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Lectures on the Conformal Bootstrap

1 Introduction

Quantum Field Theories generically become scale-invariant at long distances. Under very general conditions, scale invariance actually implies invariance under conformal transformations, which are transformations that locally look like a rescaling + rotation.[1]

We can think of a UV-complete QFT as a renormalization group flow between conformal fixed-points,

$$\left.\begin{array}{c}\mathrm{CFT}_{UV}\\ \downarrow\\ \mathrm{CFT}_{IR}\end{array}\right\}\mathrm{QFT}. \tag{1.1}$$

By studying CFTs, we can map out the possible beginnings and endings of RG flows, and thus understand the space of QFTs. (You'll see many other reasons that CFTs are interesting over the course of the summer.)

RG-flows can be nonperturbative. A simple example is ϕ^4 theory in 3-dimensions,

$$S = \int d^3x \left((\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4\right) \tag{1.2}$$

This theory is free in the UV. For generic values of m^2, it becomes massive in the IR — in this case, we can think of it as a flow between the free boson CFT and the trivial CFT. However, for a special value of m^2, the theory flows to a nontrivial interacting CFT. It is hard to learn about this CFT using Feynman diagrams. Note that g has mass-dimension 1, so that perturbation theory leads to an expansion in xg, where x is a distance scale. At distances $x \gg 1/g$, this expansion breaks down. The best perturbative tool we have is the ϵ-expansion, where we start with the theory in $4-\epsilon$ dimensions and then continue $\epsilon \to 1$.

In the example above, the UV theory is a continuum QFT (the free boson). However, interesting IR CFTs can arise from very different microscropic systems. An example is the 3d Ising model, which is a lattice of classical spins $s_i \in \{\pm 1\}$ with nearest-neighbor interactions. The partition function is given by

$$Z_{\mathrm{Ising}} = \sum_{s_i} \exp\left(\sum_{\langle ij\rangle} -J s_i s_j\right). \tag{1.3}$$

[1] In 2d and 4d, it is enough that the theory be Lorentz-invariant and unitary. In 3d or $d > 4$, the appropriate conditions are not known, but conformal invariance appears in myriad examples.

We can think of this sum as a discretized path integral, where the spins s_i form a $\mathbb{Z}_2$-valued field on the lattice that we integrate over. In the continuum limit, for a special value of J, this theory also flows to a nontrivial CFT. Actually *it is the same CFT as for ϕ^4 theory.* The 3d Ising CFT also arises in water at the critical point on its phase diagram, and uni-axial magnets at their critical temperature. This IR equivalence of different microscopic theories is called "critical universality." IR equivalences show up all over high-energy and condensed-matter physics, where they are sometimes called "dualities."

The lattice description is relatively easy to simulate on a computer, but all of the above realizations of the 3d Ising CFT (especially boiling water) make it difficult to do computations. The main reason is that none of them fully exploit the symmetries of the IR theory.

The conformal bootstrap philosophy is to:

1. Focus on the CFT itself and not a specific microscopic description,
2. Determine the full consequences of symmetries,
3. Impose consistency conditions,
4. Combine 2 and 3 to constrain or even solve the theory nonperturbatively.

The merits of this strategy for the 3d Ising model will become clear during this course. However, sometimes bootstrapping is the *only* known strategy for understanding the full dynamics of a theory. An example is the $6d$ CFT describing the IR limit of a stack of M5 branes in M-theory. This theory has no known Lagrangian description, but is amenable to a bootstrap analysis.[2] A beautiful and ambitious goal of the bootstrap program is to eventually provide a fully nonperturbative and constructive definition of Quantum Field Theory, obviating the need for a Lagrangian. We are not there yet, but you can help!

2 QFT Generalities

2.1 The Stress Tensor

The first step of the conformal bootstrap is to understand the full consequences of symmetries. For concreteness, let us imagine that we've taken the long distance/continuum limit of ϕ^4 theory or the 3d Ising model and focus on the structures that are present nonperturbatively.

[2]At large central charge, this theory is essentially solved using the AdS/CFT correspondence. In general, supersymmetry is an extremely powerful tool for learning about this theory, and many interesting supersymmetrically-protected quantities can be computed using non-renormalization theorems, localization, and other techniques. However, the bootstrap is currently the only known tool for studying non-protected quantities at small central charge.

A local quantum field theory has a conserved stress tensor,

$$\partial_\mu T^{\mu\nu}(x) \;=\; 0 \qquad \text{(operator equation)} \tag{2.1}$$

This holds as an "operator equation," meaning it is true away from other operator insertions. In the presence of other operators, there can be contact terms. In this case, we have the Ward identity

$$\partial_\mu \langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle \;=\; -\sum_i \delta(x-x_i)\partial_i^\nu \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle. \tag{2.2}$$

Exercise 1. *Consider coupling our QFT to a background metric g. For concreteness, you may imagine that a correlator is given by the path integral*

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g \;=\; \int D\phi\, \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\, e^{-S[g,\phi]} \tag{2.3}$$

The stress tensor is defined by

$$\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g \;=\; -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}(x)}\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g \tag{2.4}$$

Derive (2.2) by demanding that the theory is coupled to g in a diffeomorphism-invariant way (i.e., the action should depend only on the metric and not on a choice of coordinates). Generalize (2.2) to the case of operators with spin.

Let us choose a ball B surrounding some of the operators $\mathcal{O}_i$. Integrating (2.2) over B and using Stokes' theorem, we get

$$\int_\Sigma dS_\mu \langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \;=\; -\sum_{i \text{ inside } \Sigma} \partial_i^\nu \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \tag{2.5}$$

where $\Sigma = \partial B$. Thus, the integral of $T^{\mu\nu}$ over a closed surface Σ,

$$P^\nu[\Sigma] \;\equiv\; -i\int_\Sigma dS_\mu T^{\mu\nu}(x) \tag{2.6}$$

is topological: we are free to deform Σ however we want as long as we don't cross any operator insertions.[3] We say that $P^\nu[\Sigma]$ is a "topological surface operator." Whenever Σ surrounds $\mathcal{O}$ and no other operators, we have

$$\langle P^\mu[\Sigma]\mathcal{O}(x)\dots\rangle \;=\; i\partial^\mu\langle \mathcal{O}(x)\dots\rangle. \tag{2.7}$$

[3] The word "surface" usually refers to a 2-manifold, but here we will abuse terminology and use it to refer to a $d-1$-manifold in a d-dimensional theory (codimension-1 manifold).

2.2 Quantization

A single path integral can be interpreted in terms of different time evolutions in different Hilbert spaces. We refer to such interpretations as "quantizations" of the theory. The example you're all familiar with is that in a Lorentz invariant theory, we can consider time evolution in reference frames that are boosted with respect to one other. In a rotationally-invariant Euclidean theory on $\mathbb{R}^d$, we are free to choose any direction as "time" and think of states living on $\mathbb{R}^{d-1}$-slices orthogonal to that direction.

In general, to specify a quantization, we pick a foliation of our spacetime by spatial slices. A state $|\psi\rangle$ lives on a slice. A correlation function $\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_2)\rangle$ gets interpreted as a time ordered expectation value

$$\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \;=\; \langle 0|T\{\mathcal{O}_1(t_1,\mathbf{x}_1)\cdots\mathcal{O}_n(t_n,\mathbf{x}_n)\}|0\rangle \tag{2.8}$$

where the time ordering $T\{\dots\}$ is with respect to our choice of foliation, and $|0\rangle$ refers to the vacuum in the Hilbert space living on a spatial slice. Other choices of initial and final state correspond to different boundary conditions for the path integral.

