Lecture 1

- These lectures deal with some of the recent progress in the study of SUSYc QFT's.
  - 1 perturbative theory.
  - 2 IR fixed pts after long RG flow
- This work has lead to:
  - Exact computation of new and old observables.
  - New insights into the non-perturbative dynamics
  - Surprising connections with other areas of physics and mathematics
  - Beatiful and

Common theme: Study of SUSYc QFT's in non-trivial spacetimes

⇒ New window into non-perturbative dynamics of QFTs.

Classic example: Witten index

Addresses the question of whether SUSY is spontaneously broken in strongly coupled theories.

UV

strong dynamics. Is SUSY broken?

IR

Order parameter for unbroken SUSY is the energy of the vacuum:

$$E_{vac} = 0 \qquad \text{SUSY}$$

$$E_{vac} \neq 0 \qquad \cancel{\text{SUSY}}$$

Hard to compute as E_{vac} can be of non-perturbative nature

$$E_{vac} \simeq e^{-1/g^2}$$

Witten introduced an observable that determines whether SUSY is unbroken

$$\mathrm{Tr}\,(-1)^F e^{-\beta H} = Z_{T^4} = N^B_{E=0} - N^F_{E=0}$$

Can be computed exactly

$\Rightarrow$ Lead to important insights into non-perturbative ~~SUSY breaking~~ dynamics.

Example: $D=4$ $N=1$ SYM w/ gauge group G

$$\mathrm{Tr}\,(-1)^F e^{-\beta H} = h = \text{dual Coxeter number of } G \neq 0$$

Even though the theory is strongly coupled in the IR. $\Rightarrow$ gaugino condensation

$$\langle \lambda^\alpha \lambda_\alpha \rangle = \Lambda^3 e^{\frac{2\pi i k}{h}} \qquad k = 0, 1, \ldots, h-1$$

Recent progress:
- More refined observables
- probe details of the QFT
- Non-trivial functions of parameters of theory

New observables constructed by considering SUSYc QFTs on curved spaces (not topologically twisted)

$D=2$	S^2	
$D=3$	S^3, $S^2 \times S^1$	lots of CFTs.
$D=4$	S^4, $S^3 \times S^1$ — critical dimensions	
$D=5$	$\exists$ SUSYc CFTs	?
$D=6$	Special one: $D=6$ (0,2) SCFT	

· Important insights are also obtained by considering squashed spheres S^D_b = e.g. S^2_b.

$$x_1^2 + x_2^2 + b^2 x_3^2 = r^2 \qquad b: \text{squashing parameter}$$

. Broad classes of SUSYc QFT's in $D<6$ constructed by compactifying the $D=6$ (0,2) SCFT

<u>Example</u>: $Z(S^1 \times S^{D-1})$ = "superconformal index"

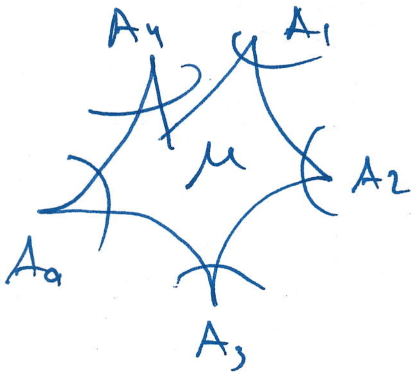
- Vast generalization of Witten index
- Probes short representations of SUSY (not just vacuum)
- Non-trivial function of chemical potentials
- Involve's non-trivial functions "elliptic Γ-functions"

~~(see Kim's lectures)~~

Some Applications :

1. Dualities :

(a) S-dualities in $D=4$ (strong-weak coupling dualities)



parameter space of the theory

Establish that :

- $Z_{A_i} = Z_{A_j}$
- Mapping of operators under dualities.

Ising : $\sigma \longleftrightarrow \mu$

spin $\quad$ disorder

$D=4$, $\mathcal{N}\geq 2$ QFT's : ($D=4$) Wilson loops $\longleftrightarrow$ 't Hooft loops

$D=3$

(b) IR dualities ($D=3$) Wilson loops $\longleftrightarrow$ Vortex loops.

