In the continuum:

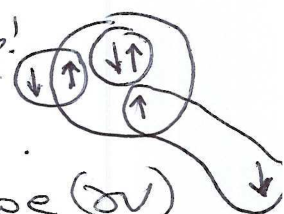
$$S(V) = \frac{g_{d-2}}{\epsilon^{d-2}}(\partial V) + \dots + \frac{g_1(\partial V)}{\epsilon} + g_0(\partial V)\log\epsilon + S_0(V)$$

the EE admits an expansion of the form

d - space-time dimension

(it could be $\neq$ the exp. in ϵ for interactive theories)

divergences!



- in the limit ~~$\epsilon \to 0$~~ EE is divergent
- g_i are [non universal, depend on the regularization.
  local and extensive functions of the edge (∂V)
  due to the UV origin of divergences
- g_0 universal: anomaly in even d and smooth ~~edge~~ boundary
- Area law leading divergent term $\frac{A}{\epsilon^{d-2}} \longleftrightarrow \frac{A}{4G}$ - black hole entropy

$\neq$ language in QFT, this term is non universal, depends on details microscopic

from a different point of view the area law tells: $S \sim L^{d-2}$

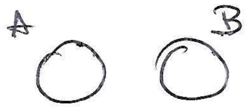
implies a reduction in the Hilbert space dimension from $2^{Vol} \to 2^{Area}$

(many body syst

- Finite terms → thermal entropy $\sim V$
  - $-\gamma$ topol.
  - $(k_F L)^{d-2}\log(k_F L)$ — sup Fermi

↓ depend on [shape, state, theory

- **Mutual information** $I(A,B)$



$= S(A) + S(B) - S(A \cup B)$

↓ local terms in the boundary are substructed

and Relative entropy.

To eliminate ambiguities we have two options

Relative entropy ← (A) substructing $S_{\rho_1}(V) - S_{\rho_2}(V)$ for two $\neq$ states

Mutual inf. ← (B) substracting $S_\rho(V_1) + S_\rho(V_2) - S_\rho(V_1 \cup V_2)$

for $\neq$ regions.

figure

Mutual Information properties.

- $I(A,B) \geq 0$

$\downarrow$

due to subadditivity of S

$S(AB) \leq S(A) + S(B)$ in general

- $I(A B) \leq I(A B C)$

" increasing with n° of degrees of freedom "

" order by inclusion "

$\Longleftrightarrow \quad S \leq A \qquad S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$

- I is an upper bound for correlations

$$I \geq \frac{1}{2} \frac{|\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle|^2}{|O_A|^2 \; |O_B|^2}$$

$O_A, O_B \rightarrow$ bounded operators.

Ex scalar

$\langle \phi \phi \rangle^2 \sim 1/x^2$

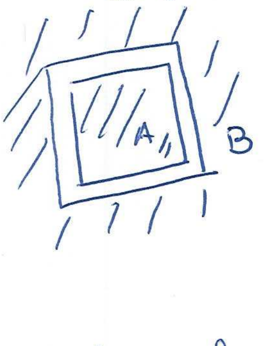
$\langle F_{\mu\nu}(0) F_{\mu\nu}(x) \rangle^2 \sim 1/x^6$.

(Cardy 2013)

- I as a "regularized entropy"

$$I(A,B) \sim S(A) + S(B) - \underbrace{S(A \cup B)}_{\sim 0} \sim 2 S(A)$$

$$S(B) = S(A^c) = S(A)$$



divergencies of $S \Longleftrightarrow$ A approaches B

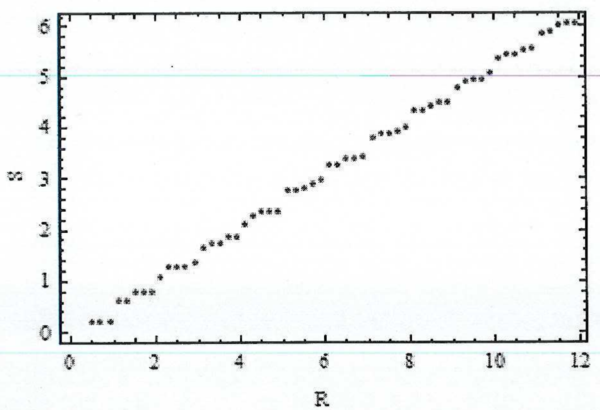
Example of ordering:

