + \text{small corrections}$$

$\Rightarrow$ λ_0 scale does not appear

- The value of λ_1 does not appear as well so long as $t \lesssim \delta t$

- Therefore the only scales in the problem are
$$(\delta\lambda) \, , \, (\delta t)$$

Since the term is already proportional to $(\delta\lambda)$, the full perturbation series must be of the form

$$\langle \mathcal{O} \rangle (t) - \langle \mathcal{O} \rangle_{\lambda_0}$$
$$= (\delta t)^{-\Delta} f(g)$$
where g is the dimensional coupling
$$g = (\delta\lambda)(\delta t)^{d-\Delta}$$

- By definition $f(g)$ has a power series expansion in g – the first term should be $O(g)$.

$\Rightarrow$
$$\langle \mathcal{O} \rangle (t) - \langle \mathcal{O} \rangle_{\lambda_0}$$
$$= (\delta t)^{-\Delta} \left[a_1 g + a_2 g^2 + \cdots \right]$$

- Therefore to lowest order the renormalized

$$\delta \langle O \rangle \sim (\delta\lambda)(\delta t)^{d-2\Delta}$$

Note:
- This result does not depend on the IR fate of the theory
- The result is entirely controlled by the UV fixed point

~~UV~~ $\delta t \to 0$ LIMIT

- The predictions
$$\langle \mathcal{O} \rangle \sim (\delta t)^{d-2\Delta}$$
$$\langle \mathcal{E} \rangle \sim (\delta t)^{d-2\Delta}$$
seem to preclude a $\delta t \to 0$ limit for any
$$d > \Delta > d/2$$

- However there seem to be perfectly well-defined answers in abrupt quench

- Note our quench protocols have
$$\Lambda_{UV}^{-1} \ll \delta t \ll [\lambda_0]^{-\frac{1}{d-\Delta}}.$$

- Strictly abrupt quench of course means that δt is small compared to all scales — including the scale of Λ_{UV}

- Nevertheless one would expect that for long distance quantities abrupt quench would be a good approximation.

- To examine the validity we now look at UV FINITE quantities — no need to add counterterms etc.

- One such quantity is EXCESS ENERGY

$$\Delta \mathcal{E} = \mathcal{E}_{\infty} - \mathcal{E}_{\text{ground}}.$$

where $\mathcal{E}_{\text{ground}}$ = ground state energy of the final theory

For any finite δt
This is UV finite — we already saw that counter-terms needed to render $\mathcal{E}$ finite ~~are~~ involve $\partial_t^2 m^2$ etc.
— given by the adiabatic expansion.

However at late times the time derivative terms vanish
⇒ left with the leading adiabatic answer — $\mathcal{E}_{\text{ground}}$

- For free bosonic fields

$$\Delta \mathcal{E} = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \, \omega_{out} \, |\beta_k|^2$$

- Results.

$d=2$ $\qquad \Delta \mathcal{E}^{\delta t \to 0} \sim \frac{m^2}{16\pi} + c_1 m^4 (\delta t)^2$

$d=3$ $\qquad \Delta \mathcal{E}^{\delta t \to 0} \sim \frac{m^3}{24\pi} + c_2 m^4 (\delta t)$

$d=4$ $\qquad \Delta \mathcal{E}^{\delta t \to 0} = c_3 m^4 \log(m \delta t)$

$d \geq 5$ $\qquad \Delta \mathcal{E}^{\delta t \to 0} \sim m^4 (\delta t)^{4-d}$

- The finite result for $d = 2,3$ is in fact the result for abrupt quench

$$\Delta \mathcal{E}^{inst} = \frac{\Omega_{d-2}}{(2\pi)^{d-1}} \int_0^\infty dk \, k^{d-1} \frac{\left(\sqrt{k^2+m^2} - k\right)^2}{4k\sqrt{k^2+m^2}}$$

- For $d \geq 4$ $\Delta \mathcal{E}$ diverges as $\delta t \to 0$
Likewise
$\Delta \mathcal{E}^{inst}$ diverges in the UV

- While these results are for free fields we expect similar results for interacting theories

$\Delta < d/2$ $\quad$ ΔE agrees with abrupt. Subleading $(\delta t)^{d-2\Delta}$

