Lecture 2:

- Putting QFT's on curved manifold $M \to Z[M]$.
  - IR finite
  - UV ambiguities (parametrized by counterterms constructing out of background fields
    $\left\{ \begin{array}{l} \text{- metric on } M. \\ \text{- coupling constant} \end{array} \right.$

Goal is to compute $Z[M]$ exactly. First we need to understand how to put SUSY theories on curved background. Two approaches:

Ⓐ ~~For~~ S^D : ① Realize SCA_D on the fields. (canonical)

$$\delta\varphi \longrightarrow \delta\varphi = \delta_0\varphi + \underbrace{\frac{1}{r}\,\delta_1\varphi}_{\text{uniquely fixed by Weyl covariance}}$$

$$\eta_{\mu\nu} \qquad\qquad g_{\mu\nu}$$

$$\nabla_\mu \epsilon = 0 \qquad\qquad \nabla_\mu \epsilon = \frac{1}{D}\gamma_\mu \not{\nabla}\epsilon$$

i.e ~~[illegible]~~ $\left\{ \begin{array}{l} g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu} \\ \varphi \to e^{-\Delta\Omega}\varphi \end{array} \right. \qquad \delta(e^{-\Delta\Omega}\varphi) = e^{-\Delta\Omega}\,\delta\varphi$

e.g: ~~[illegible]~~ $\varphi \quad \not{\partial}\varphi\,\epsilon + \frac{2\Delta}{D}\varphi\,\not{\partial}\epsilon$

$$\epsilon\,\not{\partial}\chi + \frac{2\Delta+1-D}{D}\,\cancel{\chi}\,\not{\partial}\epsilon$$

δ_ϵ realizes superconformal algebra SCA_D.

2. Restrict to transformations that generate the isometry subalgebra G on M.

3. Find action invariant under G

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{r}\mathcal{L}_1 + \frac{1}{r^2}\mathcal{L}_2$$

a. SUSY algebra is deformed. away from Poincaré

b. " transformation "

c. $\mathcal{L}$ "

Ⓑ SUGRA approach:

- couple SUSYc QFT to SUGRA (background, not dynamical)

$$\mathcal{L}(\phi, g_{\mu\nu})$$

QFT multiplets (spin ≤ 1)

- vector multiplet
- chiral multiplet

SUGRA multiplet $(g_{\mu\nu}, \psi_{\mu\alpha}, \underbrace{\ldots}_{\text{auxiliary fields.}})$

$\mathcal{L}$ is invariant under SUGRA transformations. $\epsilon(x)$ arbitrary.

Idea is to couple QFT to a <u>fixed</u> <u>SUSYc</u> <u>SUGRA</u> background

$\delta\psi_\mu = 0$

δ fermi =

Generalization of conformal killing spinor equation. 2d $N=(2,2)$ SUGRA.

$$(D_\mu - iA_\mu)\epsilon = -\frac{1}{4}(1+\gamma_3)H\gamma_\mu\epsilon - \frac{1}{4}(1-\gamma_3)\bar{H}\gamma_\mu\epsilon$$

$$(D_\mu + iA_\mu)\bar{\epsilon} = -\frac{1}{4}(1-\gamma_3)H\gamma_\mu\epsilon - \frac{1}{4}(1+\gamma_3)\bar{H}\gamma_\mu\bar{\epsilon}$$

Localization of Path Integrals :

Consider the partition function of a SUSYc QFT given by a path integral

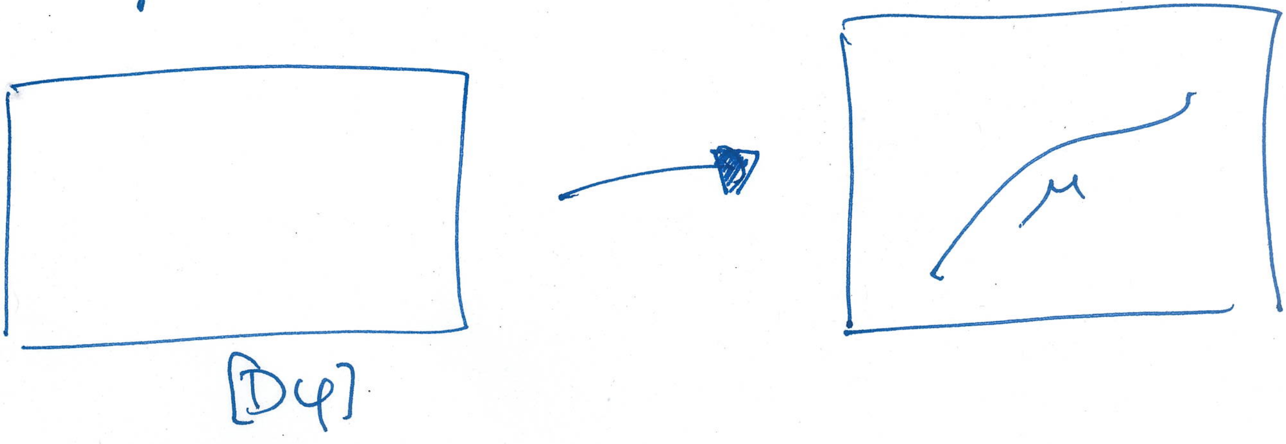
$$Z = \int [D\varphi]\, e^{-S[\varphi]}$$

- Consider inserting operators $\mathcal{O}$, disorder singularities or boundary conditions invariant under some supercharge Q. C G

Want to compute :

$$\int [D\varphi]\, e^{-S[\varphi]}\, \mathcal{O}(\varphi)$$

The magic of localization is that the path integral localizes from



How does this happen

$$Z(t) = \int [D\varphi]\, e^{-S[\varphi] + t\{Q, V\}}\, \mathcal{O}(\varphi)$$

Such that $Q^2 = B$

1. $\{Q, S\} = \{Q, \mathcal{O}\} = 0$
2. $\{Q, \{Q, V\}\} = 0$ $\quad \{B, V\} = 0$
3. $\{Q, V\}$ is non-degenerate and does not change the behavior at ∞ in the space of fields.
4. $\{Q,V\}_{\text{bosonic}} \geq 0$