Let Σ_t be a spatial slice at time t and consider the operator $P^\nu[\Sigma_t]$. Because $P^\nu[\Sigma_t]$ is topological, we are free to move it forward or backward in time from one spatial slice to another $P^\nu[\Sigma_t] = P^\nu[\Sigma_{t'}]$, as long as we don't cross any operator insertions. Thus, we can often neglect to specify the surface Σ_t and just write P^ν (though we should keep in mind where the surface lives with respect to other operator insertions). We call P^ν "momentum," and the fact that it's topological is the path integral version of the statement that momentum is conserved.

We must take care when we move P^ν past an operator insertion. Consider a local operator $\mathcal{O}(x)$ at time t. If Σ_2 is a spatial surface at time $t_2 > t$ and Σ_1 is a spatial surface at time $t_1 < t$, then the difference $P^\nu[\Sigma_2] - P^\nu[\Sigma_1]$ becomes a commutator because of time ordering,

$$\langle (P^\mu[\Sigma_2] - P^\mu[\Sigma_1])\mathcal{O}(x)\dots\rangle = \langle 0|T\{[P^\mu,\mathcal{O}(x)]\dots\}|0\rangle. \tag{2.9}$$

Because P^μ is topological, $\Sigma_2 - \Sigma_1$ can be freely deformed to a sphere Σ' surrounding $\mathcal{O}(x)$,

$$\begin{aligned}
\langle 0|T\{[P^\mu,\mathcal{O}(x)]\dots\}|0\rangle &= \langle P^\mu[\Sigma_2-\Sigma_1]\mathcal{O}(x)\dots\rangle \\
&= \langle P^\mu[\Sigma']\mathcal{O}(x)\dots\rangle \\
&= i\partial^\mu\langle\mathcal{O}(x)\dots\rangle \\
&= i\partial^\mu\langle 0|T\{\mathcal{O}(x)\dots\}|0\rangle,
\end{aligned} \tag{2.10}$$

where in the third line we've used the Ward identity (2.5). Figure **??** makes it clear that this result is independent of how we choose to quantize our theory. Thus, we often write

$$[P^\mu,\mathcal{O}(x)] \;=\; i\partial^\mu\mathcal{O}(x) \tag{2.11}$$

without specifying which quantization we are using. In general, a commutator $[Q,\mathcal{O}(x)]$ is shorthand for surrounding $\mathcal{O}(x)$ with a topological surface operator $Q[\Sigma]$.

Equation (2.11) can be integrated to give

$$\mathcal{O}(x) \;=\; e^{-iP\cdot x}\mathcal{O}(0)e^{iP\cdot x} \tag{2.12}$$

This statement is also true in any quantization of the theory. In fact, in path integral language, $e^{-iP\cdot x}[\Sigma]$ can be thought of as a topological surface operator given by inserting many P^ν's on a given surface Σ. When we surround $\mathcal{O}(0)$ with $e^{-iP\cdot x}[\Sigma]$, it becomes conjugation $\mathcal{O}(0) \to e^{-iP\cdot x}\mathcal{O}(0)e^{iP\cdot x}$ in any quantization.

Consider the time-ordered correlator (2.8) with $t_n > \cdots > t_1$. This is

$$\begin{aligned} \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle &= \langle 0|e^{-it_nP^0}\mathcal{O}_n(0,\mathbf{x}_n)e^{it_nP^0}\cdots e^{-it_1P^0}\mathcal{O}_n(0,\mathbf{x}_1)e^{it_1P^0}|0\rangle & (2.13)\\ &= \langle 0|\mathcal{O}_n(0,\mathbf{x}_n)e^{i(t_n-t_{n-1})P^0}\cdots e^{i(t_2-t_1)P^0}\mathcal{O}_1(0,\mathbf{x}_1)|0\rangle & (2.14)\end{aligned}$$

In other words, the path integral between spatial slices separated by time t computes the action of $U = e^{itP^0}$. In a reflection-positive Euclidean theory, P^0 is anti-hermitian, $P^0 = iH$, where H has positive spectrum. Thus, the factors $e^{i(t_k-t_{k-1})P^0} = e^{-(t_k-t_{k-1})H}$ lead to suppression at large time separation. Note that a non-time ordered Euclidean correlator doesn't even make sense because it would have insertions of $e^{\text{positive}\times H}$ and the spectrum of H is generally unbounded above.

2.3 More Symmetries

Given a conserved current $\partial_\mu J^\mu = 0$ (away from other operator insertions), we can always define a topological surface operator by integration.[4] For momentum P^ν, the corresponding currents were simply $T^{\mu\nu}(x)$. More generally, given a vector field $\epsilon = \epsilon^\mu(x)\partial^\mu$, the charge

$$Q_\epsilon[\Sigma] \;=\; \int_\Sigma dS_\mu \epsilon_\nu T^{\mu\nu}(x) \tag{2.15}$$

will be conserved whenever

$$0 \;=\; \partial_\mu(\epsilon_\nu T^{\mu\nu}) \;=\; \partial_\mu\epsilon_\nu T^{\mu\nu} + \epsilon_\nu\partial_\mu T^{\mu\nu} \;=\; \frac{1}{2}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)T^{\mu\nu}, \tag{2.16}$$

where we've used that $T^{\mu\nu}$ is symmetric and conserved. Thus, we should demand

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu \;=\; 0. \tag{2.17}$$

This is the Killing equation. In flat space, it has solutions

$$\begin{array}{lllll} \epsilon &=& p_\mu &=& -i\partial_\mu \quad \text{(translations)} \\ \epsilon &=& m_{\mu\nu} &=& i(x_\mu\partial_\nu - x_\nu\partial_\mu) \quad \text{(rotations).} \end{array} \tag{2.18}$$

The corresponding charges are momentum $P_\mu = Q_{p_\mu}$ and angular momentum $M_{\mu\nu} = Q_{m_{\mu\nu}}$.

[4]It is an interesting question whether the converse is true. In the case of continuous symmetries, Noether's theorem gives a way to construct a current when a theory has a Lagrangian description. However, it is not clear whether a current must exist when a microscopic Lagrangian is absent.