$\Delta > d/2$ $\quad$ Diverges as $(\delta t)^{d-2\Delta}$

Late time Correlators

- Result for momentum space correlators

$$\langle \phi(k,t)\,\phi(-k,t)\rangle_{\text{smooth}} \rightarrow \text{abrupt}$$

when $\quad k\delta t\,,\; m\delta t \ll 1$

- In position space this should imply that

$$\langle \phi(r,t)\,\phi(0,t)\rangle \rightarrow \text{abrupt}$$

when $\quad mt \gg 1$
$\quad\quad\quad r/\delta t \gg 1$

We always have $m\delta t \ll 1$

- However they should also agree in the small r limit. In fact

$$C(t,r) \xrightarrow[r\to 0]{} \frac{a_1}{r^{d-2}} + \frac{a_2 m^2(t)}{r^{d-4}}$$

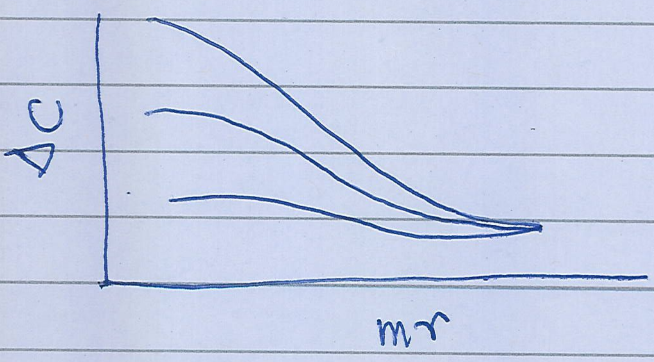
Same as counterterms $\longrightarrow$

$$+ \frac{a_3(3m^4 + \partial_t^2 m^2)}{r^{d-6}}$$

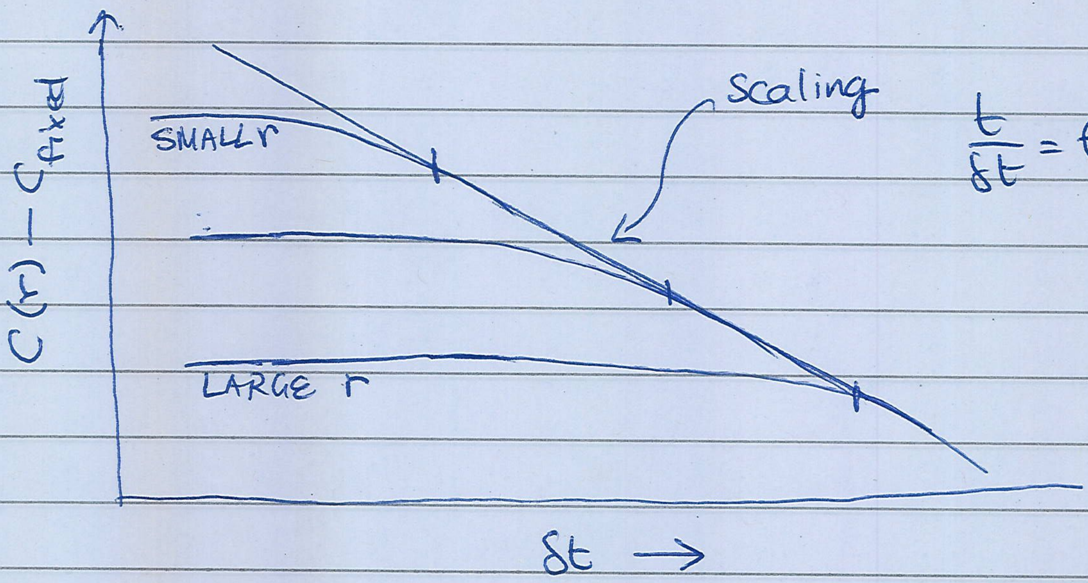
At late times $\partial_t^2 \to 0$ and result same as for a constant mass.

Reflects the fact that the leading singularity does not care about variations of m^2