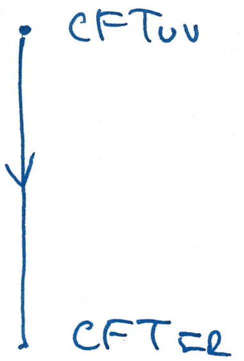
$UV_1 \rightarrow$ IR SCFT $\leftarrow UV_2$

strongly coupled

- Seiberg dualities
- mirror symmetry (example : SQED $N_F = \bar{N}_F = 1$ $\updownarrow$ XYZ model)

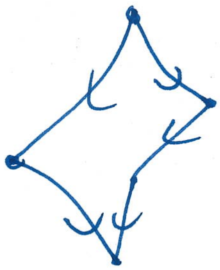
⋮

2. Fundamental questions in QFT: Nature of RG flows



$\exists$ a quantity such that $C_{UV} > C_{IR}$?

Important in mapping out the space of QFTs.



$(D=2)$ $(D=4)$ $(D=3)$

$\Rightarrow$ Study of c-theorem, a-theorem, F-theorem, ...

$\exists$ a unifying formulation. The partition function of ~~the~~ the CFT on S^D!

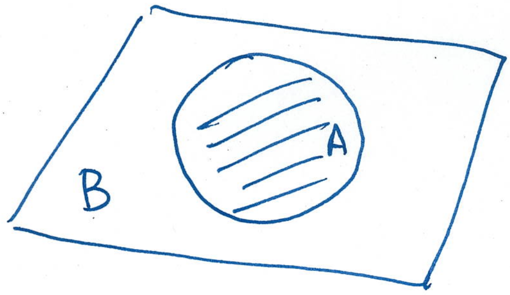
$$Z_{S^D} = e^{+F_{S^D}}$$

Candidate monotonic quantity: $|F_{UV}| > |F_{IR}|$.

Z_{S^D} is a fundamental object in QFT!

(more suitable for counting DoF than the thermal partition function)

Furthermore, F computes the entanglement entropy of the CFT



A : S^{D-2} of radius r

$R^{1,D-1}$

$t=0$ slice

$$S_{\text{von-Neumann}} = -\mathrm{Tr}\,\rho_A \log \rho_A = -\mathrm{Tr}\,\rho_B \log \rho_B = F \;!!$$

3. Z has a geometrical interpretation

It computes exactly quantities of physical and mathematic interest using physics based-methods!

Example: $$Z_{S^2} = e^{-K_M}$$

K_M : Kahler potential of Calabi-Yau moduli space

K_M encodes important physical information: string compactifications $R^{1,3} \times M$

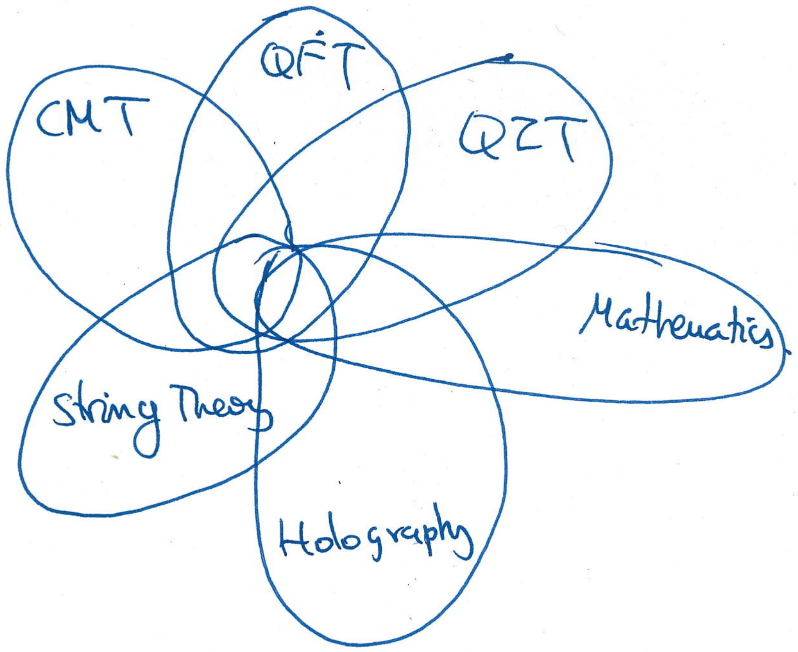
- Yukawa couplings in heterotic string compactification
- kinetic term for moduli in Type II compactification
- Exact in α', including all worldsheet instanton corrections

$Z_{S^2} \longleftrightarrow$ string theory

Gromov-Witten invariants (topological invariants associated to instantons)

