AdS/CFT SETUP

- Consider AdS_{d+1} — this is assumed to be dual to some CFT which is d dimensional

- The bulk theory has several fields — gravity, scalars, vectors etc. The AdS/CFT correspondence asserts a correspondence ~~unit~~ of each of the fields with a specific operator on the CFT of the boundary

$$g_{\mu\nu} = g_{\mu\nu}^{AdS} + h_{\mu\nu}$$

$$h_{\mu\nu} \Leftrightarrow T_{\mu\nu}$$

$$\phi \Leftrightarrow \mathcal{O}$$

$$A_\mu \Leftrightarrow J_\mu$$

- Now consider deformations of the AdS_{d+1} obtained by turning on one or more of these fields.

- In the regime where the bulk theory is classical, evaluate the action on-shell.

This is a function of the boundary values of the fields — as in Hamilton Jacobi theory

$$ds^2 = \frac{R^2}{z^2}\left[-dt^2 + d\vec{x}^2 + dz^2\right]$$

$(d-1)$

Boundary is at $z=0$

Consider regulated boundary at $z=\epsilon$ and impose

$$\phi_0(x,t) = \cancel{\lim}\ \phi(x,t,\epsilon)$$

Then on-shell action is a function of $\phi_0(x,t)$

- In fourier space. $\phi_0(k) = \phi(k,\epsilon)$.

Suppose $\tilde{\Phi}(k,z)$ is a solution of the wave ~~fn.~~ equation. In these coordinates

$$\tilde{\Phi}(k,z) \underset{z\to 0}{\sim} z^{\Delta_-}\left[A_0(k) + z^2 A_1(k) + \cdots\right] + z^{\Delta_+}\left[B_0(k) + z^2 B_1(k) + \cdots\right.$$

Soln which obeys the boundary condition is

$$\phi(z,k) = \frac{\tilde{\phi}(z,k)}{\tilde{\phi}(\epsilon,k)} \phi_0(k)$$

$$\Delta_{\pm} = \frac{d}{2} \pm \nu \qquad \nu = \sqrt{\left(\frac{d}{2}\right)^2 + m^2}.$$

· On-shell action is.

$$S_{os} = \frac{R^{d-1}}{\epsilon^d} \Delta_- \int d^d k \, \phi_0(k) \phi_0(-k).$$
$$+ R^{d-1} \cdot 2\nu \int d^d k \, \frac{B_0(k)}{A_0(k)} \phi_0(k) \phi_0(-k) \, \epsilon^{-2\Delta_-}$$

The solution itself is

$$\phi(z,k) = z^{\Delta_-} \epsilon^{-\Delta_-} \phi_0(k) + \cdots$$
$$+ z^{\Delta_+} A(k) + \cdots$$

where $A(k) = \dfrac{B_0(k)}{A_0(k)} \epsilon^{-\Delta_-} \phi_0(k)$

· GKPW dictionary states

$$\exp\left[\int dk \, \phi_0(k) \epsilon^{-\Delta_-} \mathcal{O}_+\right] = e^{-S_{os}}.$$

- $S_{\text{on-shell}}$ is of course divergent as $\epsilon \to 0$
  $\Rightarrow$ Need to define a RENORMALIZED ACTION by adding suitable counterterms.

$$S_{ct} = \frac{R^{d-1}}{\epsilon^{d}} \Delta_- \int d^d k \, \phi_0(k) \phi_0(-k)$$

which is, however, just the local counterterm

$$S_{ct} = \frac{\Delta_-}{R} \int d^d x \sqrt{\gamma} \, \phi^2(x)$$

γ : Induced metric

$$ds^2_{ind} = \gamma_{ab} dx^a dx^b = \left(\frac{R}{\epsilon}\right)^2 \left[-dt^2 + d\vec{x}^2\right]$$

This is the simplest example of holographic renormalization

- If we consider deformations of the AdS metric itself we would similarly have counterterms

AdS_5 $$S_{ct} = -\frac{3}{\ell} \sqrt{-\gamma} \left(1 - \frac{\ell^2}{12} R\right)$$

AdS_3 $$-\frac{1}{\ell} \sqrt{-\gamma}$$

After adding the counterterm one gets

$$\boxed{\langle \mathcal{O}_+ \rangle = 2\nu R^{d-1} A(k)}$$

- Similar considerations for other fields.