- To look at agreement in large r useful to remove this small r singularity



- Most interesting behavior actually happens at INTERMEDIATE TIMES

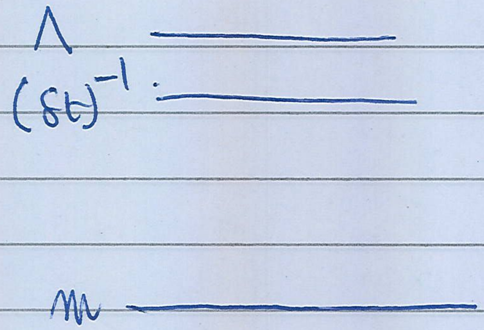


$\frac{t}{\delta t} = \text{fixed}$

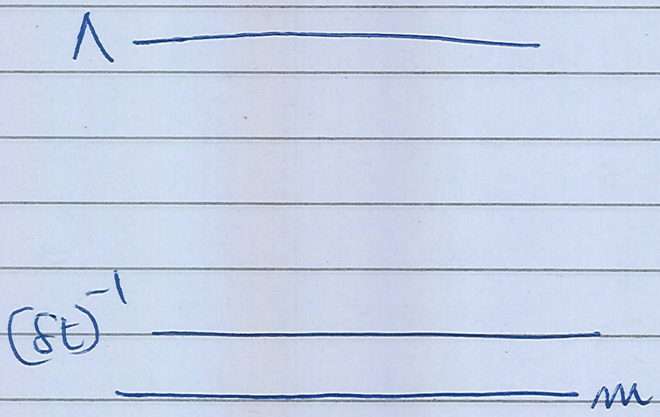
$\Rightarrow$ Intermediate regime of r where this scaling holds.

Finite Cutoff.

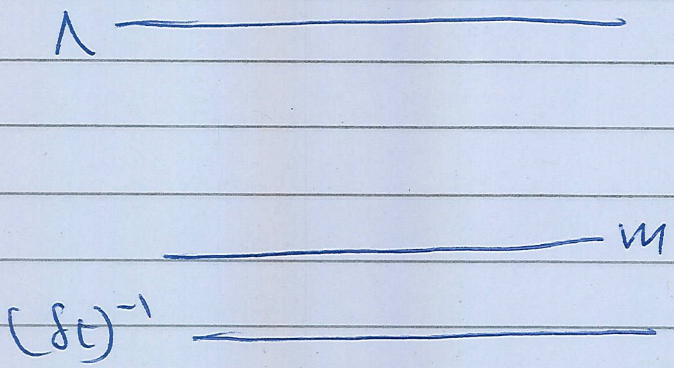
- Our results have implications for local quantities at finite cutoff



$\rightarrow$ Approach to Abrupt Quench (for some quantities)



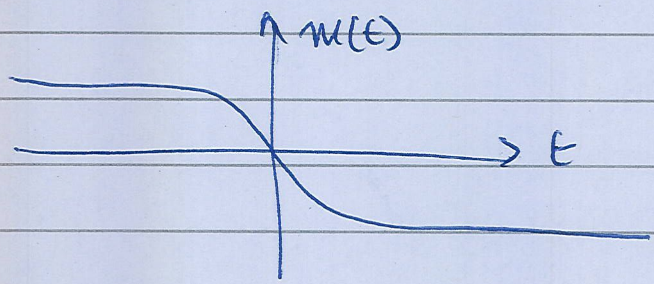
$\rightarrow$ New Scaling regime



Kibble Zurek in suitable situation.

- For KZ to our scaling

$$\bar{\Psi} \not{\partial} \Psi + m(t) \bar{\Psi} \Psi$$



- In this case we can analyze both the KZ regime and the fast quench regime

KZ regime $\qquad m\delta t \gg 1$

$$E_{gap} \sim m(vt)$$

$$\frac{1}{E_{gap}^2} \frac{dE_{gap}}{dt} \sim \frac{1}{m^2 v^2 t^2} m v = (m v t^2)^{-1}$$

Thus $\qquad t_{KZ} \sim (mv)^{-1/2}$

$$\langle \bar{\Psi} \Psi \rangle \quad m(t_{KZ}) = m v t_{KZ} = (mv)^{1/2}$$

Thus

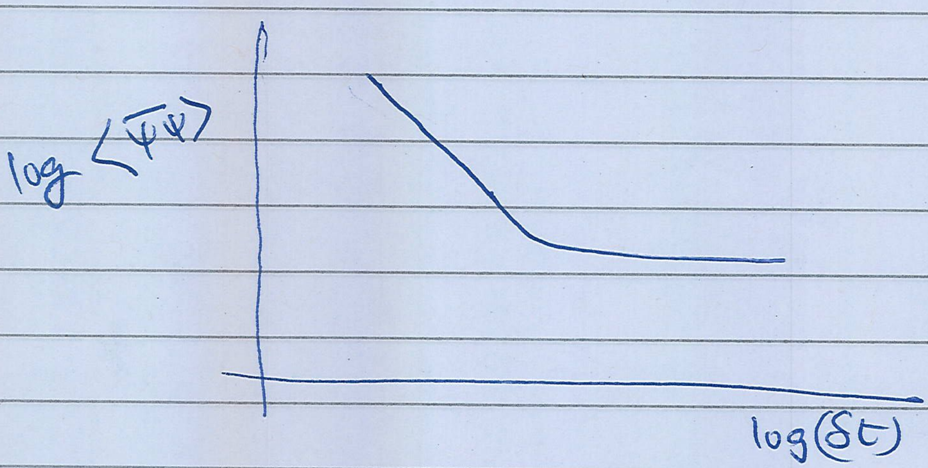
$$\langle \bar{\Psi} \Psi \rangle \sim \left[m(t_{KZ}) \right]^{\Delta}$$