3 Conformal Symmetry

In a conformal theory, the stress tensor satisfies an additional condition: it is traceless,

$$T^{\mu}_{\mu}(x) \;=\; 0 \qquad \text{(operator equation).} \tag{3.1}$$

This is equivalent to the statement that the theory is insensitive to position-dependent rescalings of the metric $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}$ (near flat space).[5] In the presence of a traceless stress tensor, we can relax the requirement on ϵ further to

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = c(x)\eta_{\mu\nu} \tag{3.2}$$

where $c(x)$ is some scalar function.[6] Equation (3.2) is the *conformal* Killing equation. It has two additional types of solutions in $\mathbb{R}^d$,

$$\begin{array}{lcll} d & = & -ix^\mu \partial_\mu & \text{(dilatations)} \\ k_\mu & = & i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) & \text{(special conformal transformations).} \end{array} \tag{3.3}$$

The corresponding symmetry charges are $D = Q_d$ and $K_\mu = Q_{k_\mu}$.[7]

3.1 Finite Conformal Transformations

Before discussing the charges $P_\mu, M_{\mu\nu}, D, K$, let us take a moment to understand the geometrical meaning of the conformal Killing vectors (2.18) and (3.3). The vector fields $p_\mu, m_{\mu\nu}, d, k_\mu$ generate infinitesimal spacetime transformations $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$. The conformal Killing equation implies

$$\frac{\partial x'^\mu}{\partial x^\nu} \;=\; \delta^\mu_\nu + \partial_\nu \epsilon^\mu \;=\; \left(1 + \frac{1}{2}c(x)\right)\left(\delta^\mu_\nu + \frac{1}{2}\left(\partial_\nu \epsilon^\mu - \partial^\mu \epsilon_\nu\right)\right) \tag{3.4}$$

This is an infinitesimal rescaling times an infinitesimal rotation. Exponentiating gives a transformation of the same form,

$$\frac{\partial x'^\mu}{\partial x^\nu} \;=\; \Omega(x) O_\nu{}^\mu(x), \qquad O^T O = I_{d\times d}. \tag{3.5}$$

where $\Omega(x)$ and $O_\nu{}^\mu(x)$ are now finite (position-dependent) rescalings and rotations. Such transformations define the conformal group. It is a finite-dimensional subgroup of the diffeomorphism group of $\mathbb{R}^d$. (We'll see exactly which group in a moment.)

[5] In curved space, there can by Weyl anomalies.

[6] By contracting both sides with the metric, we find $c(x) = \frac{2}{d}\partial \cdot \epsilon(x)$.

[7] The above solutions are present in any spacetime dimension. In two dimensions, there are an infinite set of additional solutions to the conformal Killing equation, leading to an infinite set of additional conserved quantities. This is an extremely interesting subject which we unfortunately won't have time for in this course.

The finite versions of translations and rotations are familiar. Exponentiating d gives a scale transformation $x \to \lambda x$ which rescales the metric by a constant factor. The nontrivial case is k_μ. We can understand what it does by first considering an inversion

$$I : x^\mu \to \frac{x^\mu}{x^2} \tag{3.6}$$

I is a conformal transformation (though it is not continuously connected to the identity, and thus can't be obtained by exponentiating the conformal algebra).

Exercise 2. *Verify that $I \circ p_\mu \circ I = k_\mu$. Conclude that $e^{ia\cdot k} = I \circ e^{ia\cdot p} \circ I$, or*

$$x \quad \to \quad x'(x) = \frac{x^\mu + a^\mu x^2}{1 + 2(a\cdot x) + a^2 x^2}. \tag{3.7}$$

We can think of k_μ as a "translation that moves infinity and fixes the origin" in the same sense that the usual translations move the origin and fix infinity.

3.2 The Conformal Algebra

The charges Q_ϵ give a representation of the conformal algebra

$$[Q_{\epsilon_1}, Q_{\epsilon_2}] \quad = \quad Q_{[\epsilon_1,\epsilon_2]} \tag{3.8}$$

where $[\epsilon_1, \epsilon_2]$ is a commutator of vector fields.[8] As usual, (3.8) is true in any quantization of the theory. (In path integral language, it tells us how to move the topological surface operators $Q_{\epsilon_i}[\Sigma]$ through each other.)

Exercise 3. *Show that*

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i(\delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu) && (3.9)\\
[M_{\mu\nu}, K_\rho] &= i(\delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu) && (3.10)\\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\rho\mu} - \delta_{\mu\sigma} M_{\rho\nu}) && (3.11)\\
[D, P_\mu] &= iP_\mu && (3.12)\\
[D, K_\mu] &= -iK_\mu && (3.13)\\
[K_\mu, P_\nu] &= -2i(\delta_{\mu\nu} D - M_{\mu\nu}) && (3.14)
\end{aligned}$$

(all other commutators vanish).

The first three commutation relations just say that $M_{\mu\nu}$ generates the algebra of Euclidean rotations SO(d) and that P_μ, K_μ transform as vectors. The last three are more interesting. We'll see shortly that the eigenvalues of D are of the form $i\Delta$ where the dimension Δ is a positive real number. Equations (3.12, 3.13) thus say that P_μ is a raising operator for dimension and K_μ is a lowering operator. We will be more precise about this idea shortly.

[8] This is not obvious and deserves proof, which we leave as an exercise. In fact, in 2-dimensions, the algebra of charges and the algebra of conformal killing vectors *do not* coincide (the former is a central extension of the latter).

Exercise 4. *Define the generators*

$$\begin{aligned} L_{\mu\nu} &= M_{\mu\nu} \\ L_{-1,0} &= D \\ L_{0,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\ L_{-1,\mu} &= \frac{1}{2}(P_\mu + K_\mu) \end{aligned} \tag{3.15}$$

Where $L_{ab} = -L_{ba}$ and $a, b \in \{-1, 0, 1, \ldots, d\}$. Show that L_{ab} satisfy the commutation relations of the algebra $\mathrm{SO}(d+1,1)$ (the algebra of linear transformations that preserve the metric on $d+2$-dimensional Minkowski space $\mathbb{R}^{d+1,1}$).

The fact that the conformal group is $\mathrm{SO}(d+1,1)$ suggests that it might be a good to think about its action in terms of $\mathbb{R}^{d+1,1}$ instead of $\mathbb{R}^d$. This is the idea behind the "embedding formalism," which provides a simple and powerful way of understanding the constraints of conformal invariance. We will be more pedestrian in this course, but worry not — Joao will tell you about the embedding formalism next week.

4 Primaries and Descendants

Now that we have our conserved charges, we can classify operators into representations of those charges. We will do this in steps — first we classify them into Poincare representations, then scale+Poincare representations, and finally conformal representations.

4.1 Poincare Representations

In a Poincare-invariant QFT, local operators at the origin transform in irreducible representations of the rotation group,

$$[M_{\mu\nu}, \mathcal{O}^a(0)] = -(\mathcal{S}_{\mu\nu})^a{}_b \mathcal{O}^b(0) \tag{4.1}$$

where $\mathcal{S}_{\mu\nu}$ satisfies the same algebra as $M_{\mu\nu}$. The action (4.1), together with the commutation relations of the Poincare group, determines how rotations act away from the origin.

To see this, it is convenient to adopt shorthand notation where commutators of charges with local operators is implicit, $[Q, \mathcal{O}] \to Q\mathcal{O}$. This notation is valid because of the Jacobi identity (more formally, the fact that adjoint action gives a representation of a Lie algebra). Alternatively, in path integral language, $Q_n \cdots Q_1 \mathcal{O}(x)$ means surrounding $\mathcal{O}(x)$ with topological surface operators where Q_n is the outermost surface and Q_1 is the innermost. The conformal commutation relations tell us how to re-order these surfaces.

