C-Theorem : ENTANGLEMENT ENTROPY AND the RENORM. group flow

→ Pointing to the question about the existence of a c-theorem in $d > 2$ ....

ZAMOLODCHIKOV . 1986 . ↑ [general constraint on the renormalization group flow ⊛ in the theories space] (1+1) dimensions.

- there is a universal function c defined for any theory which is dimensionless, decreasing along the renormalization group trajectories and takes finite values at fixed points proportional to the Virasoro central charge

↓

. "ordering" in the space of theories

universality + Lorentz invariance

⊛ tells us about changes in the physics with scales. the RG flow interpolates between UV to IR fixed points where the theory looks the same at all scales → conformal invariance.

. "irreversibility" due to loss of degrees of freedom from the UV to the IR.

(UV) small size → large size (IR)

$C(x)$ is built from $\langle T_{\mu\nu}(0)\, T_{\alpha\beta}(x)\rangle$ and proportional to C_{Virasoro}, such that $C_{UV} > C_{IR}$

$$C(r) = \frac{3}{4\pi}\int_r^\infty d^2x\; x^2 \langle \Theta(0)\Theta(x)\rangle$$

↳ trace

the same idea in terms of the EE :

ENTROPIC c-theorem (HC y MH in 2004)

[Θ zero for CFT and drives the c-function out of the fixed point.]

$$(*) \quad \Delta c = -\frac{3}{4}\int_0^\infty R^2 \langle \Theta(R)\Theta(0)\rangle \, d(R^2)$$

$$= -12\pi \lambda^2 (1-h)^2 \int \langle \phi(r)\phi(0)\rangle \, d^2 r$$

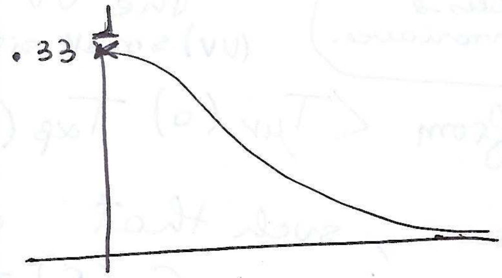
$$\Theta(z,\bar z) = 4\pi \lambda (1-h)\, \phi(z,\bar z) + \dots$$

$$S = S_0 - \lambda \int \underbrace{\phi(z,\bar z)}_{\text{dimension } h} \, d^2 z .$$

$$\Theta = T_{00} + T_{11} = 4T_{z\bar z} = 4T_{\bar z z}$$

$$z, \bar z = x^0 \pm i x^1$$

Plot$\left[\frac{1}{3} - \text{NIntegrate}\left[y^3 \left(B_K(h,y)^2 - B_K(0,y)^2\right)\right], (y,0,x)\right]$, $(x,0,3)\Big]$



C-theorem : Entanglement entropy and the Renormalization group flow.

- appears in 1986. It was introduced by Zamolodchikov for (1+1) dimensional theories.
- gives a constraint on the RG flow which tells us about changes in physics with scales, and interpolates between UV to IR fixed points where the theories look the same at all scales → conformal invariance.
- there is a universal function C defined for any theory which is dimensionless, decreasing along the RN group trajectories and takes finite values at fixed points proportional to the Virasoro central charge.

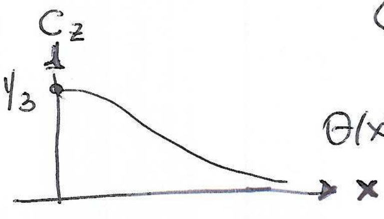
↓ this implies.

"ordering" in the space of theories : $C_{UV} > C_{IR}$
"irreversibility" due to loss of degrees of freedom from UV to IR.

- In Zamolodchikov theorem $C(x)$ is built from

two point function of the stress tensor

$$\langle T_{\mu\nu}(0)\, T_{\alpha\beta}(x) \rangle \quad ; \quad C(r) = \frac{3}{4\pi}\int_r^{\infty} d^2x\; x^2 \langle \Theta(0)\,\Theta(x) \rangle + C_{IR}$$



$\Theta(x) = T_\mu^{\ \mu}(x) \rightarrow \Theta$ zero for CFT and drives the c function out of the FP.

trace.