$$\Delta = \frac{d-1}{2}$$

$$\langle \bar{\Psi} \Psi \rangle \sim v^{\frac{d-1}{2}}$$

For fast quench $2\Delta - d = d-2$
and

$$\langle \bar{\Psi} \Psi \rangle \sim v^{d-2}$$

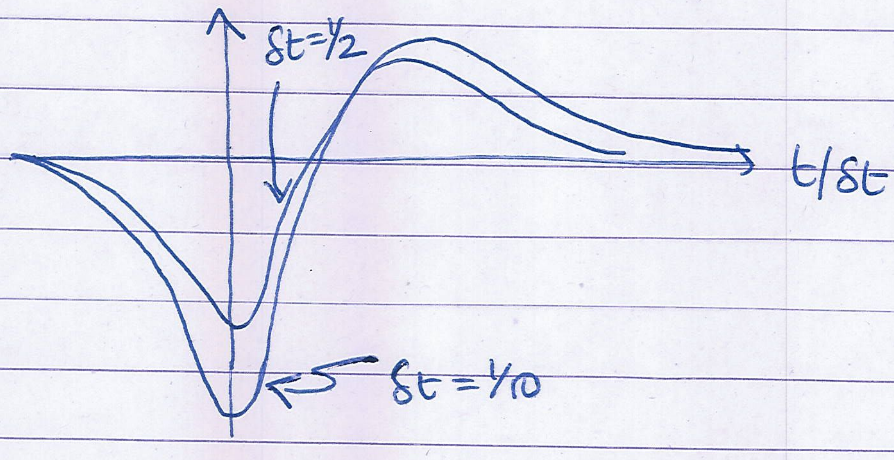


$d=5$ $\langle \bar{\Psi} \Psi \rangle \sim v^2$ KZ

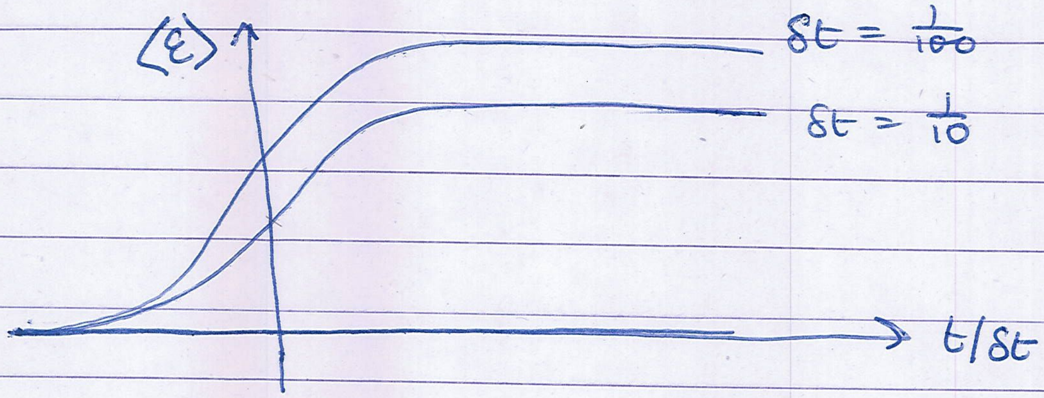
$\langle \bar{\Psi} \Psi \rangle \sim v^3$ Fast

FIGURES

1. Profile of $\langle \phi^2 \rangle_{renorm} - \langle \phi^2 \rangle_{adia}$



2. Energy profiles



Momentum space 2 pnt fns.

$$C(t,r) \sim \int dk\, k^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr) \left\{ |\alpha_k|^2 + |\beta_k|^2 + \alpha_k \beta_k^* e^{2ikt} + \alpha_k^* \beta_k e^{-2ikt} \right\}$$

$$C_{inst}(t,r) \sim \int dk\, k^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr) \cdot \frac{k^2 + m^2 \sin^2(kt)}{k\sqrt{k^2+m^2}}$$

Agree when $(\delta t)k \ll 1$ for all k.