Acting with a rotation on $\mathcal{O}(x)$, we have

$$\begin{aligned}
M_{\mu\nu}\mathcal{O}(x) &= M_{\mu\nu}e^{-iP\cdot x}\mathcal{O}(0) && (4.2)\\
&= e^{-iP\cdot x}(e^{iP\cdot x}M_{\mu\nu}e^{-iP\cdot x})\mathcal{O}(0) && (4.3)\\
&= e^{-iP\cdot x}(-x_\mu P_\nu + x_\nu P_\mu + M_{\mu\nu})\mathcal{O}(0) && (4.4)\\
&= (-i(x_\mu\partial_\nu - x_\nu\partial_\mu) - \mathcal{S}_{\mu\nu})\mathcal{O}(x) && (4.5)\\
&= -(m_{\mu\nu} + \mathcal{S}_{\mu\nu})\mathcal{O}(x) && (4.6)
\end{aligned}$$

where in the third line, we've used the Poincare algebra and the Hausdorff formula

$$e^A B e^{-A} = e^{[A,\cdot]}B = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots \tag{4.7}$$

Exercise 5. *Check that the minus sign in (4.6) is consistent with the fact that $M_{\mu\nu}$, $m_{\mu\nu}$, and $\mathcal{S}_{\mu\nu}$ all satisfy the same algebra.*

4.2 Scale+Poincare Representations

In a scale-invariant theory, it's also natural to diagonalize the dilatation operator acting on operators at the origin,

$$[D, \mathcal{O}(0)] = i\Delta\mathcal{O}(0) \tag{4.8}$$

Δ is called the "dimension" of $\mathcal{O}$.[9]

Exercise 6. *Mimic the computation above to derive the action of dilatation on an $\mathcal{O}(x)$ with dimension Δ,*

$$[D, \mathcal{O}(x)] = i(x^\mu\partial_\mu + \Delta)\mathcal{O}(x). \tag{4.9}$$

Equation (4.9) is constraining enough to fix two point functions of scalars up to a constant. By rotation and translation invariance, we must have

$$\langle\mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = f(|x-y|) \tag{4.10}$$

for some function f. In a scale-invariant theory with scale-invariant boundary conditions, the simultaneous action of D on all operators in a correlator must vanish.[10] By (4.9), this is

$$i\left(x^\mu\partial_\mu + \Delta_1 + y^\mu\partial_\mu + \Delta_2\right)f(|x-y|) = 0 \qquad \Longrightarrow \qquad f(|x-y|) = \frac{C}{|x-y|^{\Delta_1+\Delta_2}} \tag{4.11}$$

If we had an operator with negative scaling dimension, then its correlators would grow with distance, violating cluster decomposition. This is unphysical, so we expect dimensions Δ to be positive. Shortly, we will prove this fact for unitary conformal theories (and derive even stronger constraints on Δ).

[9] The dilatation operator is diagonalizable in all unitary (reflection positive) CFTs. However, there exist interesting non-unitary theories where D has a nontrivial Jordan block decomposition. In these cases, we define a local operator as a finite-dimensional representation of D.

[10] It is also interesting to consider non-conformally invariant boundary conditions. These can be interpreted as a nontrivial operator at ∞.

4.3 Conformal Representations

Note that K_μ is a lowering operator for dimension,

$$\begin{aligned} DK_\mu\mathcal{O}(0) &= ([D,K_\mu]+K_\mu D)\mathcal{O}(0) && (4.12)\\ &= i(\Delta-1)K_\mu\mathcal{O}(0) && (4.13)\end{aligned}$$

(where again, we're using shorthand notation $[Q,\mathcal{O}]\to Q\mathcal{O}$). Thus, given an operator $\mathcal{O}(0)$, we can repeatedly act with K_μ to obtain operators $K_{\mu_1}\dots K_{\mu_n}\mathcal{O}(0)$ with arbitrarily low dimension. Because dimensions are bounded from below in physically sensible theories, this process must eventually terminate. That is, there must exist operators such that

$$[K_\mu,\mathcal{O}(0)] = 0 \qquad \text{(primary operator).} \tag{4.14}$$

Such operators are called "primary." Given a primary, we can construct operators of higher dimension, called "descendants," by acting with momentum generators, which act like raising operators for dimension,

$$\begin{aligned} \mathcal{O}(0) &\to P_{\mu_1}\cdots P_{\mu_n}\mathcal{O}(0) \qquad \text{(descendant operators)}\\ \Delta &\to \Delta+n. \end{aligned} \tag{4.15}$$

For example, $\mathcal{O}(x)=e^{-ix\cdot P}\mathcal{O}(0)$ is an (infinite) linear combination of descendant operators. The conditions (4.1, 4.8, 4.14) are enough to determine how K_μ acts on any descendant using the conformal algebra. For example,

Exercise 7. *Let $\mathcal{O}(0)$ be a primary operator with rotation representation matrices $\mathcal{S}_{\mu\nu}$ and dimension Δ. Using the conformal algebra, derive*

$$K_\mu\mathcal{O}(x) = \left(-2ix_\mu\Delta - 2x^\nu\mathcal{S}_{\mu\nu} + ix^2\partial_\mu - 2ix_\mu x\cdot\partial\right)\mathcal{O}(x). \tag{4.16}$$

To summarize, a primary operator satisfies

$$\begin{aligned} [D,\mathcal{O}(0)] &= i\Delta\mathcal{O}(0)\\ [M_{\mu\nu},\mathcal{O}(0)] &= -\mathcal{S}_{\mu\nu}\mathcal{O}(0)\\ [K_\mu,\mathcal{O}(0)] &= 0 \end{aligned} \tag{4.17}$$

From these conditions, we can construct a representation of the conformal algebra out of $\mathcal{O}(0)$ and its descendants

$$\begin{array}{c|c} \text{operator} & \text{dimension} \\ \hline \vdots & \\ P_{\mu_1}P_{\mu_2}\mathcal{O}(0) & \Delta+2 \\ \uparrow & \\ P_{\mu_1}\mathcal{O}(0) & \Delta+1 \\ \uparrow & \\ \mathcal{O}(0) & \Delta \end{array} \tag{4.18}$$

The action of any conformal generator on any state follows from the conformal algebra. This should remind you of the construction of irreducible representations of SU(2) starting from a highest-weight state. In this case, our primary is a *lowest-weight* state of $-iD$, but the representation is built in an analogous way.[11] It turns out that any local operator in a unitary CFT is a linear combination of primaries and descendants. We will prove this in section **??**.

4.4 Finite Conformal Transformations

We can also act on primary operators with exponentiated charges $U = e^{-iQ_\epsilon}$ corresponding to finite conformal transformations. (As usual, in path-integral language, acting with U means surrounding $\mathcal{O}(x)$ with a topological surface operator $U[\Sigma]$.) To act on an operator $\mathcal{O}(x)$, we must find a decomposition

$$Ue^{-ix\cdot P} = e^{-ix'(x)\cdot P}e^{-i\lambda(x)D}e^{-i\omega(x)\cdot M}e^{-ib(x)\cdot K} \tag{4.19}$$

Using the primariness conditions (4.17), it follows that

$$U\mathcal{O}(x)U^{-1} = e^{\lambda(x)\Delta}e^{i\omega(x)\cdot\mathcal{S}}\mathcal{O}(x'(x)) \tag{4.20}$$

Exercise 8. *By thinking about (4.19), deduce that*

$$e^{\lambda(x)\Delta} = \Omega(x')^\Delta \tag{4.21}$$

$$(e^{i\omega(x)\cdot\mathcal{S}})^a{}_b = R[O(x')]^a{}_b \tag{4.22}$$

where the scalar factor $\Omega(x)$ and orthogonal matrix $O^\mu{}_\nu(x)$ are given by

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x')O^\mu{}_\nu(x'), \tag{4.23}$$

and $R[O(x')]^a{}_b$ denotes the action of $O(x')$ in the representation of SO(d) *associated to the operator $\mathcal{O}$. For example,*

$$R[O(x')] = 1 \qquad \textit{when } \mathcal{O} \textit{ is a scalar,} \tag{4.24}$$

$$R[O(x')]^\mu{}_\nu = O^\mu{}_\nu(x') \qquad \textit{when } \mathcal{O}^\nu \textit{ is a vector, etc.} \tag{4.25}$$

Thus, primary operators satisfy the transformation rule

$$U\mathcal{O}(x)U^{-1} = \Omega(x')^\Delta R[O(x')]\mathcal{O}(x') \tag{4.26}$$

We could have started the whole course with this relation, but the connection to the underlying conformal algebra will be crucial in what follows, so it is best to derive it.