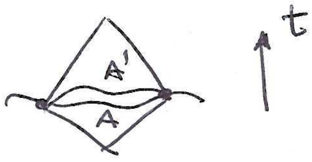
Plot [$\frac{1}{3}$ - NIntegrate [y^3 (BK $(1,y)^2$ - BK $(0,y)^2$)], (y, 0, x)], (x, 0, 3)]

Lorentz inv + unitarity (reflection posit + unit [illegible]

Causality

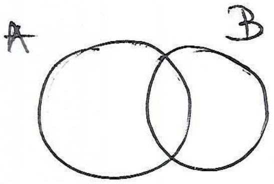
$$S(A) = S(A')$$

$$\rho_A = \rho_{A'}$$



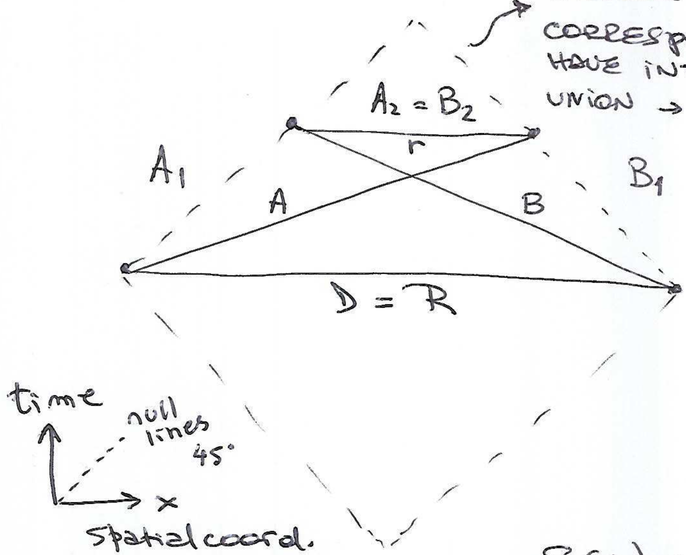
S is a function of the diamond shaped region of equivalently the region boundary

Strong subadditivity



$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

- Spatial Cauchy surface passing through A and B
- Boundaries must be spatial to each other



CAUSAL DOMAIN OF DEPENDENCE (CDD) CORRESPONDING TO A and B (spatial intervals)
HAVE INTERS → CDD OF A_2
UNION → CDD OF D

$S(A) = S(A_1 A_2)$ } CAUSALITY
$S(B) = S(B_1 B_2)$ } UNITARITY OF the CAUSAL evolution

↓

S independent of the cauchy surface.

two intervals relatively boosted to each other with diamon shaped causal domain of Dependence

relativistic geometry: $DA_2 = A.B$

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

$$S(A_1 A_2) + S(B_1 B_2) \geq S(A_2) + S(A_1 A_2 B_1)$$

chossing $A = B = \sqrt{rR} \Rightarrow$

$$2S(\sqrt{rR}) \geq S(R) + S(r)$$

setting $R = r + \mathcal{E}$

$$\rightarrow r S''(r) + S'(r) \leq 0$$

Defining $\boxed{C(r) = r S'(r)} \rightarrow C'(r) \leq 0$

↓ DIMENSIONLESS ✓ ↓ DECREASING ✓

At the conformal points

$$C(r) = r \frac{d}{dr} \left(\frac{c}{3} \log\left(\frac{R}{\mathcal{E}}\right)\right) = \frac{c}{3}$$

well defined ✓
universal ✓

⇓

same as Zamolodchicov c-theorem

$$C_{ZAMO} \neq C_{EE}$$

(see picture)

$$\boxed{S(r) = \frac{c}{3} \log\left(\frac{r}{\mathcal{E}}\right) + C_0}$$

↓

At the critical point

Entropic c-theorem, two dimensional case

The function C(r) given by

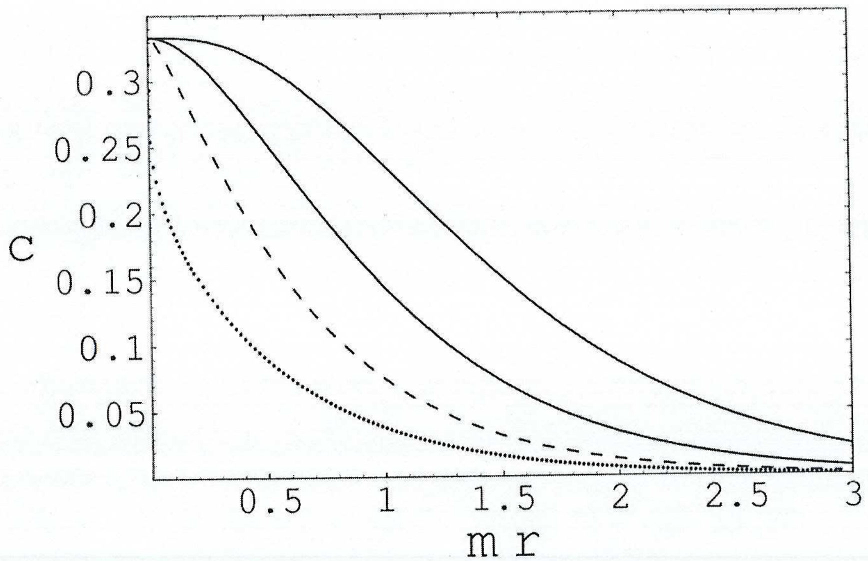
$$C(r) = rS'(r)$$

satisfies

$$C'(r) \leq 0$$

The central charge of the uv conformal point is larger than the central charge at the ir fixed point: the same result than Zamolodchikov c-theorem