- Lesson: $\phi_0(k)$: SOURCE
  $A(k)$: RESPONSE.

- ~~First to~~ Consider first the static situation.
  If $\phi_0 = 0$ but $A \neq 0$
  $\Rightarrow$ The state must be some kind of excited state

Interesting excited state : BLACK HOLE

(d+1) $$ds^2 = -r^2 f_b(r) dt^2 + \frac{dr^2}{r^2 f_b(r)} + r^2 (d\vec{x}^2)$$

$$f_b(r) = 1 - \left(\frac{r_0}{r}\right)^d$$

Interpretation: Thermal state of the dual field theory

Another interesting situation →
Spontaneous symmetry breaking

Suppose global symmetry on boundary
e.g a U(1) global symmetry
- Dual gravity has a bulk gauge field.
- Add charged scalar

There are situations when even in the absence of source
$$\langle O \rangle \neq 0$$
→ Spontaneous breaking of U(1)

MODELLING TIME DEP IN AdS/CFT

- Time dependent coupling of the dual theory $\equiv$ Time dep boundary conditions

- Response $\Rightarrow$ Calculation of the resulting bulk solution
  — Extract out the "normalizable" piece.

- Problem of ~~a~~ dynamics in a strongly coupled field theory thus reduced to solution of PDE's.

  Can be put on a computer — though non-trivial

Balasubramanian et.al.
Abajo-Arrastia, Aparicio, Lopez

HOLOGRAPHIC THERMALIZATION

· Setup : In the dual field Theory
  · Start with vacuum
  · Turn on a source for a time extent δt
  · Calculate the bulk solution
  · Extract out various quantities.

Chesler & Yaffe : Boundary metric time dependent in a suitable fashion — NUMERICAL

Bhattacharya & Minwalla : Small amplitude collapse in AdS. — boundary value of dilton changes

$$S = \int d^{d+1}x \sqrt{g} \left(R - \frac{d(d-1)}{2} - \frac{1}{2}(\partial\phi)^2 \right)$$

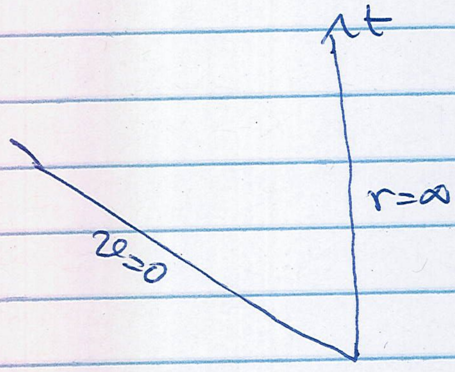
To set up the initial/boundary value problem useful to use ingoing EF coordinates in which AdS.

$$ds^2 = 2\,dr\,dv - r^2 dv^2 + r^2 d\vec{x}^2$$

$$v = t - \frac{1}{r}$$

Need to find solutions of the form

$$ds^2 = 2\,dr\,dv - g(r,v)\,dv^2 + f^2(r,v)\,d\vec{x}^2$$
$$\phi = \phi(r,v)$$



Initial conditions

$$g(r,v) = r^2 \quad (v<0)$$
$$f(r,v) = r \quad (v<0)$$
$$\phi(r,v) = 0 \quad (v<0)$$

Boundary conditions

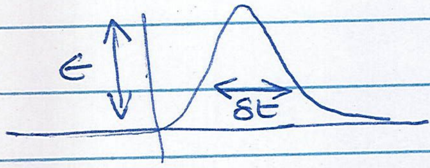
$$\lim_{r\to\infty} \frac{g(r,v)}{r^2} = 1$$
$$\lim_{r\to\infty} \frac{f(r,v)}{r} = 1$$
$$\lim_{r\to\infty} \phi(r,v) = \phi_0(v).$$