Then $\partial_t Z = 0$. Let's see that:

$$\partial_t Z = \int [D\varphi]\, \{Q, V\}\, e^{-S - t\{Q,V\}}\, \mathcal{O} \qquad \delta\langle \mathcal{O}\rangle = 0$$

$$= \int [D\varphi]\, \{Q, e^{-S - t\{Q,V\}}\, \mathcal{O}\} = \text{total derivative} = 0$$

$$Q = \int dx\, \delta_\epsilon \Phi \frac{\delta}{\delta \Phi} \Rightarrow Z(0) = Z(t \to \infty)$$

- Evaluate partition function at $t \to \infty$. Semiclassical wrt to $\hbar_{\text{eff}} = \frac{1}{t}$ $\Longrightarrow$ 1) 1-loop exact wrt t

2) Exact wrt to all other parameters

$\Longrightarrow$ μ: saddle points of $\{Q, V\}$ of $\sigma - {}^{\{Q,V\}}$action $\Longrightarrow$ $\boxed{\text{System of PDE's}}$

$$Z = \int_\mu d\varphi_0\, \underset{\uparrow}{Z_{1\text{-loop}}(\varphi_0)}\, e^{-S[\varphi_0]}\, \mathcal{O}(\varphi_0)$$

one-loop determinant of $\{Q, V\}$ around saddle points labeled by φ_0

Rough intuitive argument. Fixed pts of SUSY.

Comments:

- Freedom in the choice of Q and $\{Q, V\}$.
- Different choices can lead to completely different representations of the <u>same</u> observable

$$Z_{\mu_1} \underset{\substack{\uparrow \\ \text{non-trivial dynamics.}}}{=} Z_{\mu_2}$$

Same logic allows one to proof decoupling theorem.

- Consider an open coupling λ that multiplies an operator $\mathcal{O}$

$$\int d^D x \; \lambda \mathcal{O}$$

  - gauge coupling
  - Yukawa couplings, . . .

If $\mathcal{O} = \{Q, \mathcal{O}'\} \Longrightarrow Z$ is independent of λ !

- $D < 4$ $[g^2] \geq 0$ can prove that g^2 decouples

$$\begin{tikzcd} UV \arrow[d, squiggly] & g^2 \\ IR & g^2 \to \infty \end{tikzcd}$$

Z is a RG-invariant observable

· Lecture 2

Today we will discuss the exact computation of correlators in $D=2$ $\mathcal{N}=(2,2)$ QFT's (same SUSY as $D=4$ $\mathcal{N}=1$)

- Such theories can exhibit the exciting phenomena of $D=4$ gauge theories (e.g NLSM)
  - mass gap
  - chiral symmetry breaking
  - instantons

    ⋮

- Play an important role in string theory:

$$\begin{tikzcd} \Sigma \arrow[r] & X \\ \text{D=2 QFT on worldsheet} \arrow[u] & \text{target space} \arrow[u] \end{tikzcd}$$

- As we shall see, these yield the exact answer to
  - worldsheet correlators (Σ)

    $\updownarrow$

  - Low energy couplings (Yukawa couplings, metric in field space, ...)

$D = 2$ $\mathcal{N} = (2,2)$ SUSY on S^2

~~Consider the S^2 partition function of 2d $\mathcal{N} = (2,2)$ Q~~

$$\begin{tikzcd}
 & \mathcal{N}=(2,2) \text{ SCA} \arrow[dl, "S^2"'] \arrow[dr] & \\
SU(2|1)_A & & SU(2|1)_B
\end{tikzcd}$$

A Key observation is that :

$SU(2|1)_A$

$$\begin{cases} \nabla_\mu \epsilon = \frac{i}{2r} \gamma_\mu \epsilon \\ \nabla_\mu \bar{\epsilon} = \frac{i}{2r} \gamma_\mu \bar{\epsilon} \end{cases}$$

$SU(2|1)_B$

$$D_\mu \tilde{\epsilon} = \frac{i}{2r} \gamma_\mu \hat{\epsilon} \qquad \hat{\epsilon} = \epsilon_+ + \bar{\epsilon}_-$$

$$D_\mu \tilde{\bar{\epsilon}} = \frac{i}{2r} \gamma_\mu \bar{\hat{\epsilon}} \qquad \bar{\tilde{\epsilon}} = \bar{\epsilon}_+ + \epsilon_-$$

$D=2$ $\mathcal{N}=(2,2)$ SUSY on S^2

$$\left\{\begin{array}{l}\underline{a}\ \text{SUSY algebra}\\ \underline{b}\ \text{Supergeometry}\\ \underline{c}\ \text{Representation on fields.}\end{array}\right.$$

$$\begin{tikzcd} & SU(2|1)_A \\ -\ \mathcal{N}=(2,2)\ SCA \arrow[ur] \arrow[dr] & \\ & SU(2|1)_B. \end{tikzcd}$$

- Described by the SUSY algebra $SU(2|1)_A$. Interesting commutat

$$\{\bar{Q}_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2}\epsilon_{\alpha\beta} R. \qquad \begin{array}{l} m=1,2,3 \\ \alpha,\beta=1,2 \end{array}$$

J_m : $SU(2)$ isometry generators of

R : vector R-symmetry $[R, Q_\alpha] = -Q_\alpha$ $[R,\bar{Q}_\alpha]=\bar{Q}$

Comments:

- subalgebra of $D=2$ $\mathcal{N}=(2,2)$ superconformal algebra preserving S^2 isometries
- In the $r\to\infty \Rightarrow \mathcal{N}=(2,2)$ Super-Poincaré

$$\left(\begin{array}{ll} Q_\alpha \to \sqrt{r}\,Q_\alpha & J_i \to r\epsilon_{ij}P_j \\ \bar{Q}_\alpha \to \sqrt{r}\,\bar{Q}_\alpha & J_3 \to J_3 \end{array}\right)$$

- $SU(2|1)_A$ realised geometrically by killing vectors and the conformal killing spinors.

$$Q \longleftrightarrow \nabla_i \epsilon = \frac{1}{2r}\gamma_i \cancel{\gamma_3}\,\epsilon$$

$$\bar{Q} \longleftrightarrow \nabla_i \bar{\epsilon} = +\frac{1}{2r}\gamma_i \cancel{\gamma_3}\,\bar{\epsilon}$$

$\xi^i = \bar{\epsilon}\gamma^i\epsilon$ is a killing vector

- Symmetries of superspace $\dfrac{SU(2|1)}{U(1)_3 \times U(1)_R}$.