At the conformal point : $S(r) = \frac{c}{3}\log(r/\epsilon) + c_0 \longrightarrow$ C (r) = c / 3



Comparison with Zamolodchikov c-functions.
From top to bottom: Zamolodchikov c-functions for a real scalar and Dirac fields and entropic c-functions for Dirac and real scalar fields.

More dimensions:

- there's alot of work done since '86
- Cardy's proposal '88 for even dimensions:

  he proposed the A coeff. of the Euler anomaly
  (Euler density term in the trace anomaly at the fixed point)

But this was proved for $d=4$ recently in 2011 by K-S (Komargodski - Schwimmer) (introducing a coupling with a dilaton)

For d even? In 2010, Myers-Sinha ~~[illegible]~~ introduced the holographic c-theorems proposing the constant term of the EE of a sphere as a candidate.

In 2011, Jafferis, Pufu, Klebanov, Safdi calculate explicitly the Part. funct for spheres in $d=3$ and they conjecture the (free energy / finite term) is decreasing under the RG.

In fact, these two proposals are the same!

in $d-1$ sphere

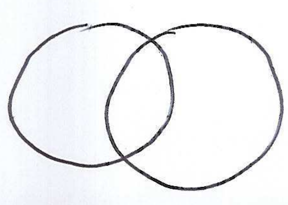
$\rightarrow$ EE constant term is the constant in $\log Z$ for a d-dim sphere

In 2012, Liu and Mezei, reinforce these ideas with the RENORMALIZED EE

How to generalize the 2-dim proof?

Starting point < Sets ○ / SSA.

two is not enough!

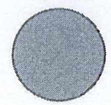


- Different shape, so we do not have a simple parameter
- new div. spoiled the ineq. to extract univ. inf.!

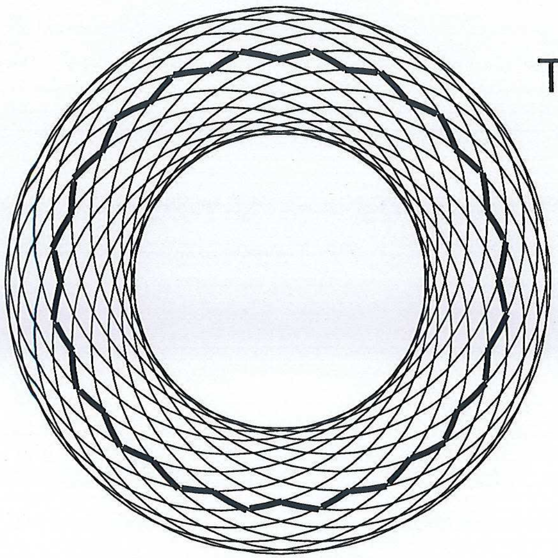
$\Rightarrow \propto$ number of ○ (picture)

Generalization: Three dimensional case

Planar construction

Starting point: Sets $\longrightarrow$ 

SSA



Two is not enough!

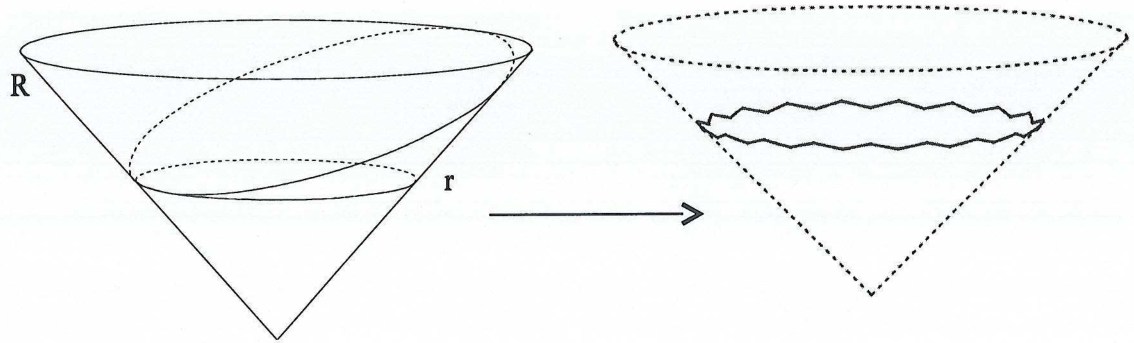
circles rotated an angle 2πk/N with k = 1, 2, ..N around a point different from the center.
In the infinite N limit, the sets look like circles centered at the same point.