[11] The representation (4.18) can be thought of as an *induced representation* $\mathrm{Ind}_H^G(R_H)$, where H is the subgroup of the conformal group generated by $D, M_{\mu\nu}, K_\mu$ (sometimes called the isotropy subgroup), R_H is the finite-dimensional representation of H defined by (4.17), and G is the full conformal group.

5 Conformal Correlators

5.1 Scalar Operators

We have already seen that scale invariance fixes two-point functions of scalars up to a constant

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \qquad \text{(SFT)} \tag{5.1}$$

For primary scalars in a CFT, the correlators must satisfy a stronger condition,

$$\begin{aligned} \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle &= \langle (U\mathcal{O}_1(x_1)U^{-1})\cdots(U\mathcal{O}_n(x_n)U^{-1})\rangle \\ &= \Omega(x_1')^{\Delta_1}\cdots\Omega(x_n')^{\Delta_n}\langle \mathcal{O}_1(x_n')\cdots\mathcal{O}_n(x_n')\rangle. \end{aligned} \tag{5.2}$$

Let us check whether this holds for (5.1).

Exercise 9. *Show that under a conformal transformation,*

$$(x-y)^2 = \frac{(x'-y')^2}{\Omega(x')\Omega(y')} \tag{5.3}$$

Hint: This is obviously true for translations, rotations, and scale transformations. It suffices to check it for inversions $I : x \to \frac{x}{x^2}$*, since* $k = IpI$*.*

Thus,

$$\frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \Omega(x_1')^{\frac{\Delta_1+\Delta_2}{2}}\Omega(x_2')^{\frac{\Delta_1+\Delta_2}{2}}\frac{C}{|x_1' - x_2'|^{\Delta_1+\Delta_2}}. \tag{5.4}$$

Consistency with (5.2) requires $C = 0$ unless $\Delta_1 = \Delta_2$. In other words,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C\delta_{\Delta_1\Delta_2}}{x_{12}^{2\Delta_1}} \qquad \text{(CFT, primary operators)}. \tag{5.5}$$

where $x_{12} \equiv x_1 - x_2$.

Exercise 10. *Recover the same result by demanding that the action of* K_μ *on* $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$ *vanish.*

Conformal invariance is also sufficiently powerful to fix a three-point function of primary scalars. Using (5.3), it's easy to check that the famous formula

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{f_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{5.6}$$

(where f_{123} is a constant) satisfies the consistency condition (5.2).

For four-point functions, there exist nontrivial conformally-invariant combinations of four points called "conformal cross-ratios,"

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \tag{5.7}$$

The reason that there are exactly two independent cross-ratios can be understood as follows.

- Using special conformal transformations, we can move x_4 to infinity.
- Using translations, we can move x_1 to zero.
- Using rotations, we can move x_3 to $(1, 0, \ldots, 0)$.
- Using rotations that fix x_3, we can move x_2 to the point $(x, y, 0, \ldots, 0)$.

There are exactly two undetermined quantities x, y, providing two independent conformal cross-ratios. Evaluating u and v for this special configuration of points gives

$$u = z\bar{z}, \qquad v = (1-z)(1-\bar{z}). \tag{5.8}$$

where $z \equiv x + iy$.

Four-point functions can depend nontrivially on the cross-ratios. In the case of identical scalars ϕ with dimension Δ_ϕ,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \quad = \quad \frac{g(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \tag{5.9}$$

is consistent with conformal invariance for a general function $g(u, v)$ of two variables.

Exercise 11. *Generalize (5.9) to the case of non-identical scalars $\phi_i(x)$ of dimension Δ_i.*

Note that the four-point function (5.9) is manifestly invariant under permutations of the points x_i. This leads to consistency conditions on $g(u, v)$,

$$g(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} g(v,u) \qquad \text{from swapping } 1 \leftrightarrow 3 \tag{5.10}$$

$$g(u,v) = g(u/v, 1/v) \qquad \text{from swapping } 1 \leftrightarrow 2 \text{ or } 3 \leftrightarrow 4 \tag{5.11}$$

(All other permutations can be generated from the ones above.) We will see shortly that $g(u, v)$ is not arbitrary, but is actually determined in terms of the dimensions Δ_i and three-point function coefficients f_{ijk} of the theory. Together with (5.10) this leads to powerful constraints on the Δ_i, f_{ijk}.

5.2 Operators With Spin

The story is similar for operators with spin. For brevity, we will simply write down the answers without doing any computations. When Joao discusses the embedding formalism next week, you will learn a totally transparent and practical way to derive all of these results, so it's not worth dwelling on them here.

Two-point functions of spinning operators are completely fixed by conformal invariance. They vanish when the two operators have different dimensions or spins. For example, a two-point function of spin-1 operators is given by

$$\langle J_\mu(x) J_\nu(y)\rangle = \frac{I_{\mu\nu}(x-y)}{(x-y)^{2\Delta}}, \qquad I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}. \tag{5.12}$$

Note that $I_{\mu\nu}$ is the orthogonal matrix associated with an inversion, via $\frac{\partial x'^\mu}{\partial x^\nu} = \Omega I^\mu{}_\nu$.

Exercise 12. *Check that (5.12) is consistent with conformal symmetry. Hint: it is enough to check inversions.*

Two-point functions of operators in more general spin representations can be constructed from the above, for example for spin-ℓ traceless symmetric tensors, we have

$$\langle J_{\mu_1\ldots\mu_\ell}(x) J_{\nu_1\ldots\nu_\ell}(0)\rangle = \text{symmetrize}\left(\frac{I_{\mu_1\nu_1}(x)\cdots I_{\mu_\ell\nu_\ell}(x)}{x^{2\Delta}}\right) - \text{traces}. \tag{5.13}$$

Three-point functions are fixed up to a finite number of coefficients. For example, a three-point function of two scalars ϕ and a spin-ℓ operator $J_{\mu_1\ldots\mu_\ell}$ is determined up to a single coefficient $f_{\phi\phi J}$,

$$\begin{aligned}\langle \phi_1(x_1)\phi_2(x_2) J^{\mu_1\ldots\mu_\ell}(x_3)\rangle &= f_{\phi\phi J}\frac{(Z^{\mu_1}\cdots Z^{\mu_\ell} - \text{traces})}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}},\\ Z^\mu &\equiv \frac{|x_{13}||x_{23}|}{|x_{12}|}\left(\frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}\right).\end{aligned} \tag{5.14}$$

6 Radial Quantization and the State-Operator Correspondence

So far, we've written a lot of commutation relations and I've been careful to point out that they are true in any quantization of the theory. Now we'll really put that idea to use. In general, it's a good idea to choose quantizations that respect symmetries of the theory. In a CFT, it's natural to foliate spacetime with spheres around the origin and consider evolving states from smaller spheres to larger spheres using the dilatation operator. This is called "radial quantization." Field configurations on S^{d-1} span a Hilbert space $\mathcal{H}$. We can act on $\mathcal{H}$ by inserting operators on the surface of the sphere. For example, to act with a symmetry generator Q, we insert the surface operator $Q[S^{d-1}]$ into the path integral.