SUSY transformations on fields

- Find representation of $SU(2|1)$ on supermultiplets
- Multiplets by dimensional reduction of $D=4$ $\mathcal{N}=1$ multiplets:

vector multiplet: $V = V^\dagger$ $\quad V \to V + i(\Phi - \Phi^\dagger)$

$(A_i, \sigma_1, \sigma_2, \lambda, \bar{\lambda}, D)$

chiral multiplet $\quad \bar{D}_+ \Phi = \bar{D}_- \Phi = 0$

$(\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F})$

Extra multiplets allowed in $D=2$ (richness in constructing model

twisted chiral multiplet $\quad \bar{D}_+ Y = D_- Y = 0$

(allowed by Lorentz invariance)

(important example: $\Sigma = \bar{D}_+ D_- V$ such that $\mathcal{L}_{v.m} = \int d^4\theta\, \Sigma \Sigma^\dagger$)

$(Y, \bar{Y}, \chi, \bar{\chi}, G, \bar{G})$

twisted vector multiplet $\quad V = V^\dagger \quad V \to V + i(Y - Y^\dagger)$.

(not today)

$Z_A(\tau, \bar{\tau})$

$Z_B(z, \bar{z})$

There are two important quantities in these theories:

$W(z, \Phi)$: superpotential

$W(\tau, Y)$: twisted superpotential

z : background c.m.

τ : " t.c.m.

- let's now start computing observables in these theories; increasing in complexity

Landau-Ginzburg Models

- Effective field theory approach to critical phenomena

$\mathcal{L}(\phi)$ ϕ : order parameter (e.g: magnetization, density, ...)

- $V(\phi)$ controls the universality class:

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \lambda \phi^4 \xrightarrow[m=0]{} \text{Ising CFT}$$

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \lambda_1 \phi^4 + \lambda_2 \phi^6 \xrightarrow[m=\lambda_1=0]{} \text{Tricritical Ising CFT}$$

$$\begin{tikzcd} \text{UV} \arrow[d] & \mathcal{L}(\phi) \\ \text{IR} & \text{CFT} \end{tikzcd}$$

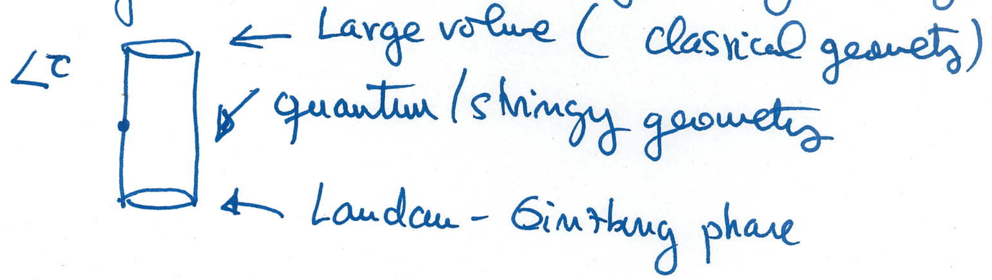
- There are $\mathcal{N}=(2,2)$ versions. Universality class controlled by:

$$W(Y) \quad \text{twisted superpotential}$$

- UV description of $\mathcal{N}=(2,2)$ SCFT Minimal Model

$$\begin{tikzcd} W = Y^{k+2} \arrow[d, squiggly] & \\ MM_k & c = \dfrac{3k}{k+2}. \end{tikzcd}$$

- Describe "non-geometrical phases" of string theory



- Dual description of certain NLSM's (tomorrow)
  $\Updownarrow$
  Mirror

SU(2|1) invariant Lagrangian

$$\mathcal{L}_{kin} = |D_i Y|^2 + i\bar{X}\gamma^i D_i X + G\bar{G}$$

Comments:

- Same as flat space Lagrangian (no $\frac{1}{r}$ corrections) (since $\mathcal{L}_{kin}$ is Weyl invariant).
- $\mathcal{L}_{kin}$ is Q-exact.
- Same is true if we allow a more general Kähler potential:

$$\mathcal{L}_{kin} = \int d^4\theta \, K(\dot{Y},\bar{Y})$$

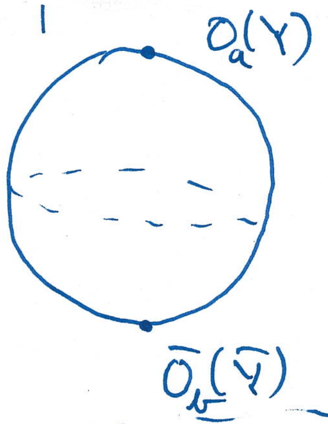
$\Rightarrow$ S^2 correlators independent of K!

$$\mathcal{L}_W = \underbrace{-i W'(Y)\, G - W''(Y)\, \bar{\chi}\, \frac{1-\gamma^3}{2}\, \chi}_{\text{same as flat space}} + \underbrace{\frac{i}{r} W(Y)}_{\frac{1}{r}\text{ correction.}}$$

$$\mathcal{L}_{\bar W} = - \qquad - \qquad +$$

Comments:

- $\mathcal{L}_W$ Not Q-exact. Correlators depend on $W(\cancel{Z},Y)$ and $\bar W(\cancel{\bar Z},\bar Y$
- $\frac{1}{r}$ associated to corrections to non-conformal couplings
- By choosing $Q^2 = J + \frac{R}{2}$. Can compute exact correlator



twisted chiral operators

not yet ↓

Computes the Zamolodchikov metric: Turn on $W = i \sum_a \tau_a O_a(Y$

~~$\mathcal{L} \to \mathcal{L} +$~~ $$\frac{\langle O_a(N)\, \bar{O}_{\bar b}(S) \rangle}{Z} = G_{a\bar b}$$

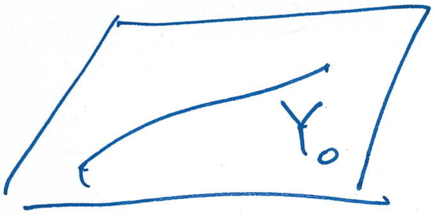
where in $\mathcal{N} = (2,2)$ SCFT

$$G_{a\bar b} = -\partial_a \partial_{\bar b} \log K(\tau_a, \bar\tau_a)$$

← Kähler potential on CFT moduli space

Saddle point of $\mathcal{L}\{Q, V\} = \mathcal{L}_{kin}$

$D_i Y = 0$ ⟹ localize to the 0-modes.