- Z_{S^2}, new, physics based approach to these (only method)

Z_{S^D} is an object at the intersection of various research areas



SUSY in curved spacetimes

Recall the logic in $\mathbb{R}^D$:

1. Suitable notion of SUSY parameters

$$\nabla_\mu \epsilon = 0 \qquad \xi^\mu = \epsilon_1 \Gamma^\mu \epsilon_2 \text{ killing vector}$$

Super-Poincaré

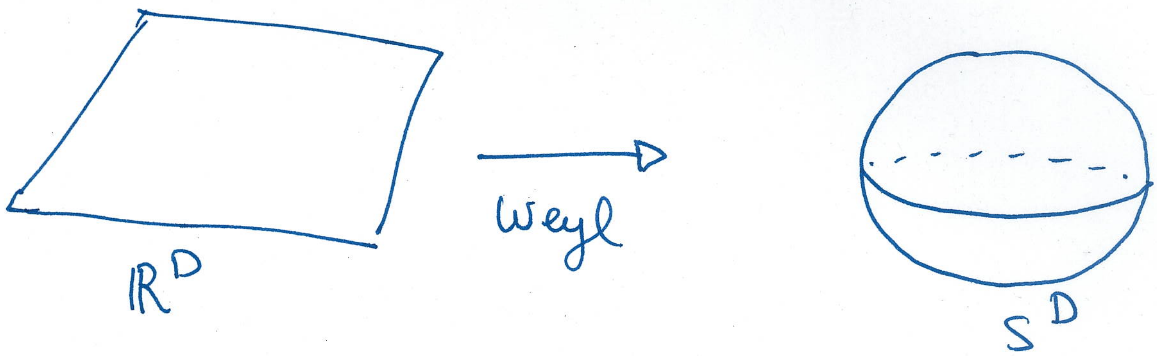
killing vectors and spinors realize the SUSY algebra.

2. Realize SUSY algebra on fields (appear in supermultiplet)

3. Find an invariant action. ; $\mathcal{L}$

Consider first SUSYc QFT's on conformally flat spaces such as S^D and $S^{D-1} \times S^1$

E.g:



$$ds^2(S^D) = \frac{1}{\left(1 + \frac{x_i x_i}{4r^2}\right)^2} dx_i dx_i$$

- Start with D-dimensional superconformal algebra (SCA)

SUSY transformations realized by conformal killing spinors:

$$\nabla_\mu \epsilon = \gamma_\mu \tilde{\epsilon} = \frac{1}{D} \gamma_\mu \nabla\!\!\!/ \, \epsilon$$

Comments:

- Conformally invariant: $\begin{cases} g_{\mu\nu} \to e^{2\rho(x)} g_{\mu\nu} \\ \epsilon \to e^{\frac{1}{2}\rho} \epsilon \end{cases}$
- $\xi^\mu = \epsilon_1 \Gamma^\mu \epsilon_2$ is a conformal killing vector
- In flat space $\epsilon = \epsilon_P + x^\mu \Gamma_\mu \epsilon_c$

$$\begin{tikzcd}[column sep=small]
 & SCA_D \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{R}^D & S^D & S^1 \times S^{D-1}
\end{tikzcd}$$

SUSY algebra: G_1 $\quad G_2$ $\quad G_3$

Relevant SUSY algebra is the subalgebra of SCA_D that generates the isometries of the spacetime, and no conformal transformations

Example: $D=4$ $\quad N=1$

$$\begin{tikzcd}[column sep=small]
 & SU(2,2|1) \arrow[dl, "\mathbb{R}^4"'] \arrow[d, "S^4"] \arrow[dr, "S^1\times S^3"] & \\
N=1 \text{ Super-Poincaré} & OSp(1|4) & SU(2|1)\times SU(2)\times U(1) \\
\cup & \cup & \cup \\
ISO(4) & Sp(4) \simeq SO(5) & SO(4)\times U(1)_H \times U
\end{tikzcd}$$

$\Rightarrow$ Generically get an R-symmetry as part of SUSY algebra.

Construction of QFT's:

1. Realize SCA_D on fields. Transformations completely determined by Weyl covariantization the flat spacetime transformations

$$\underset{\eta_{\mu\nu}}{\delta_{flat}\varphi} \longrightarrow \underset{g_{\mu\nu}}{\delta\varphi} = \delta_0 \varphi + \frac{1}{r}\delta_1\varphi \quad \text{is Weyl covariant}$$

unique!