In radial quantization, a correlation function becomes a radially ordered product,

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle &= \langle 0|\mathcal{R}\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\}|0\rangle & (6.1)\\ &= \theta(|x_n|-|x_{n-1})\cdots\theta(|x_2|-|x_1|)\langle 0|\mathcal{O}(x_n)\cdots\mathcal{O}(x_1)|0\rangle \\ &\quad +\text{permutations} & (6.2)\end{aligned}$$

Of course, we can perform radial quantization around different points. In this way, the same correlation function gets interpreted as an expectation value of differently ordered operators in different vacuum states. This is totally analogous to changing reference frames in Lorentz invariant theories. It is consistent because operators at the same radius but at different directions on the sphere commute, just as spacelike-separated operators commute in the usual quantization.

6.1 Operator $\Longrightarrow$ State

The simplest way to prepare a state in $\mathcal{H}$ is to perform the path integral over the interior B of the sphere, with no operator insertions inside B. This gives the vacuum state $|0\rangle$ on ∂B. It's easy to see that $|0\rangle$ is invariant under all symmetries because a topological surface on the boundary of B can be shrunk to zero inside B.

A more exciting possibility is to insert an operator $\mathcal{O}(x)$ inside B and then perform the path integral. This defines a state which we call $\mathcal{O}(x)|0\rangle \in \mathcal{H}$. In general, by inserting many operators inside B, we can prepare a variety of states on the boundary ∂B. In this language, $|0\rangle$ is prepared by inserting the unit operator.

6.2 State $\Longrightarrow$ Operator

This construction also works backwards. Given a state $|\psi\rangle$ in radial quantization, it's natural to decompose it into eigenstates of the dilatation operator D

$$|\psi\rangle = |\mathcal{O}_1\rangle + |\mathcal{O}_2\rangle + \dots \qquad (6.3)$$

$$D|\psi\rangle = i\Delta_1|\mathcal{O}_1\rangle + i\Delta_2|\mathcal{O}_2\rangle + \dots \qquad (6.4)$$

These eigenstates $|\mathcal{O}_i\rangle$ can themselves be used as operators. We can cut small spherical holes out of the path integral centered around positions x_i and glue in the states $|\mathcal{O}_i\rangle$. This procedure gives a quantity that behaves exactly like a correlator of local operators $\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle$.

6.3 State $\Longleftrightarrow$ Operator

So far I've been vague about what I mean by a local operator. But now, we can give a rigorous definition: we will simply *define* the space of local operators to be the space of states on the sphere in radial quantization.[12] Its action is given by cutting and gluing as described above. With this definition, the two constructions above are inverse to each other, with the identification

$$\mathcal{O}(0) \quad \longleftrightarrow \quad |\mathcal{O}\rangle \tag{6.5}$$

We call this the "state-operator correspondence."

It is straightforward to see how the conformal group acts on states in radial quantization. A primary operator creates a state that is killed by K_μ and transforms in a finite-dimensional representation of D and $M_{\mu\nu}$,

$$[K_\mu, \mathcal{O}(0)] = 0 \qquad \longleftrightarrow \qquad K_\mu|\mathcal{O}\rangle = 0, \tag{6.6}$$

$$[D, \mathcal{O}(0)] = i\Delta\mathcal{O}(0) \qquad \longleftrightarrow \qquad D|\mathcal{O}\rangle = i\Delta|\mathcal{O}\rangle, \tag{6.7}$$

$$[M_{\mu\nu}, \mathcal{O}(0)] = -\mathcal{S}_{\mu\nu}\mathcal{O}(0) \qquad \longleftrightarrow \qquad M_{\mu\nu}|\mathcal{O}\rangle = -\mathcal{S}_{\mu\nu}|\mathcal{O}\rangle. \tag{6.8}$$

One can verify this by acting with the operator equations on $|0\rangle$ and using the fact that K, D, M kill $|0\rangle$. A conformal multiplet in radial quantization is given by acting with momentum generators on a primary state

$$|\mathcal{O}\rangle, P_\mu|\mathcal{O}\rangle, P_\mu P_\nu|\mathcal{O}\rangle, \ldots \qquad \text{(conformal multiplet)}. \tag{6.9}$$

This is equivalent to acting with derivatives of $\mathcal{O}(x)$ at the origin, for example

$$i\partial_\mu\mathcal{O}(0)|0\rangle \quad = \quad [P_\mu, \mathcal{O}(0)]|0\rangle \quad = \quad P_\mu|\mathcal{O}\rangle. \tag{6.10}$$

$\mathcal{O}(x)$ creates a linear combination of descendant states,

$$\mathcal{O}(x)|0\rangle \quad = \quad e^{-iP\cdot x}\mathcal{O}(0)e^{iP\cdot x}|0\rangle \quad = \quad e^{-iP\cdot x}|\mathcal{O}\rangle \quad = \quad \sum_{n=0}^{\infty}\frac{1}{n!}(-iP\cdot x)^n|\mathcal{O}\rangle. \tag{6.11}$$

As with the classification of operators, the action of the conformal algebra on a multiplet in radial quantization is determined by the commutation relations of the algebra. In fact the required computations look *exactly identical* to the computations we did to determine the action of conformal generators on operators! This is because by surrounding operators with charges supported on spheres, we were secretly already doing radial quantization.

6.4 Another View of Radial Quantization

To study a conformal Killing vector ϵ, it is useful to perform a Weyl rescaling of the metric $g \to \Omega(x)^2 g$ so that ϵ becomes a regular Killing vector (isometry). We can turn a dilatation

[12] The dilatation operator is diagonalizable in all unitary (reflection positive) CFTs. However, there exist interesting non-unitary theories where D has a nontrivial Jordan block decomposition. In these cases, we define a local operator as a finite-dimensional representation of D.

into an isometry by performing a Weyl rescaling from $\mathbb{R}^d$ to the cylinder $\mathbb{R} \times S^{d-1}$,

$$\begin{aligned} ds^2_{\mathbb{R}^d} &= dr^2 + r^2 ds^2_{S^{d-1}} \\ &= r^2\left(\frac{dr^2}{r^2} + ds^2_{S^{d-1}}\right) \\ &= e^{2\tau}(d\tau^2 + ds^2_{S^{d-1}}) = e^{2\tau} ds^2_{\mathbb{R}\times S^{d-1}}, \end{aligned} \tag{6.12}$$

where $r = e^\tau$.

Dilatations $r \to \lambda r$ become shifts of radial time $\tau \to \tau + \log\lambda$. Radial quantization in flat space is equivalent to the usual quantization on the cylinder. States live on spheres and time evolution is generated by acting with $e^{iD\tau}$.

Let us build a more detailed dictionary between the two pictures. Under a Weyl rescaling, correlation functions of local operators transform as[13]

$$\frac{\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_g}{\langle 1\rangle_g} = \left(\prod_i \Omega(x_i)^{\Delta_i}\right)\frac{\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{\Omega^2 g}}{\langle 1\rangle_{\Omega^2 g}}. \tag{6.13}$$

This is a nontrivial claim — if we implement the Ising model in flat space, compute expectation values and take the continuum limit, it's non-obvious that the answer should be related in a simple way to the same lattice theory on the cylinder. In general it's not, but at the critical value of the coupling when the theory becomes conformal, tracelessness of the stress tensor implies insensitivity to Weyl rescalings, and the answers become related.