- 1-loop determinant of $\{Q, V\}$ around Y_0 is Y_0 independent

⟹

$$Z = \int dY d\bar{Y} \, e^{-4\pi i r W(Y) - 4\pi i r \overline{W(\bar{Y})}}$$

Comments:

- Reduce to a nice oscillatory integral
- Captures exact dynamics of LG theory.

Example: $W = Y^{k+2}$

$$Z = \frac{\Gamma\left(\frac{1}{k+2}\right)}{\Gamma(1 - 1/k+2)}$$

Comment:

1 On general grounds, can derive that for a CFT

$$Z = \underset{\uparrow \text{ Weyl anomaly}}{r^{c/3}} Z_0$$

get $c = \frac{3k}{k+2}$!

c for MM_k

Parenthesis: This follows from:

1) $\delta Z = - \int \sqrt{g} \, \langle T^{ij} \rangle \, \delta g_{ij}$

2) $r \to r + \delta r$ $\quad \delta g_{ij} = 2 g_{ij} \, \delta \log r$, $\delta Z = -2 \delta \log r \int \sqrt{g} \langle T_i^{\,i}$

3) $\langle T_i^{\,i} \rangle = \dfrac{-c}{24\pi} R$ $\quad C_{scalar} = 1.$

$\int_{S^2} \sqrt{g} R = 4\pi$

$\Rightarrow \quad Z \propto r^{c/3}$

From Z for $W = Y^{k+2} \Rightarrow c = \dfrac{3k}{k+2}$

2. Can we Z to give a LG realization of the CFT identity:

$$\begin{tikzcd}
MM_k \arrow[r, phantom, "\simeq"] & MM_k/\mathbb{Z}_{k+2} \\
Ⓐ \; W(Y) = Y^{k+2} \arrow[u] & Ⓑ \; W(\Phi) = \Phi^{k+2} \arrow[u] \\
 & \text{see later} \arrow[u]
\end{tikzcd}$$

$Z_A = Z_B$!

Simplest example of mirror symmetry! (more later)

3. Extend to mirrors of NLSM's including Calabi-Yau's.

$\Rightarrow$ Elementary computation of worldsheet instantons
(Gromov-Witten invariants

Gauge Theories:

- Filld content:

vector multiplet V $\longleftrightarrow$ gauge group G

chiral multiplet Φ $\longleftrightarrow$ representation R of G

- $SU(2|1)$ transformations obtained by Weyl covariantization

V: Weyl weight 0

Φ: " " q

- $SU(2|1)$-invariant Lagrangian: $\mathcal{L} = \mathcal{L}_0 + \frac{1}{r}\mathcal{L}_1 + \frac{1}{r^2}\mathcal{L}_2$

$$\mathcal{L}_1 = \underbrace{\frac{1}{g^2}\mathrm{Tr}(\sigma F_{\hat{1}\hat{2}})}_{\text{v.m.}} \underbrace{+ i(q-1)\bar{\phi}\sigma_2\phi - \frac{iq}{2}\bar{\psi}\gamma^3\psi}_{\text{c.m.}}$$

$$\mathcal{L}_2 = \underbrace{\frac{2q-q^2}{4}\bar{\phi}\phi}_{\text{c.m}} + \underbrace{\frac{1}{2g^2}\mathrm{Tr}\,\sigma_1^2}_{\text{v.m.}}$$

Comments

- $\mathcal{L}_{\mathcal{W}}$ has no $\frac{1}{r}$ corrections. (unlike $\mathcal{L}_W$)

- $\mathcal{L}_{\mathcal{W}}$ susyc iff $R[\mathcal{W}]=2$. (unlike $\mathcal{L}_W$)

Parameters

- g_Z : super-renormalizable gauge couplings for each gauge group factor G_Z

- Complexified marginal couplings τ for each $U(1)$ factor

$$\tau = \frac{\theta}{2\pi} + i\zeta$$

$$\mathcal{L}_{FZ} = -i\zeta \operatorname{Tr} D \qquad \mathcal{L}_{top} = -\frac{i\theta}{2\pi} \operatorname{Tr} F$$

Recast as a twisted superpotential coupling for $\Sigma = \overline{D}_+ D_-$

$$\widetilde{W}(\Sigma) = \frac{i}{2}\tau\,\Sigma$$

- $\mathcal{W}$ parameters
- W parameters
- Parameters in the Cartan of G_F, the flavour symmetry

↑ "Mass"

Holomorphic combination:

$$M_Z = \underset{\substack{\uparrow \\ \text{masses}}}{m_I} + \frac{i}{2r}\underset{\substack{\uparrow \\ \text{R-charges}}}{q_Z}$$

Decoupling Theorems:

- $\mathcal{L}_{v.m} = \{Q, V\} \Longrightarrow Z$ independent of g_Z

$$\begin{tikzcd} \mathrm{UV} \arrow[d, squiggly, "g_Z \to \infty"] \\ \mathrm{IR} \end{tikzcd}$$

captures IR, strongly coupled dynamics

- $\mathcal{L}_{\mathcal{W}} = \{Q, V'\} \Rightarrow Z$ independent of parameters in $\mathcal{W}$

BUT

- Depends crucially, however, on

  W: twisted superpotential (in particular τ).

  M_Z: complexified masses.

- This has important consequences for CY dynamics.

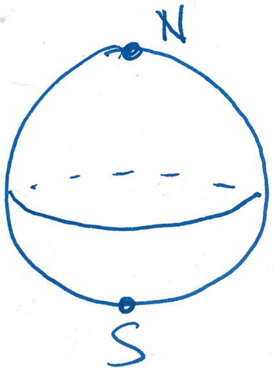
  parameters in W $\longleftrightarrow$ Kahler moduli of CY.

  parameters in $\mathcal{W}$ $\longleftrightarrow$ complex moduli of CY

  Z_{S^2} captures exact dynamics of Kahler moduli of CY!