(Need to fix additional gauge freedom by requiring $f(r,v) = r + O(1/r)$

This implies $g(r,v) = r^2 + O(1)$

$$\phi(r,v) = \phi_0(v) + 1/r$$

- The boundary condition has the following form



- Perturbation expansion in ϵ

RESULT : A black brane always forms

$$ds^2 = 2\,dr\,dv - \left(r^2 - \frac{M(v)}{r^{d-2}}\right) dv^2 + r^2 d\vec{x}^{\,2}$$

where to leading order

$$M(v) \approx O\left(\frac{\epsilon^2}{(\delta t)^d}\right)$$

Odd d : $M(v) = M$ for $v > \delta t$

even d $M(v) \to M$ as a power in $\left(\frac{\delta t}{v}\right)$

Temperature of brane : $T \sim \left[M\cancel{(v)}\right]^{1/d}$

$$\sim \frac{\epsilon^{2/d}}{(\delta t)}$$

Apparent horizon : $r = \left[M(v)\right]^{1/d}$

Event horizon

$$r_H = \begin{cases} M^{1/d} & v > 0 \\ \dfrac{M^{1/d}}{1 - M^{1/d} v} & v < 0 \end{cases}$$

- Corrections to this Vaidya metric may be computed as a power series in ϵ
  - Naive pert. theory diverges
  - need resummation

Note: Vaidya details.

Ingoing null geodesics $v = \text{const.}$
Outgoing

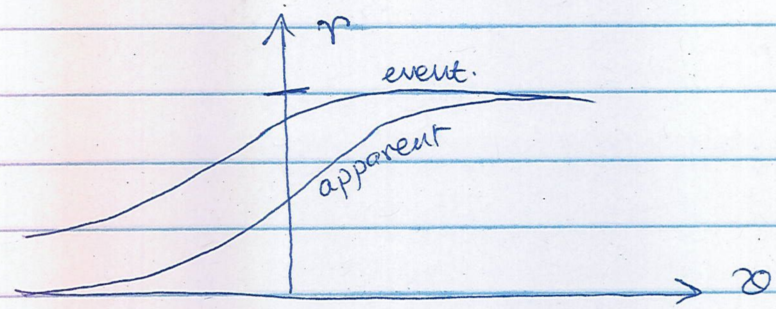
$$\frac{dr}{dv} = \frac{1}{2}\left(r^2 - \frac{M(v)}{r^{d-2}}\right)$$

For $r^2 < \frac{M(v)}{r^{d-2}}$ even "outgoing" has $\frac{dr}{dv} < 0 \Rightarrow r^2_{\text{ap hor}} = \frac{M(v)}{r^{d-2}}$

For event horizon: This is the "outgoing" null geodesic

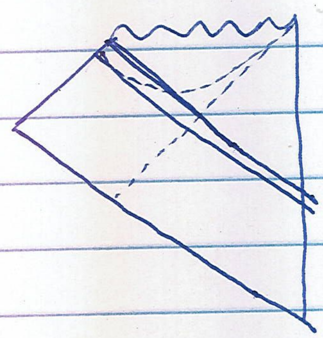
$$\frac{dr_e}{dv} = \frac{1}{2}\left(r_e^2 - \frac{M(v)}{r_e^{d-2}}\right)$$

With condition $\lim_{v \to 0} r_e(v) = M^{1/d}$.



Lin & Shuryak

Probes of thermalization



Clearly local quantities will look thermal right after the pulse.

$\Rightarrow$ Thermalization time

Time scale natural $\tau = \frac{1}{\text{Temp}} \sim \frac{\delta t}{\epsilon^{2/d}}$

In $\epsilon \ll 1$ limit $\tau \gg \delta t$

In fact $\delta t \to 0$ $\epsilon \to 0$, τ fixed

$\Rightarrow$ thermalization time still goes to zero.

· The time scale for thermalization is however different for non-local observables.