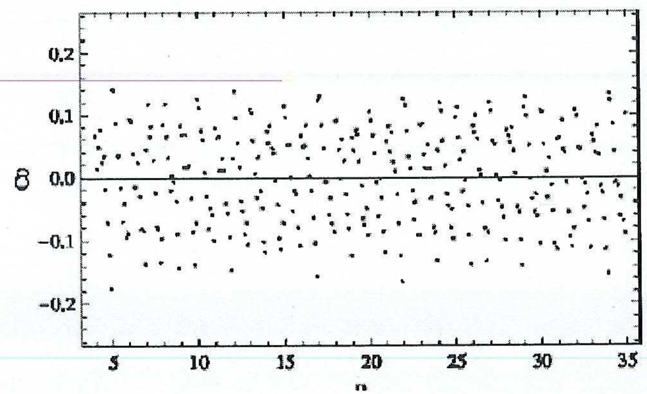
$\underbrace{\text{Scalar } m=0}_{\phi_i,\ \Pi_i}$ versus $\underbrace{\text{gauge } (2+1)\text{ dim.}}_{\substack{E_i = \partial_i \phi \\ B = \Pi = \partial_0 \phi}}$ $\rightarrow$ only derivatives

$\Downarrow$

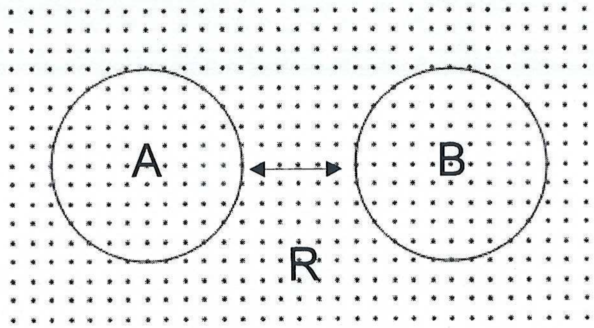
$I_{gauge} < I_s$

How to extract unambiguous information from the finite term?





Circles in a square lattice (no log term): $S(R) = c_1 R + c_0$

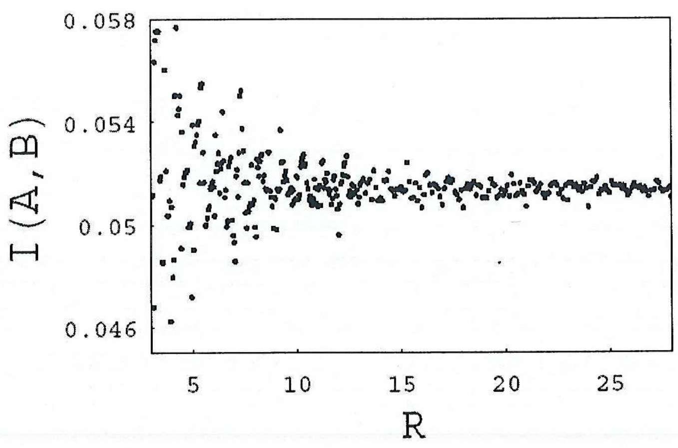


$R_A = R_B = R$

Mutual information $I(A,B) = S(A) + S(B) - S(A \cup B)$

The boundary divergences cancel out in the combination.

$I(A,B) \neq 0 \longrightarrow$ S cannot be finite



Very little large distance entanglement
Compare I(A,B)=0.05 with log(2)=0.69
Less than 1/10 bit for infinitely many degree of freedom!