Computation of partition function

- choose a Q such that $Q^2 = J + \frac{R}{2}$

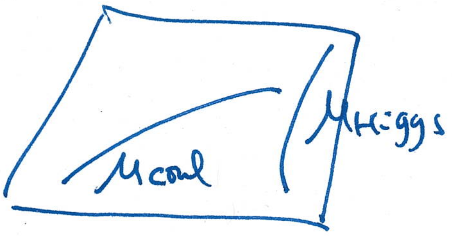


N, S pole are fixed points of J.

- Localize path integral by choosing a suitable deformation term $\{Q, V\} \Longrightarrow$ saddle points.

Rather interestingly, we find that different choice of $\{Q, V\}$ lead to inequivalent representations of the partition function:

$$\begin{tikzcd} Z = Z_{coul} \arrow[d] & = Z_{Higgs} \arrow[d] \\ Q\cdot V_{coul} & Q\cdot V_{Higgs} \end{tikzcd}$$



disconnected saddle points.

Z_{coul}: Localizes to field configurations for vector multiplet

$$\mathcal{L}_{v.m} \Longrightarrow \quad F = \frac{B}{2} vol(S^2)$$

$$\sigma_2 = a$$

$$\sigma_1 = -\frac{B}{2r}$$

$$[a, B] = 0$$

B: quantized magnetic flux ( )

a: zero mode on S^2

- Can compute one-loop determinant around saddle points by using monopole spherical harmonics.

$$Z_{coul} = \underbrace{\frac{1}{|W(G)|}\sum_B \int_{\mathfrak{t}} da}_{\text{integral over all saddle points}}\; e^{-4\pi i r \xi_{ren} \mathrm{Tr}\, a + i\theta \mathrm{Tr} B} \underbrace{\prod_{\alpha>0}\left[(\alpha\cdot a)^2 + \frac{\alpha\cdot B}{4r^2}\right]}_{\text{vector multiplet}} \underbrace{\prod_{\omega\in R} \frac{\Gamma(-ir\;\omega\cdot(a+M) - \frac{\omega\cdot}{2}}{\Gamma(1+ir\;\omega\cdot(a+M)) - \frac{\omega\cdot}{2}}}_{\text{chiral multiplet}}$$

where $\xi_{ren} = \xi - \frac{1}{2\pi}\sum_i Q_i \log(rM)$

Comments:

- Mellin-Barnes representation.
- Explicitly computable by residues (enclose poles of Γ-functions)
- Expression factorizes into a sum of blocks; Roughly

$$Z_{coul} = \sum_{v} |Z_v(\tau, M)|^2$$

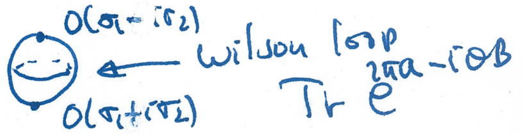
- $Z_v(\tau, M)$ are generalised hypergeometric functions

- Z_{coul} actually annihilated by a system of differential operators:

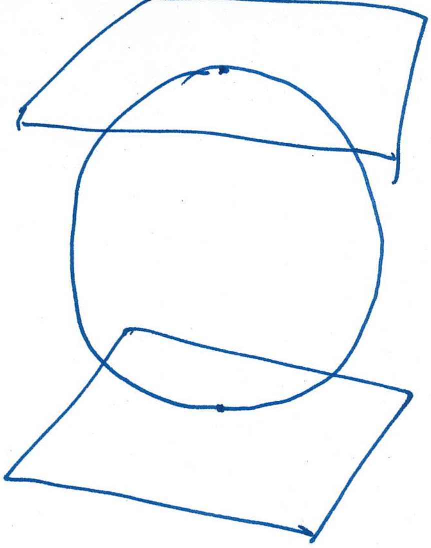
$$\mathcal{L}(\partial/\partial\tau)\cdot Z = 0 \qquad \bar{\mathcal{L}}(\partial/\partial\bar{\tau})\cdot Z = 0$$

There will have an elegant interpretation in terms of mirror symmetry (next lecture)

- Can insert other operators ⟶



Z_{Higgs} : Localizes to field configurations for chiral multiplet.



vortex : $D_{\bar{z}}\phi = 0$

$F = (\phi\bar{\phi} - \xi)$

$[(\sigma + m)\phi = 0]$.

anti-vortex: $D_z\phi = 0$

$F = -(\phi\bar{\phi} - \xi)$

$[(\sigma + m)\phi = 0]$.

- Sum over all vortices at the north pole and anti-vortices at the south pole
- Defines vortex partition function.

$$Z_{\text{vortex}}(\underset{\substack{\uparrow \\ \text{choice of vacuum}}}{\sigma}, m, e^{2\pi i\tau}) = \sum_{k=0}^{\infty} e^{2\pi i\tau k} \cdot \underbrace{\int_{M^k_{\text{vortex}}} e^{\omega}}_{\text{volume}}$$

- Putting them together

$$Z_{\text{Higgs}} = \sum_{\substack{\sigma \in \text{Higgs} \\ \text{vacua}}} e^{-S_0} \operatorname*{res}_{a=\sigma} Z_{1\text{-loop}} \left| Z_{\text{vortex}}(\tau, M) \right|^2$$

Indeed, explicitly $Z_{\text{coul}} = Z_{\text{Higgs}} = Z.$

Next we turn to some interesting applications of these results

. Lecture 3 :

- We now discuss some applications of these results.