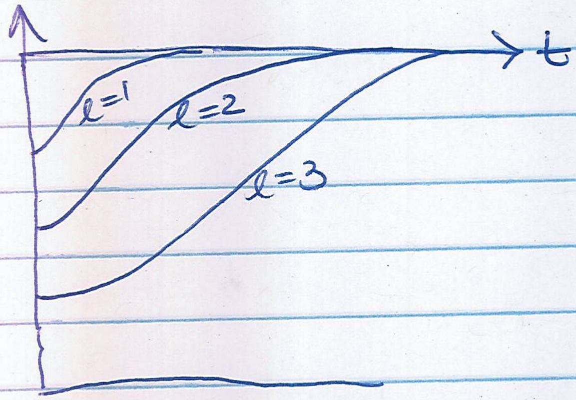
- Two point functions
- entanglement entropy

Two point function:

— Calculate by geodesic approx. (Note geodesic regularized by subtracting $2\log z_0/2$

Plot

$$\mathcal{L}_{renor} - \mathcal{L}_{thermal}$$ for Thin shell

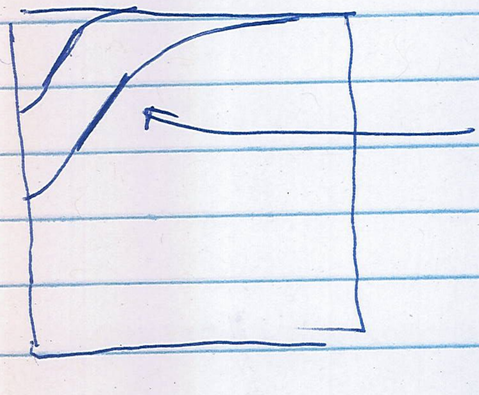


longer distance correlators take longer to thermalize

For large distance correlators the geodesics cross the shell - but effect becomes less pronounced at late times

For AdS_3 this also gives the entanglement entropy

For higher AdS need to calculate extremal surface



a regime of linear growth

Hubeny, Rangamani & Tonni 1

Liu & Shu

Entanglement Tsunamis

- Consider thin shell Vaidya metric

$$ds^2 = \frac{1}{z^2}\left[-f(v,z)\,dv^2 - 2\,dv\,dz + dz^2\right]$$

$$f(v,z) = 1 - \theta(v)\,g(z).$$

$h(z) \equiv 1 - g(z)$ parametrizes different equilibrium states

- $h(z)$ assumed to have the properties
  - (a) $h(z_h) = 0$
  - (b) $z \to 0$ $\quad h(z) \to 1 - Mz^d$
  - (c) $z < z_h$ $\quad h(z) > 0$ and monotonically decreasing
  - (d) Metric satisfies bulk null energy condition

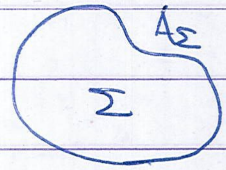
- Now calculate $S_\Sigma(t) = \frac{A_\Sigma}{4G_N}$ $\quad A_\Sigma$: area of $(d-1)$ dimensional 'extremal' surface.

Size of Σ: R = height of future domain of dependence

$$\Delta S_\Sigma = S_\Sigma(t) - (S_\Sigma)_{vacuum}.$$

- The scale which enters in the time evolution of ΔS_Σ is

$$l_{eq} \sim z_h \sim \left(\frac{1}{s_{eq}}\right)^{\frac{1}{d-1}}$$



where long after the quench

$$\Delta S_\Sigma \to \Delta S_\Sigma^{eq} = s_{eq} V_\Sigma$$

- For $t \sim l_{eq}$ local thermodynamics already applies at scales smaller than l_{eq}
However $S_\Sigma(t)$ remain far for equilibrium when $R \gg l_{eq}$

- ~~[illegible]~~ For $R \lesssim l_{eq}$ evolution lasts to $t \sim l_{eq}$ and extremal surfaces remain outside horizon

- For $R \gg l_{eq}$ evolution controlled by geometry around or INSIDE horizon

- Evolution of $\Delta S_\Sigma(t)$ has several regimes

(1) Pre-local-equilibration growth

$t \ll l_{eq}$. $\quad \Delta S_\Sigma = \frac{\pi}{d-1} A_\Sigma \mathcal{E} t^2 + \cdots$ INDEPENDENT OF SHAPE