~~e.g.:~~ $\cancel{\delta\psi \ni \not{D}\phi\, \epsilon + \frac{\Delta}{2}\phi \not{D}\epsilon}$

Imagine φ is a field of Weyl weight Δ:

$g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}$

$\phi \to e^{-\Delta\Omega}\phi$

Boson: $$D_\mu \varphi \, \gamma^\mu \epsilon + \frac{2\Delta}{D} \varphi \, \gamma^\mu D_\mu \epsilon$$

Fermion: $$\epsilon \gamma^\mu D_\mu \chi + \frac{2\Delta + 1 - D}{D} D_\mu \epsilon \, \gamma^\mu \chi$$

$\delta\varphi$ realizes, by construction, the superconformal algebra SCAD

2. Restrict to those transformations that generate the isometry subalgebra G. $\Rightarrow$ $\delta\varphi$ realize G.

3. Find action that is invariant under G.

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{r}\mathcal{L}_1 + \frac{1}{r^2}\mathcal{L}_2$$

covariantized flat space Lagrangian

related to conformal breaking

Find by Noether procedure.

$\Longrightarrow$
1. SUSY algebra is deformed
2. SUSY transformations " "
3. $\mathcal{L}$ is deformed

- Will finish with a couple of remarks about the problem of constructing SUSYc QFT's on general backgrounds:

~~[illegible]~~ S_6^M

- If there is an R-symmetry one can perform "topological twist"

$$(\nabla_\mu - A_\mu)\epsilon = 0$$

background gauge field for the R-symmetry

By choosing: $\omega_\mu = A_\mu \Longrightarrow$ Find suitable spinors

$$\partial_\mu \epsilon = 0$$

For some S_6^D, milder modification: $(\nabla_\mu - A_\mu)\epsilon = \partial_\mu \tilde{\epsilon}$

- Study within SUGRA. Analyze which background fields in the supergravity multiplet ($g_{\mu\nu}$, auxilliary) fields give rise to SUSYc backgrounds:

$$\delta\psi_\mu = 0$$

$$\Downarrow$$

$$\nabla_\mu \epsilon = B_\mu \epsilon$$ generalized killing spinor equation

SUGRA Approach

- Couple SUSY QFT to SUGRA (not dynamical, just background)

$$\mathcal{L} = \mathcal{L}(\phi, g_{\mu\nu})$$

$\rightarrow$ SUGRA multiplet

$(g_{\mu\nu}, \psi_{\mu\alpha}, \underbrace{\ldots}_{\text{auxiliary fields}})$

QFT ~~matt~~ multiplets (spin ≤ 1)
- vector multiplet
- chiral multiplet (twisted chiral multiplet)
- hypermultiplet

$\mathcal{L}$ is invariant under SUGRA transformations: $\epsilon(x)$ arbitrary

Couple QFT to off-shell SUGRA background
$\nearrow$ SUSY

<u>bosonic backgrounds:</u> $\delta\psi_\mu = 0$ $\quad$ δ fermi $= 0$

Get a generalization of conformal killing spinor equation:
e.g: 2d $N = (2,2)$ SUGRA

$$(\nabla_\mu - iA_\mu)\epsilon = -\frac{1}{4}(1+\gamma_3) H \gamma_\mu \epsilon - \frac{1}{4}(1-\gamma_3)\bar{H}\gamma_\mu\epsilon$$

$$(\nabla_\mu + iA_\mu)\bar{\epsilon} = -\frac{1}{4}(1-\gamma_3) H \gamma_\mu \bar{\epsilon} - \frac{1}{4}(1+\gamma_3)\bar{H}\gamma_\mu\bar{\epsilon}$$

Immediate application:

$$MM_k \simeq MM_k / \mathbb{Z}_k$$

$$W = Y^{k+2} \stackrel{?}{=} W = \Phi^{k+2} \quad ?$$

Since $R[W] = 2 \implies R[\Phi] = \frac{2}{k+2}$

$\implies$ From one-loop determinants of chiral multiplets $M = \frac{i}{r} \frac{1}{k+}$

$$Z_W = \frac{\Gamma\left(\frac{1}{k+2}\right)}{\Gamma\left(1 - \frac{1}{k+2}\right)} \quad !$$