Thus, given an operator $\mathcal{O}(x)$ in $\mathbb{R}^d$, it is natural to define a cylinder operator

$$\mathcal{O}_{\text{cyl.}}(\tau, \mathbf{n}) \equiv e^{\Delta\tau}\mathcal{O}_{\text{flat}}(x = e^\tau \mathbf{n}) \tag{6.14}$$

We often omit the subscript "cyl." or "flat" and indicate which type of operator to use by its coordinate.

Exercise 13. *Using (6.13), compute a two-point function of cylinder operators*

$$\langle \mathcal{O}(\tau_1, \mathbf{n}_1)\mathcal{O}(\tau_2, \mathbf{n}_2)\rangle. \tag{6.15}$$

Verify that it is time-translationally invariant on the cylinder. Show that in the limit of large time separation $\tau = \tau_2 - \tau_1 \gg 1$, the two-point function has an expansion in terms of the form $e^{-(\Delta+n)\tau}$ with integer $n \geq 0$. Interpret these as coming from the exchange of states in the conformal multiplet of $\mathcal{O}$.

6.5 Reflection Positivity

In Lorentzian signature, we are usually interested in unitary theories — that is, theories for which the symmetry generators, including the Hamiltonian, are Hermitian operators so that

[13] In even dimensions, the partition function itself can transform with a Weyl anomaly $\langle 1\rangle_g = \langle 1\rangle_{\Omega^2 g} e^{S_{\text{Weyl}}[g]}$. This will not be important for our discussion, so we have divided through by the partition function.

they become unitary transformations when exponentiated. Unitarity in Lorentzian signature is equivalent to a property called "reflection positivity" in Euclidean signature. Suppose we have a Lorentzian theory with energy-momentum generators $(H, \mathbf{P})$. Local operators at points are defined as

$$\mathcal{O}(t, \mathbf{x}) = e^{iHt - i\mathbf{x}\cdot\mathbf{P}} \mathcal{O}(0,0) e^{-iHt + i\mathbf{x}\cdot\mathbf{P}} \tag{6.16}$$

If $\mathcal{O}(0,0)$ is Hermitian, then it follows that $\mathcal{O}(t, \mathbf{x})$ is Hermitian. Now, let us Wick-rotate to Euclidean signature $t \to -it_E$,

$$\mathcal{O}(t_E, \mathbf{x}) = e^{Ht - i\mathbf{x}\cdot\mathbf{P}} \mathcal{O}(0,0) e^{-Ht_E + i\mathbf{x}\cdot\mathbf{P}} \tag{6.17}$$

The Euclidean operator satisfies

$$\mathcal{O}(t_E, \mathbf{x})^\dagger = \mathcal{O}(-t_E, \mathbf{x}) \tag{6.18}$$

Thus, the natural notion of complex-conjugation in a Euclidean theory involves a reflection in the time-direction. This means that whether an operator is Hermitian or not depends on how we choose to quantize the theory! For example, consider the momentum generators

$$P^\mu = -i \int d^{d-1}\mathbf{x}\, T^{\mu 0}(0, \mathbf{x}) \tag{6.19}$$

Hermitian conjugation includes a reflection $\Theta^\mu{}_\nu = \delta^\mu_\nu - 2\delta^\mu_0 \delta^0_\nu$ in the time direction,

$$\mathcal{O}^{\mu_1 \cdots \mu_\ell}(t_E, \mathbf{x})^\dagger = \Theta^{\mu_1}{}_{\nu_1} \cdots \Theta^{\mu_\ell}{}_{\nu_\ell} \mathcal{O}^{\nu_1 \cdots \nu_\ell}(-t_E, \mathbf{x}) \tag{6.20}$$

which acts nontrivially on operators with spin. In particular,

$$T^{i0}(0, \mathbf{x})^\dagger = -T^{i0}(0, \mathbf{x}) \tag{6.21}$$

$$T^{00}(0, \mathbf{x})^\dagger = T^{00}(0, \mathbf{x}) \tag{6.22}$$

It follows that P^0 is *anti-Hermitian*, while P^i are Hermitian (this is the reason for the conventional factor of i in front of the momentum charges). We may write $P^0 = iH$, where H is Hermitian, and then we recover the same formula as we got from Wick rotation (6.17).

To reiterate, *the correct notion of conjugation depends on how we quantize our theory.* This makes sense, because Hermitian conjugation is something that makes sense for operators on a specific Hilbert space, and different quantizations have different Hilbert spaces.

This raises the question — given a Euclidean path integral, how do we tell it admits a consistent notion of conjugation? One important condition is that norms of states should be positive. Consider some in-state at time $t_E = 0$, given by acting on the vacuum with a bunch of operators at negative Euclidean time

$$|\psi\rangle = \mathcal{O}(t_{E1}) \cdots \mathcal{O}(t_{En})|0\rangle \tag{6.23}$$

For brevity, I'm suppressing the spatial positions $\mathbf{x}_i$ of the operators. The conjugate state is given by

$$\langle\psi| = (\mathcal{O}(t_{E1}) \cdots \mathcal{O}(t_{En})|0\rangle)^\dagger \tag{6.24}$$

$$= \langle 0|\mathcal{O}(-t_{En}) \cdots \mathcal{O}(-t_{E1}) \tag{6.25}$$

That is, the conjugate state is given by starting with the vacuum in the future and positioning operators in a time-reflected way. Thus, in path-integral language, the condition

$$\langle\psi|\psi\rangle \quad \geq \quad 0 \tag{6.26}$$

says that a time-reflection symmetric configuration of Hermitian operators should have a positive path integral. This is called "reflection positivity." It follows if the theory is a Wick-rotation of a unitary theory in Lorentzian signature. However, sometimes the definition of a theory is more natural in Euclidean space. In this case, reflection positivity is often very easy to check. For example, in the 3d Ising lattice model, reflection-positivity is obvious: by cutting the path integral at $t_E = 0$, it's clear that the expectation value of a reflection-symmetric configuration of operators is a sum of squares, and hence positive.

6.6 Reflection Positivity on the Cylinder

Reflection-positivity (or unitarity) has interesting consequences when we view our CFT on the cylinder. The first consequence, as you can easily check, is that D is anti-Hermitian in radial quantization. This is why we've been writing its eigenvalues as $i\Delta$. The conjugate of a cylinder operator is given by

$$\mathcal{O}_{\text{cyl.}}(\tau, \mathbf{n})^\dagger \quad = \quad \mathcal{O}_{\text{cyl.}}(-\tau, \mathbf{n}) \tag{6.27}$$

In flat space, this becomes

$$\mathcal{O}_{\text{flat}}(e^\tau \mathbf{n})^\dagger \quad = \quad e^{-2\Delta\tau}\mathcal{O}_{\text{flat}}(e^{-\tau}\mathbf{n}) \tag{6.28}$$

$$\mathcal{O}_{\text{flat}}(x)^\dagger \quad = \quad x^{-2\Delta}\mathcal{O}_{\text{flat}}\left(\frac{x^\mu}{x^2}\right) \qquad \text{(radial quantization)} \tag{6.29}$$

This is just the image of $\mathcal{O}(x)$ under an inversion $I : x^\mu \to \frac{x^\mu}{x^2}$. For operators with spin, we must use the full formula (4.26), so that conjugation also involves an action of $I_{\mu\nu}$ on spin-indices.