(2) Post local-equilibration linear growth

$$\Delta S_\Sigma = v_E \, S_{eq} \, A_\Sigma t$$

where v_E INDEPENDENT OF SHAPE
– depends on final equilibrium

$$v_E = \left(\frac{z_h}{z_m}\right)^{d-1} \sqrt{-h(z_m)}$$

z_m : Minimum of $\frac{h(z)}{z^{2(d-1)}}$

e.g. for a Schwarzschild final state

$$v_E^S = \frac{(\eta-1)^{\frac{1}{2}(\eta-1)}}{\eta^{\frac{1}{2}\eta}} \qquad \eta \equiv \frac{2(d-1)}{d}.$$

(3) Saturation

Now ΔS_Σ depends on shape, d and final equilibrium state

- STRIP of size R $\quad d \geq 3$

  Linear growth stays till saturation

$$t_s^{\text{strip}} = \frac{R}{2v_E}$$

$\partial_t S_\Sigma$ discontinuous at $t = t_s$

- SPHERE of radius R

  $d \geq 4$

$$S(R,t) - S^{eq}(R) \sim -(t_s - t)^\gamma$$

$$\gamma = \frac{d+1}{2} \qquad t_s - t \ll l_{eq}$$

Same exponent in $d=2$

For $d=3$

$$(t_s - t)^2 \log(t_s - t)$$

- For sphere another scaling regime

$$l_{eq} \ll t_s - t \ll t_s$$

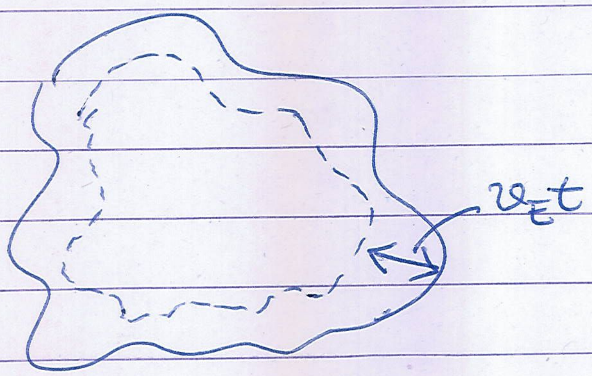
$$S(R,t) - S_{eq}(R) = -S_{eq}\, \lambda\big(t_s(R) - t\big)$$

ie depends only on $t_s(R) - t$ and not on R, t separately

- Rewrite linear growth regime

$$\Delta S_\Sigma = (v_E t)\, S_{eq} \cdot A_\Sigma$$

$$= S_{eq}\left[V_\Sigma - V_{\Sigma - v_E t} \right]$$



Interpret growth of entanglement as a TSUNAMI travelling inward.

Region covered by wave is entangled with outside

S.R. Das, T. Nishioka & T. Takayanagi

PROBE BRANES : TOY MODEL

- The important thing turns out ~~to~~ be the formation of an apparent horizon

- Study quantum quench on PROBE BRANES.

Consider N_f probe branes in a $AdS_5 \times S^5$ background.
Canonical example. D5 branes in D3 background

$$\text{D5 Probe brane tension} = \frac{N_c}{R^4 l_s^2}.$$

$$\text{For D branes tension} \sim \frac{1}{g_s} \sim \frac{1}{\sqrt{G_N}}$$

$$\text{and} \quad G_N = g_s^2 l_s^8 \sim \frac{R^8}{N_c^2} \quad \left(g_s = \frac{R^4}{N_c l_s^4}\right)$$

$$\text{For D5} \quad \frac{1}{g_s l_s^6} = \frac{N_c}{R^4 l_s^2}$$

Thus factor in front of DBI $\quad N_f N_c$
Factor in front of Einstein $\quad N_c^2$

Thus when $N_f \ll N_c$ probe approx.