A lot of short distance entanglement:
I(A,B) diverges when A and B touch each other.
This reflects the locality of the theory

The replication method

Fundamental idea : S can be express as the limit $n \to 1$ of the S_n known as Renyi entropies.

$$\longrightarrow \quad S_n = \frac{1}{1-n} \log (\operatorname{tr} \rho^n)$$

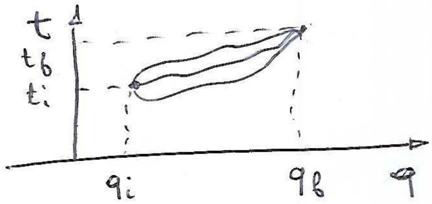
this gives an advantage : ρ^n has a functional int. representation

the final step, the limit $n \to 1$ we need an analitic continuation.

- VACUUM WAVE function in QM

path integral representation

$$\langle q_f t_f | q_i t_i \rangle = \int \mathcal{D} q(t)\, e^{i S[q(t)]/\hbar}$$



transition probability amplitude (propagator $K(q_f t_f ; q_i t_i)$)

$$\psi(q_f t_f) = \langle q_f t_f | \psi \rangle = \int dq_i \underbrace{\langle q_f t_f | q_i t_i \rangle}_{K(q_f t_f ;\, q_i t_i)} \langle q_i t_i | \psi \rangle$$

decomposition

Using eigen states of the hamiltonian:

$$\langle q_f t_f | q_i t_i \rangle = \langle q_f | e^{-i\frac{H(t_f - t_i)}{\hbar}} | q_i \rangle =$$

$$\sum_n \langle q_f | n \rangle e^{-i\frac{E_n (t_f - t_i)}{\hbar}} \langle n | q_i \rangle$$

From here we see that

Taking $t_f = 0$ and $t_i = t = i\tau$ with $\tau \to \infty$ (more general $t_f - t_i = -i\infty$)

$$\langle q_f | 0 \rangle e^{-E_0 \tau} \langle 0 | q_i \rangle$$

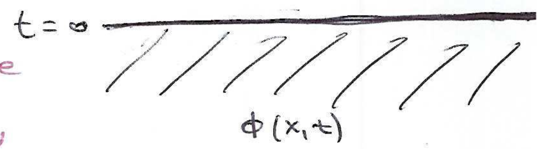
we select the vacuum state only!

$$\Rightarrow \boxed{\psi_0(q) \sim \int_{-\infty}^{t=0,\, q} \mathcal{D} q\, e^{-S_E(q)}}$$

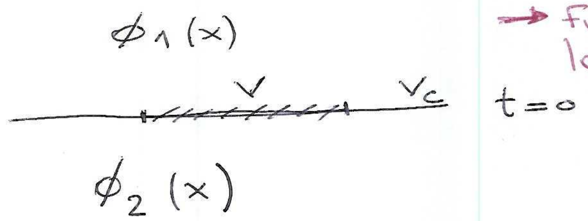
Analogously in QFT, the wave functional of the vacuum is

$$\Phi_0(\phi(x,t=0)) = N \int_{\phi(-\infty,x)=0}^{\phi(t=0,x)} \mathcal{D}\phi \; e^{-S_E[\phi]}$$

(1+1)



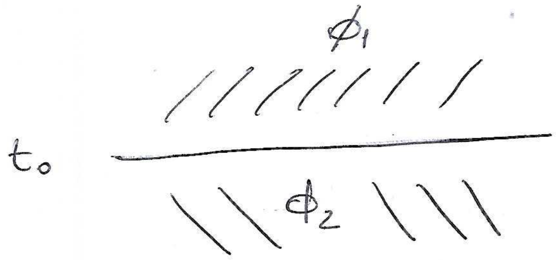
→ Functional int over the lower half plane and euclidean action



(basis of eigenvectors at $t=0$. $\hat{\phi}(x,0)|\phi_i\rangle = \phi_i(x)|\phi_i\rangle$)