$$
\begin{aligned}
& S(A)+S(B)+S(C) \\
\geq\ & S(A\cap B)+S(A\cup B)+S(C) \\
\geq\ & S(A\cup B\cup C)+S((A\cup B)\cap C)+S(A\cap B) \\
\geq\ & S(A\cup B\cup C)+S(((A\cup B)\cap C)\cup(A\cap B))+S(A\cap B\cap C) \\
=\ & S(A\cup B\cup C)+S((A\cap C)\cup(A\cap B)\cup(B\cap C))+S(A\cap B\cap C)
\end{aligned}
$$

- Equal number of regions on both sides of the inequality
- Regions on right hand side ordered by inclusion
- Totally symmetrical with respect to permutation of the regions

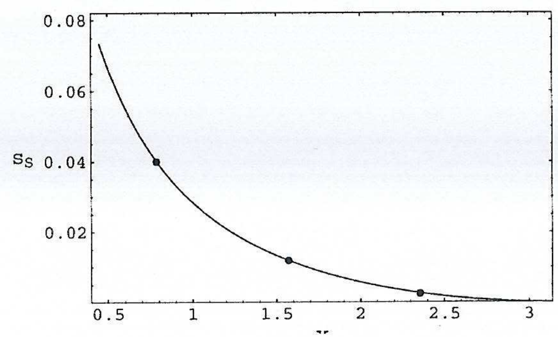
$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i\cap X_j)) + S(\cup_{\{ijk\}}(X_i\cap X_j\cap X_k)) + \ldots + S(\cap_i X_i)$$

N rotated circles on the light cone



As we approach the light cone the angles go to π and the perimeters of the wiggled regions approach the ones of circles of the same radius

$$S(\sqrt{Rr}) \geq \frac{1}{\pi}\int_0^\pi dz\, S\left(\frac{2rR}{R+r-(R-r)\cos(z)}\right)$$



Coefficient of the logarithmically divergent term for a free scalar field

In the large N limit
Infinitesimal inequality $\quad S'' \leq 0$

Running of the constant term

Interpolating function $\quad c_0 = r\,S'(r) - S(r) \qquad c_0' \leq 0$

Dimensionless and decreasing

At fixed points $\quad S(r) = c_1 r - c_0$ and $c_0(r) = c_0$

This is the constant term of the entropy of the circle

$$\Delta c_0 = c_0^{uv} - c_0^{ir} = -\int_0^\infty dr\; r\, S'' \geq 0$$

C-theorem

a) A dimensionless quantity independent of regularization.
b) At the infrared fixed point must not depend on the UV nor on the particular path of the RG running from UV to IR
c) Decreasing from UV to IR

$$\boxed{\begin{array}{c}\text{EE of a circle}\\ S(R) = c_1 R - c_0\end{array}}$$

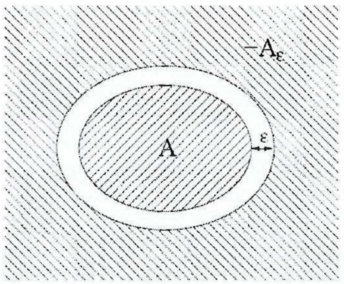
→ R has an ambiguity in its definition (cutoff dependent)

$c_1 \to \frac{a}{\epsilon} + b$ «contamination» from UV to IR

- Example in a lattice:
  $R = n\epsilon$ or $R = (n+1)\epsilon$

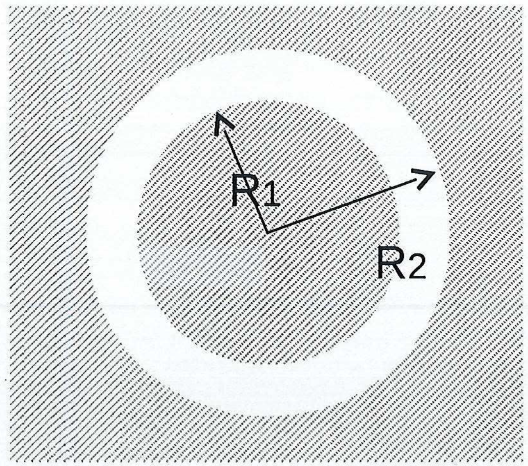
For a) and b)