- Z_{S^2} captures the exact answer to quantities of both physical and mathematical interest

- Outline these connections :

- Starting point : $D=2$ NLSM

$$\phi : \underset{\substack{\uparrow \\ \text{2D space} \\ \mathbb{R}^2, S^2, \Sigma_g, \ldots}}{\Sigma} \longrightarrow \underset{\substack{\uparrow \\ \text{target space manifold}}}{\underset{\sim}{X}} \qquad \mathcal{L} = \frac{1}{2} g_{mn} \partial_i \phi^m \partial^i$$

- Asymptotically free theory $R_{mn}(X) > 0 \Rightarrow$ shares many similarities with 4D gauge theories

- Important in the study of critical phenomena :

$D=4$

$4-\epsilon$ expansion $(\lambda\phi^4)$ $\qquad \beta = -\epsilon\lambda + |a| \cdot \lambda$

$\qquad \exists\ \beta(\lambda^*) = 0$

$D=3$ CFT

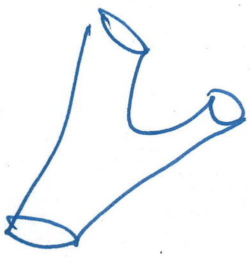
$2+\epsilon$ expansion (NLSM) $\beta = \epsilon\lambda - |b|\lambda^2$

$\qquad \lambda = \frac{1}{\xi} = \frac{1}{si}$

$D=2$

- Central to the worldsheet formulation of string theory

$$\phi : \underset{\substack{\uparrow \\ \text{wordsheet} \\ \text{QFT}}}{\Sigma} \longrightarrow X$$



Physically appealing models of particle physics:

$$X = \mathbb{R}^{1,3} \times M$$

$\Uparrow$ 6D space

⇒ UV complete, perturbative model.
$\uparrow$ quantum gravity

Rich and beautiful interplay between

2D ⟺ 4D physics.

We will see a particularly nice example of this.
2D spectrum of operators ⟷ 4D massless fields.
2D QFT correlators ⟷ 4D effective dynamics.

– NLSM's correlators can receive non-perturbative corrections
⟹ worldsheet instantons

$\bar\partial \phi = 0$ holomorphic maps from $\Sigma \longrightarrow M$

2D:

$$\langle O_1\, O_2\, O_3 \rangle = \langle O_1\, O_2\, O_3 \rangle_0 + \text{instanton contribution}$$

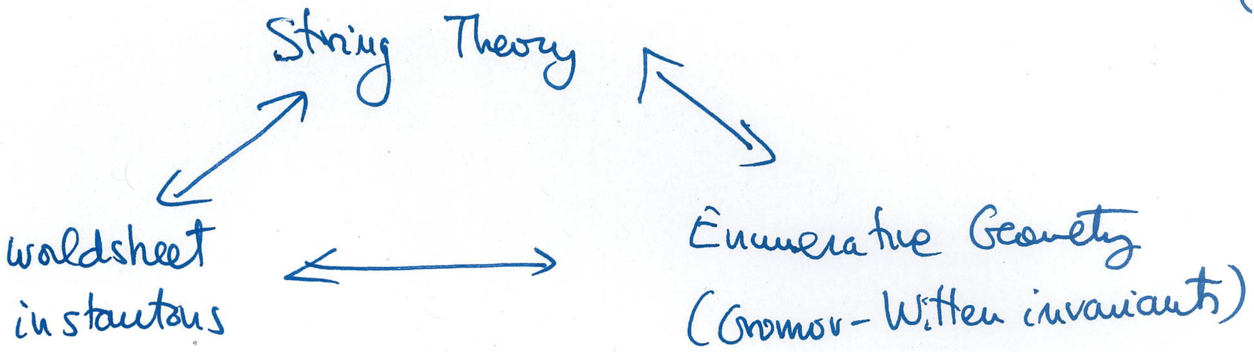
$\Uparrow$ point particle approximation (constant map)

$\Uparrow$ must sum over all homotopy classes of maps
$\Uparrow$ nonperturbative

$$e^{-S_0} = e^{-\text{Area}}.$$

4D: e.g:

– Yukawa couplings in D=4
when M is a so called Calabi-Yau manifold
$\Downarrow$
4D $\mathcal{N}=1$ or $\mathcal{N}=2$. QFTs. at low energies.



- Computation of worldsheet instantons ~~of~~ origin of remarkabl discoveries such as mirror symmetry (more later)

- Particularly interesting:

M is Calabi-Yau $\longleftrightarrow$ 2D $\mathcal{N}=(2,2)$ NLSM
(actually $\mathcal{N}=(2,2)$ SCFT)

~~Z_{S^2} computes exactly the worldsheet instantons for these !!~~

~~Basic Dictionary~~

~~Calabi-Yau has metric $ds^2=g_{m\bar{n}}\,dx^m\,dx^{\bar{n}}$~~
~~kähler~~
~~$\Updownarrow$~~
~~$J=\frac{i}{2}\,g_{m\bar{n}}\,dx^m\wedge dx^{\bar{n}}$ $dJ=0$~~
~~Kähler ~~form~~~~

Basic Dictionary :

4D massless fields $\longleftrightarrow$ moduli of CY $\longleftrightarrow$ marginal ope
in 2D SCF

$\mathbb{R}^{1,3}$ $\qquad\qquad$ M

What are the moduli?

CY admits a Kahler metric $\quad R_{mn} \doteq 0$

$\downarrow$

$g_{m\bar n}$ and $J = i\, g_{m\bar n}\, dx^m \wedge dx^n$ is closed.

Moduli are metric deformations that preserve $R_{mn}($~~g~~$+\delta g) = 0$

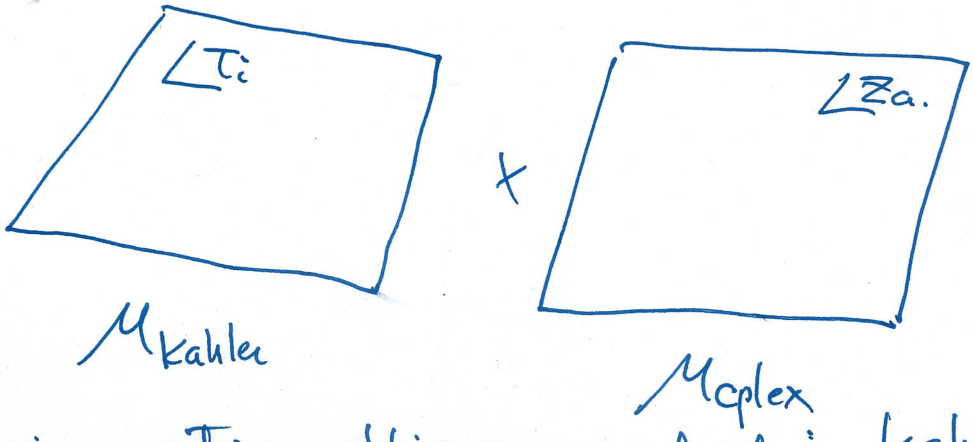
$\delta g_{m\bar n}$: Kahler deformations $\longleftrightarrow h^{1,1}(M) \longleftrightarrow$ .. $\tau_i\ \mathcal{O}_i$

$\delta g_{m\bar n}\, dx^m \wedge dx^{\bar n}$ $\qquad \{Q_A, \mathcal{O}_i\} = 0 \leftarrow$ twisted chiral.