Simplest example:

D1 brane in $AdS_5 \times S^5$

$$ds^2 = 2\,dr\,dv - r^2 dv^2 + r^2 d\vec{x}^2$$
$$+ d\theta^2 + \sin^2\theta\, d\varphi^2 + \cos^2\theta\, d\Omega_3^2$$

Consider 1 brane along r direction

Transverse fields

$$\vec{X}(r,v),\ \theta(r,v)\ \varphi(r,v)\ \Omega_3(r,v)$$

DBI action

$$S = -T\int dv\,dr\,\sqrt{\det h}. \qquad u = v + 2/r$$

$$h_{ab} = \partial_a X^M \partial_b X^N G_{MN} \qquad a,b = v,r$$

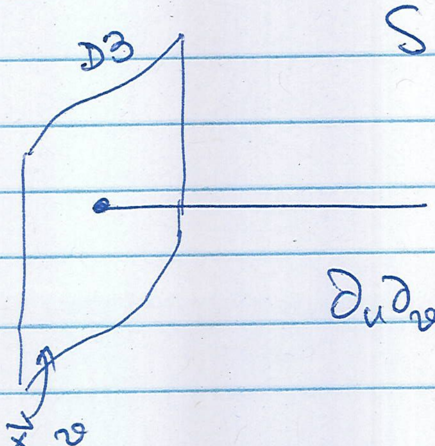
Easy to check can have solutions

$$\vec{x},\ \Omega_3 = \text{constant} \qquad \theta = \pi/2$$

$\Rightarrow$ Only dynamical variable is

$$\varphi(r,v)$$

$$S = -T\int du\,dv\; r^2\left[1 - \frac{4}{r^2}\partial_u\varphi\,\partial_v\varphi\right]^{1/2}$$



Eqn. of motion

$$\partial_u\partial_v\varphi + \frac{2}{L}\partial_v\varphi\,\partial_u\left(\frac{\partial_u\varphi\,\partial_v\varphi}{r^2}\right)$$
$$+ \frac{2}{L}\partial_u\varphi\,\partial_v\left(\frac{\partial_u\varphi\,\partial_v\varphi}{r^2}\right) = 0$$

- Any fn $\varphi(v)$ or $\varphi(u)$ is a solution. Choose $\varphi(v)$.

Induced metric

$$ds^2_{ind} = 2\,dr\,dv - (r^2 - (\partial_v\varphi)^2)\,dv^2$$

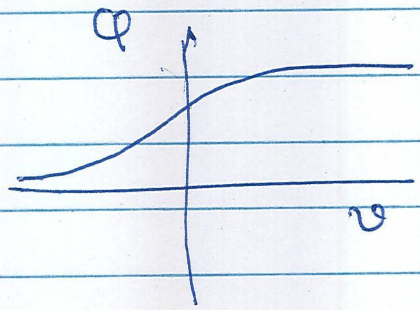
Represents a D1 brane whose end point rotates on the S^5.

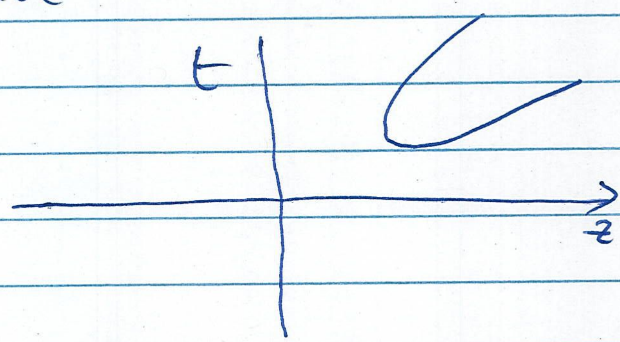
- Worldsheet metric has apparent horizon at

$$r = \varphi'(v)$$

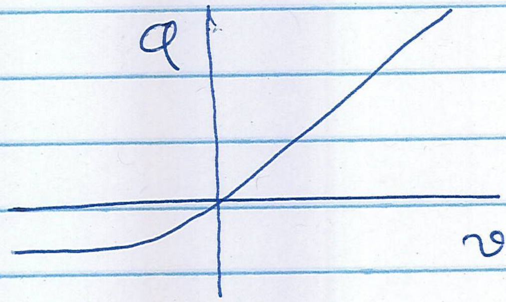
- The apparent horizon need not be associated with an event horizon — no dynamical metric

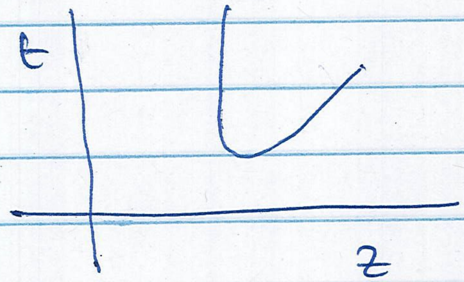
$$\varphi(v) = \varphi_0(1 + \tanh(kv))$$