Mutual information → a)



$$S(A) \overset{\epsilon \to 0}{=} \frac{1}{2} I(A, -A_\epsilon)$$ For pure global states

Regularized entropy: all coefficients on the expansion are universal



Parametrization:

$$R_1 = R - (1/2 - \alpha)\epsilon \text{ and } R_2 = R + (1/2 + \alpha)\epsilon$$

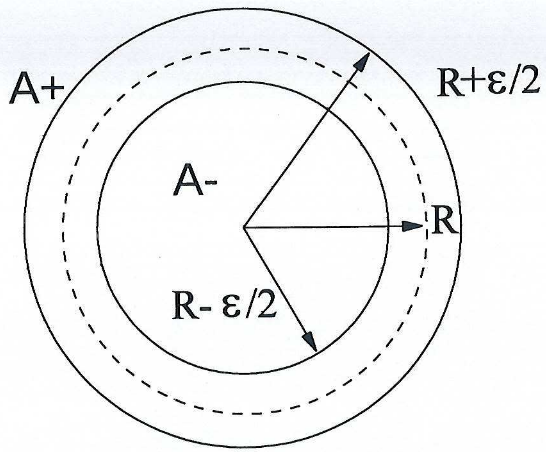
R2 – R1 =

$$I(A,B) \sim 2\left((\frac{\tilde{a}}{\epsilon} + \tilde{b})R - \tilde{c}_0 \right) + \mathcal{O}(\epsilon)$$

Mutual information as a geometric regulator for EE: all coefficients on the expansion are universal and well defined

The symmetric choice

$\boxed{\alpha = 0}$



$R_2 = R + \epsilon/2$

$R_1 = R - \epsilon/2$

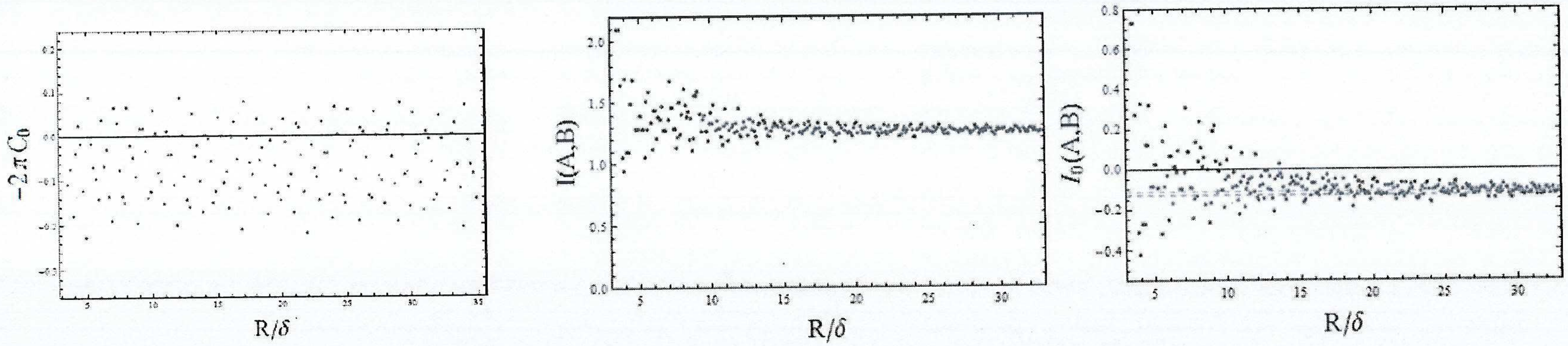
- b) IR , UV depend only on the CFT
- Precise prescription for the regularization of EE

$$S(r) = \left(\frac{a}{\delta} + b\right) r + c_0$$

$$I(R+\epsilon/2, R-\epsilon/2) = S_+(R+\epsilon/2) + S_-(R-\epsilon/2) - \cancel{S(R+\epsilon/2, R-\epsilon/2)} \longrightarrow I(A^+, A^-) = 2\left(\frac{\tilde{a}}{\epsilon} + \tilde{b}\right) R + 2C_0$$

c_0 (annular strip) = 0

H.Casini., M. H, R.C. Myers, A. Yale



$$c_0 = \frac{1}{2^4}\left(-2\log 2 + \frac{3\zeta(3)}{\pi^2}\right) \approx -0.0638070547\!8$$

Scalar field

C charge well defined through mutual information
IR , UV values depend only on the CFT
This is a physical quantity calculable with any regularization, including lattice
The constant term coincides with the one in the entropy of a circle for «good enough» regularizations