$\delta g_{\bar m\bar n}$: Complex structure deformations $\longleftrightarrow h^{1,2}(M) \longleftrightarrow$ $\quad z_a\ \tilde{\mathcal{O}}_a$

$\Omega_{mnp}\, g^{p\bar q}\, \delta g_{\bar q\bar r}\, dx^m \wedge dx^n \wedge dx^{\bar r}$ $\qquad \{Q_B, \tilde{\mathcal{O}}_a\} = 0$ chiral ring.

Space of deformations



SUSY
$\Downarrow$
There is a metric on this spaces, which is also kahler :

$$\Rightarrow \quad G_{i\bar j} = \frac{\partial}{\partial \tau_i}\frac{\partial}{\partial \bar\tau_j} K_{\text{kahler}} \; ; \qquad G_{a\bar b} = \frac{\partial}{\partial z_a}\frac{\partial}{\partial \bar z_{\bar b}} K_{\text{cplex}}$$

- K_{kahler} and K_{cplex} are the so called Kähler potentials

- They capture some of the most important low energy coupling in 4D.

$\mathcal{L}_{4D} = C_{i\bar{j}k}\, \psi^i \bar{\psi}^{\bar{j}} \phi^k$ Yukawa couplings

$$C_{ijk} = \partial_i \partial_j \partial_k K_{kahler}.$$

Goal : Compute kahler potentials!

They are very different in nature :

1) $e^{-K_{cplex}} := i \int_M \Omega \wedge \bar{\Omega}$: classical, no α'-corrections.

2) $e^{-K_{kahler}} = \underbrace{\frac{i}{3!} \int_M J \wedge J \wedge J + c.c}_{} + \alpha'$ corrections

$\uparrow$ point particle approximation

large volume expansion, ∃ a semi-classical expansion (weak coupling)

∃ a canonical large volume expansion

$$e^{-K_{kahler}} = -\frac{i}{6} \sum_{i,j,k} C_{ijk} (\tau^i - \bar{\tau}^i)(\tau^j - \bar{\tau}^j)(\tau^k - \bar{\tau}^k) + \underbrace{\frac{\zeta(3)\, \chi(M)}{4\pi^2}}_{\text{4-loop correction}} + \frac{2i}{(2\pi i)^3} \sum_{\eta \in H_2(M)} N_\eta \left(\sum_{n=1}^{\infty} \frac{e^{2\pi i n \int_\eta J}}{n^3} + c \right.$$

$$+ \ldots$$

Gromov-Witten invariants

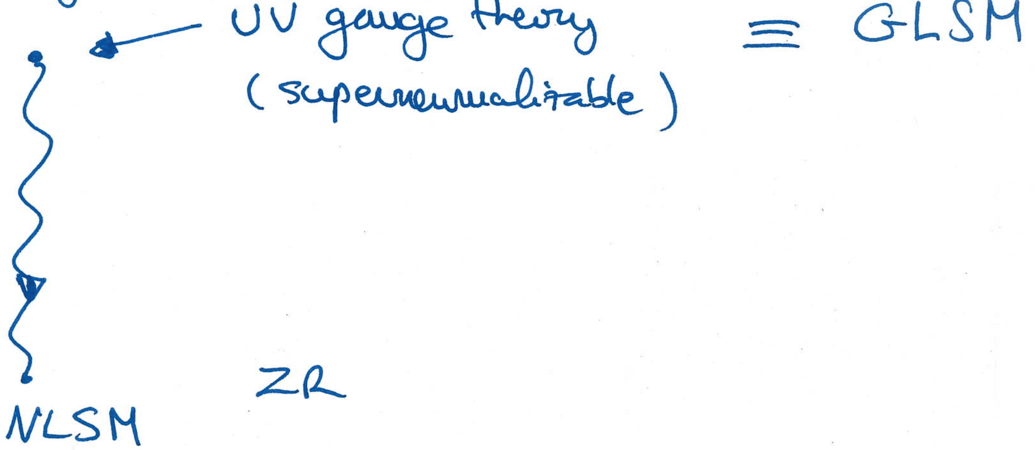
instanton action.

How can we compute $e^{-\text{kahler}}$?

Surprising answer:

$$\boxed{e^{-K_{\text{kahler}}(M)} = Z_{S^2}^{\text{NLSM}}} \; !$$

How to compute $Z_{S^2}^{\text{NLSM}}$? This hard directly. We will use the power of the RG



Using the g^2-invariance of Z_{S^2} we arrive at the eminently computable relation:

$$\boxed{e^{-K_{\text{kahler}}(M)} = Z_{S^2}^{\text{NLSM}} = Z_{S^2}^{\text{GLSM}}}$$

Compute by residues

GLSM primer: Pedagogical example is the GLSM representation of CP^{N-1} model. All other examples are extensions of this case

$$CP^{N-1}: \quad (\phi_1, \dots, \phi_N) \simeq \lambda(\phi_1 \dots \phi_N)$$
$$\vec{\phi} \neq 0 \qquad \lambda \in \mathbb{C}^*$$

Fix gauge: $\left\{ \sum_{i=1}^{N} |\phi_i|^2 = \xi \right\} \Big/ \left\{ \phi_i \to e^{i\alpha}\phi_i \right\}.$

Idea: Construct gauge theory that has CP^{N-1} as the space of vacua in some phase.

$$\begin{array}{cccccc} U(1) & \phi_1 & \dots & \phi_N & & W = 0 \\ & 1 & & 1 & & \end{array}$$

$$V = g^2\Big(\underbrace{\sum_{i=1}^{N} |\phi_i|^2 - \underset{\uparrow \text{ FI parameter}}{\xi}}_{D^2} \Big)$$

Space of vacua:

$$\sum_{i=1}^{N} |\phi_i|^2 = \xi \Big/ U(1)$$

- massless fields parametrize CP^{N-1}
- massive fields have $m^2 \sim g^2 \xi$, Decouple as $g^2 \to \infty$!