The vacuum density matrix $\rho_0 = \Phi_0(\phi_1)^* \Phi_0(\phi_2)$

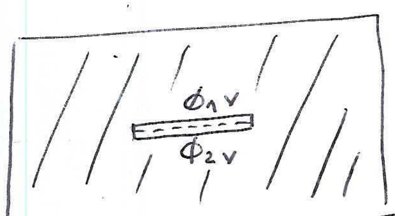
$\rho_0 = |0\rangle\langle 0|$



Now suppose you are interested in the local density matrix associated to a region V ⇒ you need a partial trace over −V or V_c

$$\rho_V[\phi_1,\phi_2] = \int_{\substack{t=0\\ x\in V^c}} \mathcal{D}\phi \; \Phi(\phi_1)^* \Phi(\phi_2) = \int_{\phi(x,0^-)=\phi_2^V}^{\phi(x,0^+)=\phi_1^V} \mathcal{D}\phi \; e^{-S_E(\phi)}$$

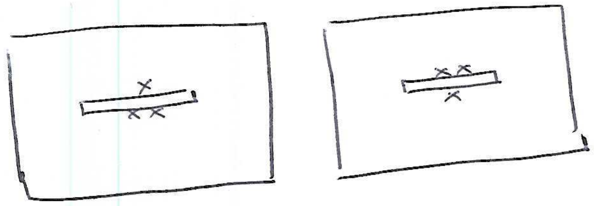
two copies of the half plane, glued on V_c



$\equiv \rho_V$

boundary condit on V → the arguments of ρ_V are in fact the boundary conditions on each side of the cut

$$\rho^2_V[\phi_1,\phi_2] = \int \mathcal{D}\phi' \; \rho_V[\phi_1\phi'] \, \rho_V[\phi'\phi_2]$$

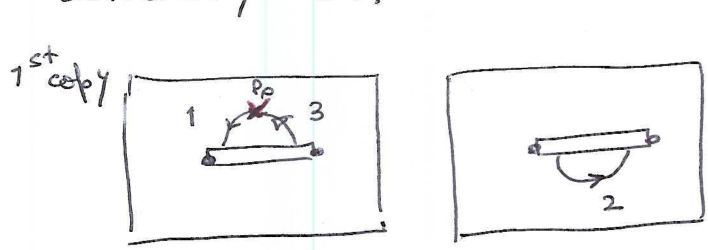


2 copies.

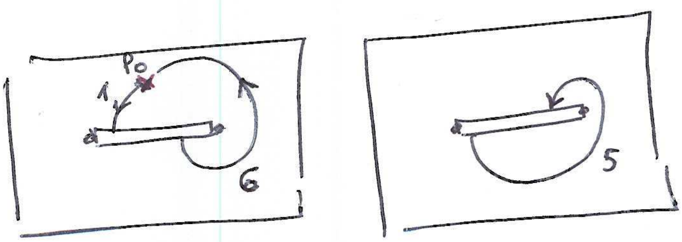
$$\mathrm{tr}\, \rho^2_V = \int \mathcal{D}\phi'' \mathcal{D}\phi' \; \rho_V[\phi''\phi'] \rho[\phi',\phi'']$$

In general, the representation of the traces $\mathrm{tr}\rho^n_V$ by a functional integral is realized by the "replication method" which consists on taking n copies of the Euclidean manifold (a plane cut along V) and sewing together the upper side of the cut in the k^{th} copy with the lower one of the $(k+1)^{th}$ for $k = 1,\dots,n$. the $n+1$ coincides with the first one

$\Rightarrow$ the resulting space is a n-sheeted $d+1$ dim. Euclidean space with conical singularities of angle $2\pi n$ located at the boundary ∂V.



1º Path:
no singularity
the cut is not detected



2º path:
if the path contains ∂V $\Rightarrow$ we have to turn twice to arrive to
...