- When the φ' saturates asymptotically there is indeed an event horizon





$$\varphi(v) = \varphi_0\left(v + \frac{1}{k}\log(\cosh kv)\right)$$

As $v \to \infty$ $\varphi' = 2\varphi_0$

$$ds^2_{in} = 2\,dr\,dv - (r^2 - 4\varphi_0^2)\,dv^2$$

$\Rightarrow$ Event horizon at $r = 2\varphi_0$.

- Presence of apparent/event horizons on the worldsheet means that the fluctuations on the D1 brane behave as fields in such a 2d metric — not necessarily minimally coupled.

- This results in BROWNIAN MOTION of the end-point

Let $\varphi(v,r) = \underset{\substack{\uparrow \\ \text{rotating} \\ \text{string}}}{\varphi(v)} + y^{\varphi}(r,v)$

Using the dynamics of y^{φ} one can compute fluctuations

$$\langle 0,v| : \left(y^{\varphi}(\bar{t}) - y^{\varphi}(\bar{t}')\right)^2 : |0,v\rangle$$

$$= \frac{1}{2\pi} \log \left(\frac{\sinh \frac{\pi(\bar{t}-\bar{t}')}{\beta}}{\pi \frac{\bar{t}-\bar{t}'}{\beta}} \right)$$

$$= \begin{cases} \dfrac{\pi(\bar{t}-\bar{t}')^2}{12\beta^2} & \pi(\bar{t}-\bar{t}') \ll \beta \\[2ex] \left(\dfrac{\bar{t}-\bar{t}'}{2\beta}\right) - \dfrac{1}{2}\log\left[\dfrac{2\pi(\bar{t}-\bar{t}')}{\beta}\right] & \pi(\bar{t}-\bar{t}') \gg \beta \end{cases}$$

Notation:

$$ds^2_{ind} = 2\,dr\,dv - (r^2-\omega^2)\,dv^2$$
$$= -(r^2-\omega^2)\,d\bar{u}\,dv$$

where

$$\bar{u} = v - 2\int \frac{dr}{r^2-\omega^2} \qquad \bar{t} = \frac{1}{2}(\bar{u}+v).$$

On boundary at $r=\infty$ $\quad \bar{u} = v = t$

- What does this represent in the dual theory

Dual theory: quantum mechanics of monopoles in the $N=4$

Global AdS:

Now there is another scale R :
Radius of boundary sphere
In $R_{AdS}=1$ units.

$$ds^2 = -(1+r^2)\,dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_3^2$$

Define dimensionless ratio

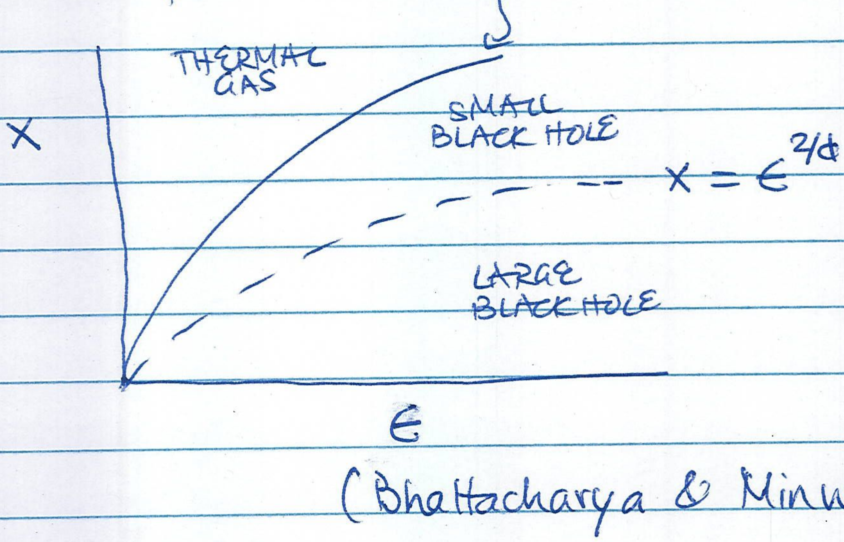
$$x = \frac{\delta t}{R}$$

$x \to 0$: basically same as Poincare
— formation of large black hole in AdS

This works for

$$x \ll \epsilon^{2/d}$$

The whole picture



(Bhattacharya & Minwalla)