- GLSM's for compact Calabi-Yaus require introducing a suitable W in the gauge theory.

⇓

Q: Where is the info about kahler and cplex moduli of C encoded in the gauge theory

kahler moduli : $W(\tau_i)$ complexified FI param

cplex moduli $W(z_a)$ parameters of superpoten

Z_{S^2} depends on $W(\tau_i)$ and not $W(z_a) \Rightarrow Z(\tau_i, \bar{\tau}_i)$

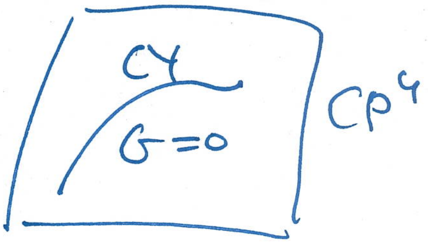
↑ except R-charge constraint

<u>Example</u>: Quintic hypersurface in $CP^4 \Rightarrow \phi_1 \dots \phi_5$

$h^{1,1}(M) = 1$

$h^{1,2}(M) = 101$

$G(z_i, \vec{\phi}) = 0$

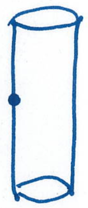


GLSM:

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	P
$U(1)$	1	1	1	1	1	-5

$W = P G$

study space of vacua



$\xi \gg 0$ → Massless dof parametrize CY_3 (Large Vol phase

$W = G(X)$

$\xi \ll 0$ → Massless L.G. ~~[illegible]~~ orbifold model (LG phase)

⑨

- Sketch of proof of why $Z_{S^2} = e^{-K_{kahler}}$:

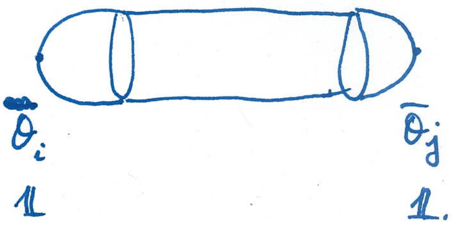
$$\Uparrow$$

tt^*-equations

- Z_{S^2} computes a very specific inner product between ground states

- Consider first the squashed S^2

$$x_1^2 + x_2^2 + b^2 x_3^2 = r^2$$



~~Comments~~:

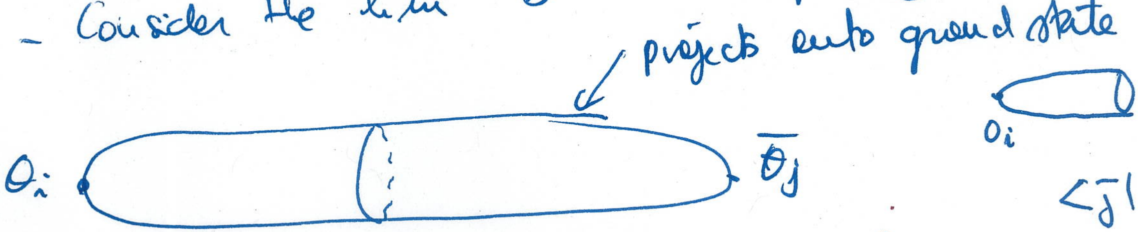
- Turn on a background gauge field for U(1) R-symmetry

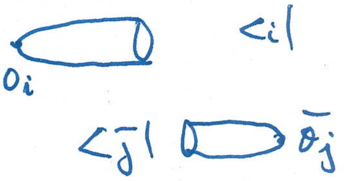
$$(\nabla_i - A_i)\epsilon = \frac{1}{2r}\gamma_i\gamma_3\,\epsilon$$

- Proof that partition function is independent of squashing parameter

$$Z_{S^2} = Z_{S^2_b}.$$

- Consider the limit $b \to 0$: Infinitely stretched S^2





$A_N = +\frac{1}{2}\omega_N$ $\quad A = \frac{1}{2}d\varphi \quad$ $A_S = -\frac{1}{2}\omega_S$

- This path integral computes the overlap of R ground state

$\langle i|\bar{j}\rangle$ and tt^* ~~⇒~~ know that from

$$\langle 0|\bar{0}\rangle = e^{-K}\ !$$
$$\overset{\Updownarrow}{\mathbb{1}} \quad \overset{\Downarrow}{\mathbb{1}}$$

$$\frac{\langle i|\bar{j}\rangle}{\langle 0|\bar{0}\rangle} = G_{i\bar{j}}$$
$\parallel$
Zamoldch
met

Problems

① Find the equation that makes $\nabla_\mu \epsilon = 0$ Weyl invariant

~~$\nabla_\mu \epsilon = 0$~~

② Embed the isometry of S^D inside conformal $SO(1,D+1)$ w/ generators:

$$P_\mu, M_{\mu\nu}, k_\mu, D.$$

③ Show that $\mathcal{L}(\phi, \ldots ; g_{\mu\nu}, \ldots)$ is invariant under

↓
SUGRA invariant

rigid susy for a background $(g^*_{\mu\nu}, \ldots)$ obeying.

$$\delta g_{\mu\nu}\big|_{g^*_{\mu\nu}} = 0 \qquad \delta \psi_\mu\big|_{g^*_{\mu\nu}} = 0 .$$

④ Given the ~~supera~~ Poincaré transformations. of a 2D $N=(2,2)$ chiral multiplet

$$\delta\phi = \bar\epsilon\psi$$
$$\delta\psi = i\,\partial_m\phi\,\gamma^m\epsilon + \bar\epsilon F$$
$$\delta F = -i\,\nabla_m\psi\,\gamma^m\epsilon$$

1) Construct the superconformal transformations when ϕ has Weyl weight w

2) Show that $\int d^2x\, F$ is a superconformal invariant if $w = 1$.