Finally we have

$$\mathrm{tr}\,\rho^n = \frac{Z(n)}{Z(1)^n}$$

$Z(n)$ → functional integral on the n-sheeted manifold

$Z(1)^n$ → normalization factor $\mathrm{tr}\,\rho_V = 1$

$$\boxed{S_n = \frac{\log Z(n) - n \log Z(1)}{1-n}}$$

Good news!

Free fields

The calculation of $Z(n)$ in this complicated manifold can be avoided.

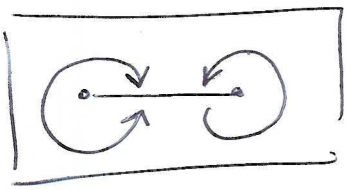
Instead of considering n copies you can introduce a vector field $\begin{pmatrix}\phi_1\\ \vdots \\ \phi_n\end{pmatrix}$ with special boundary conditions living in one copy. The price is that the field is multivalued

$$\rightarrow \quad \bar{\phi} = \begin{pmatrix}\phi_1\\ \vdots \\ \phi_n\end{pmatrix}$$

ϕ_1 → lives in the 1st copy

In this picture the conical singularities appear as the multivaluedness of the fields.

In crossing V,



the field gets multiplied by a matrix T o T^{-1}

$$T = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ & & 0 & 1 \\ (\pm 1)^{n+1} & & & 0 \end{pmatrix}$$

Diagonalizing in the replica space

$$T = \begin{pmatrix} e^{i\frac{2\pi k}{n}} & & \\ & \ddots & \\ & & \ddots \end{pmatrix} \qquad k_s = 0, \dots, n-1 \qquad k_f = -\frac{(n-1)}{2} \dots \frac{n+1}{2}$$

↓

we obtain $\underline{\underline{n}}$ decoupled fields which satisfy

$$\tilde{\phi}_k(\bar{x}, 0^+) = e^{i\frac{2\pi k}{n}} \tilde{\phi}_k(\bar{x}, 0^-)$$

limits of the field as the variables approach ∀ from above and below the cut

In this picture

$$\boxed{S_n = \frac{1}{1-n} \sum_{k=0}^{n-1} \log Z[e^{i2\pi k/n}]}$$

↓

PARTITION FUNCT. corresponding to a field which acquires a phase $e^{i2\pi a}$

¿how ~~we~~ can we calculate Z?

A. ~~bosoniz~~ Direct calculation Dirac field - - - - multisegment

B. for quadratic actions (massive and massless) → heat kernel

problems with singularities / low order exp gives div. terms.

C. Green function $\frac{d}{dm^2} \log Z = -\frac{1}{2} \mathrm{tr}\, G$

$G = (-\nabla^2 + m^2)^{-1}$ on the manifold

(no general method)

(1+1) Massive scalar and fermion . one segment

D Holography → Using the CFT - AdS correspondence
minimal area Ryu-Takayanagi

E Conformal transformation → thermal entropy.
Modular Hamiltonian

Spheres, CFT.

EXAMPLE: Fermion $m=0$ in $1+1$ — CASINI - FOSCO - H. (2005)

multicomponent set

T — — — n intervals (a_i, b_i)
T^{-1}

. $\mathcal{L} = \sum_k \bar{\psi}^k i \not\partial \psi^k$ sum over copies

After diagonalization: $\tilde{\psi}_k \to e^{i\frac{k}{n}x}\tilde{\psi}_k$
$k = -\frac{(n-1)}{2}, \dots, \frac{n-1}{2}$

→ I have to find $Z[\psi_k]$ in a space (one copy) with boundary condit along the cut