Field Theory meaning:

CFT on S^{d-1}

Low temperatures : gas of glueballs

High temperature : strongly interacting plasma

When we excite the system over a time δt homogeneously

$x \ll \epsilon^{1/d}$ ends up in the plasma phase

$x \gg \epsilon^{\frac{1}{d-1}}$ ends up in the almost free particle phase

equilibration in plasma phase - quick

for free particle phase - takes a while

- Bizon and Rostworowski: Even if the perturbation is arbitrarily weak
  - leads to collapse
  - usually after many scatterings from the boundary.

Basic physics: Anharmonic oscillator

$$\frac{d^2x}{dt^2} + x + \epsilon x^3 = 0$$

Solve perturbatively in ϵ

$$x(t) = x_0(t) + \epsilon x_1(t) + \cdots$$

$$x_0(t) = \cos t \qquad \begin{array}{l} x(0) = 1 \\ \dot{x}(0) = 0 \end{array}$$

Substitute

$$\frac{d^2x_1}{dt^2} + x_1 = -\frac{1}{4}\cos 3t - \frac{3}{4}\cos t$$

Solution is

$$x_1 = \frac{1}{32}\left(\cos 3t \cancel{-\cos t}\right) - \frac{3}{8}t\sin t$$

Last term is dangerous — grows with time — SECULAR TERM.

However in this case — to $O(\epsilon)$ this is same as

$$x = \cos\left(1+\frac{3\epsilon}{8}\right)t + \frac{\epsilon}{32}\cos\left(3\left(1+\frac{3\epsilon}{8}\right)t\right)$$

Check

$$\cos\left(1+\frac{3\epsilon}{8}\right)t = \cos t \cos\frac{3\epsilon t}{8} - \sin t \sin\frac{3\epsilon t}{8}$$

$\Rightarrow$

$$x(t) = \cos t - \frac{3\epsilon t}{8}\sin t + \frac{\epsilon}{32}\cos 3t$$

Thus this "resummed" soln is okay
— no growth
But frequency is now corrected.

- For this example this procedure works.

- However for coupled systems — particularly for a field theory

- It turns out that potential instabilities arise when

Suppose the field has discrete normal modes

$$\omega_n = d + 2n$$

Consider small amplitude perturbations amplitude ϵ

$$\phi = \epsilon\,\phi_1 + \epsilon^3\phi_3 + \cdots$$

$$\phi_1 = \sum_n a_n \cos(\omega_n t + b_n)\, e_n(r).$$

$$\phi_3 = \sum C_n(t)\, e_n(r)$$

Then C_n has an eqn. of the form

$$\ddot{C}_n + \omega_n^2 C_n = C_{ijkn}\, a_i a_j a_k \cos\left[(\omega_i \pm \omega_j \pm \omega_k)t + (b_i \pm b_j \pm b_k)\right] + \dots$$

May be shown if

$$\pm\omega_n = \omega_i \pm \omega_j \pm \omega_k$$

secular terms arise, giving rise to

$$C_n(t) = C_{ijkn}\, a_i a_j a_k\, t \sin(\omega_n t + (b_i \pm b_j \pm b_k))$$

They become important for $t \sim 1/\epsilon^2$.

- Initial claim:
  Since spectrum $\omega_n = d + 2n$
  this will almost always happen
  $\Rightarrow$ leads to collapse.

- However our example shows that the conclusion could be hasty.
  - could be resummed suitably

In fact several initial conditions have been discovered which DO NOT lead to collapse

(Buchel, Liebling, Lehner)

- Recent work on resummation

(Dias, Horowitz Santos ;
Balasubramanian, Buchel
Green, Lehner, Liebling ;
Basu, Krishnan Saurabh
Basu, Krishnan, Bala
Craps, Evnin, Vanhoof).

- No clear conclusion yet.