You can mimic the phases by coupling the field to gauge field: pure gauge everywhere except in the singularities

$$\oint_{C_a} dx^\mu A_\mu^k = -\frac{2\pi k}{n} \quad ; \quad \oint dx^\mu A_\mu^k = \frac{2\pi k}{n}$$

$$\psi_k \to e^{-i\int_{x_0}^{x} dx'_\mu A^\mu(x')}\psi_k$$

vortex like
$$\varepsilon^{\mu\nu}\partial_\nu A_\mu^k = \frac{2\pi k}{n}\sum_{i=1}^{p}\left(\delta(x-a_i) - \delta(x-b_i)\right)$$

$$\Delta\mathcal{L} = \bar{\psi}^k \gamma^\mu A_\mu \psi^k$$

⇓

$$Z[e^{i2\pi k/n}] = \langle e^{i\int A_\mu^k j_k^\mu d^2x} \rangle$$

vacuum exp-values in the free
Dirac theory

where J_μ is the Dirac current

<u>Bosonization</u> $\quad J^\mu_\mu \to \frac{1}{\sqrt{\pi}} \varepsilon^{\mu\nu} \partial_\mu \phi$

$$e^{i \int A_\mu^k \frac{1}{\sqrt{\pi}} \varepsilon^{\mu\nu} \partial_\nu \phi \, d^2x}$$

$$\downarrow$$

$$e^{-i\sqrt{4\pi}\, \frac{k}{n} \sum_{i=1}^{p} (\phi(a_i) - \phi(b_i))} \to \boxed{\text{vertex operators}}$$

Now $\langle \; \rangle$ correspond to the theory of a scalar field $\quad \mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$

For quadratic Lagrangians we have

$$\langle e^{-i\int f(x)\phi(x) d^2x} \rangle = e^{-\frac{1}{2}\int f(x) G(x-y) f(y) d^2x d^2y}$$

$$G(x-y) = -\frac{1}{2\pi} \log|x-y|$$

$$\to \log Z_k = -\frac{2k^2}{n} \Big(\overset{\ominus(\nu)}{\sum_{i,j}} \log|a_i - b_i| - \sum_{i<j} \log|a_i - a_j|$$
$$- \sum_{i<j} |b_i - b_j| - p \log(\varepsilon) \Big)$$

cutoff to split coincidence points $|a_i - a_i|$ $|b_i - b_i|$

Summing over k

$$S_n = \frac{1}{6}\left(\frac{n+1}{n}\right) \ominus \to S = \frac{1}{3} \ominus$$

For one interval this is $\quad S = \frac{1}{3} \log\left(\frac{L}{\varepsilon}\right)$

In general for any CFT $\quad S = \frac{c}{3} \log\left(\frac{L}{\varepsilon}\right)$

<u>Comments</u> A. $S = \frac{1}{3} \Xi(V)$

cannot be generalized to any CFT just replacing 1 by c

B. For non intersecting sets A, B, C and massless field

$$I(A, B \cup C) = I(A,B) + I(A,C) \quad \underline{\underline{\text{EXTENSIVE!}}}$$

$$I(A,B) = S(A) + S(B) - S(A \cup B) \quad (\text{definition})$$

C. <u>massive case</u> $\mathcal{L}_\phi = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + \lambda \cos(\sqrt{4\pi}\, \phi)\right)$

$\Rightarrow$ sine-gordon correlators of vertex operator $\langle V_a V_{-a} \rangle$

$e^{i\sqrt{4\pi}\, a \phi}$

Form factor expansion

long distance expansion in terms of intermediate particle states.

This is consistent with result in terms of Painlevé diff. equations in ref [ ] used to solve massive cases for one interval (scalar and fermion

D. Real time approach : multicomponent massive case

$$S(V) = -\int_{1/2}^{\infty} d\beta \, \mathrm{tr}\left[\left(\beta - \tfrac{1}{2}\right)\left(R(\beta) - R(-\beta)\right) - \frac{2\beta}{\beta + \frac{1}{2}} \right.$$

$$R = \left(C - \tfrac{1}{2} + \beta\right)^{-1}$$

$$C = \frac{1}{2}\delta(x-y) + \frac{m}{2\pi} K_0(m|x-y|)\gamma^0 + \frac{im}{2\pi} K_1(m|x-y|)\gamma^3$$