Exercise 14. *Check that the 2-point function of spin-1 operators (5.12) satisfies reflection-positivity.*

The action of conjugation on the conformal charges is

$$Q_\epsilon^\dagger \quad = \quad Q_{I\epsilon I} \tag{6.30}$$

In particular, we have

$$\begin{aligned} M_{\mu\nu}^\dagger &= M_{\mu\nu} \\ D^\dagger &= -D \\ P_\mu^\dagger &= K_\mu \end{aligned} \tag{6.31}$$

These facts let us calculate properties of correlation functions purely algebraically. As an example, consider a two-point function.

$$\begin{aligned}
\langle\mathcal{O}(y)\mathcal{O}(x)\rangle &= \langle 0|(y^{-2\Delta}\mathcal{O}(y/y^2))^\dagger\mathcal{O}(x)|0\rangle \\
&= y^{-2\Delta}\langle 0|(e^{-i(y/y^2)\cdot P}\mathcal{O}(0)e^{i(y/y^2)\cdot P})^\dagger e^{-ix\cdot P}\mathcal{O}(0)e^{ix\cdot P}|0\rangle \\
&= y^{-2\Delta}\langle 0|(e^{-i(y/y^2)\cdot K}\mathcal{O}(0)^\dagger e^{i(y/y^2)\cdot K}e^{-ix\cdot P}\mathcal{O}(0)e^{ix\cdot P}|0\rangle \\
&= y^{-2\Delta}\langle 0|\mathcal{O}(0)^\dagger e^{i(y/y^2)\cdot K}e^{-ix\cdot P}\mathcal{O}(0)|0\rangle \\
&= y^{-2\Delta}\langle\mathcal{O}|e^{i(y/y^2)\cdot K}e^{-ix\cdot P}|\mathcal{O}\rangle
\end{aligned} \tag{6.32}$$

Where we've defined $\langle 0|\mathcal{O}(0)^\dagger \equiv \langle\mathcal{O}|$. By expanding the exponentials, we can evaluate this using the conformal algebra. For example, the first couple terms are

$$\langle\mathcal{O}(y)\mathcal{O}(x)\rangle = y^{-2\Delta}\left(\langle\mathcal{O}|\mathcal{O}\rangle + \frac{y^\mu}{y^2}x^\nu\langle\mathcal{O}|K_\mu P_\nu|\mathcal{O}\rangle + \dots\right) \tag{6.33}$$

Here we've used that $K|\mathcal{O}\rangle = \langle\mathcal{O}|P = 0$ because $\mathcal{O}$ is primary. We need to compute

$$\begin{aligned}
\langle\mathcal{O}|K_\mu P_\nu|\mathcal{O}\rangle &= \langle\mathcal{O}|[K_\mu, P_\nu]|\mathcal{O}\rangle \\
&= \langle\mathcal{O}| - 2i(D\delta_{\mu\nu} - M_{\mu\nu})|\mathcal{O}\rangle \\
&= 2\Delta\delta_{\mu\nu}\langle\mathcal{O}|\mathcal{O}\rangle
\end{aligned} \tag{6.34}$$

since $|\mathcal{O}\rangle$ is a scalar. Thus, we get

$$= y^{-2\Delta}\langle\mathcal{O}|\mathcal{O}\rangle\left(1 + 2\Delta\frac{y\cdot x}{y^2} + \dots\right) \tag{6.35}$$

Here we've used that $K|\mathcal{O}\rangle = \langle\mathcal{O}|P = 0$ because $\mathcal{O}$ is primary. We've also used that $M|\mathcal{O}\rangle = 0$ (for a scalar) and $D|\mathcal{O}\rangle = i\Delta|\mathcal{O}\rangle$. This exactly matches the expansion of $1/(x-y)^{2\Delta}$ in small $|x|/|y|$! The overlap $\langle\mathcal{O}|\mathcal{O}\rangle$ is the normalization coefficient of our two-point function. You can imagine similarly computing all the commutators for higher terms and matching the whole series expansion.

Let us also prove our earlier claim that a two-point function of operators in different irreducible spin representations must vanish. Consider a primary operator $\mathcal{O}^a$ transforming in a nontrivial representation of SO(d). The dual state transforms in the complex-conjugate representation, so we will write it with a lowered index $(|\mathcal{O}^a\rangle)^\dagger = \langle\mathcal{O}_a|$. Consider

$$\langle\mathcal{O}_a|M_{\mu\nu}|\mathcal{O}^b\rangle = \langle\mathcal{O}^a| - (\mathcal{S}_{\mu\nu})^b{}_c|\mathcal{O}^c\rangle \qquad = -(\mathcal{S}_{\mu\nu})^b{}_c\langle\mathcal{O}_a|\mathcal{O}^c\rangle \tag{6.36}$$

$$= \langle\mathcal{O}_c|(-\mathcal{S}_{\mu\nu})^c{}_a|\mathcal{O}^b\rangle \qquad = -(\mathcal{S}_{\mu\nu})^c{}_a\langle\mathcal{O}_c|\mathcal{O}^b\rangle \tag{6.37}$$

Where we've acted with $M_{\mu\nu}$ first on the right, and then on the left (and also used that $M_{\mu\nu}$ and $\mathcal{S}_{\mu\nu}$ are Hermitian. As a matrix equation, this is

$$\mathcal{S}_{\mu\nu}N = N\mathcal{S}_{\mu\nu} \tag{6.38}$$

where $N^a{}_b \equiv \langle\mathcal{O}_b|\mathcal{O}^a\rangle$. By Schur's lemma, $N^a{}_b$ must vanish if a and b index different irreducible representations. If a, b index a single irreducible representation, then N is proportional to the identity.

Exercise 15. *This computation is not directly relevant to the course, but it is instructive for getting used to radial ordering. Consider a three-point function of scalars*

$$\begin{aligned}\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle &= \langle 0|\mathcal{R}\{\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\}|0\rangle &(6.39)\\ &= \theta(|x_3|-|x_2|)\theta(|x_2|-|x_1|)\langle 0|\mathcal{O}_k(x_3)\mathcal{O}_j(x_2)\mathcal{O}_i(x_1)|0\rangle \\ &\quad +\text{permutations} &(6.40)\end{aligned}$$

Consider the operator $e^{2\pi\mathcal{D}_1}$ where

$$\mathcal{D}_1 = i(x_1\cdot\partial_1+\Delta_1). \tag{6.41}$$

Using the fact that $e^{2\pi\mathcal{D}_1}\mathcal{O}_i(x_1) = e^{2\pi D}\mathcal{O}_i(x_1)e^{-2\pi D}$, compute the action of $e^{2\pi\mathcal{D}_1}$ on each of the terms above. You will get different answers for each of the different operator orderings.

Now determine the action of $e^{2\pi\mathcal{D}_1}$ on the known answer for the scalar three-pt function (5.6). Check that the two answers agree.

6.7 Unitarity Bounds

Thinking about the theory on the cylinder thus gives us a natural inner product on states in radial quantization. Reflection positivity implies that the norms of states with respect to this inner product should be nonnegative. For a nonzero primary operator $\mathcal{O}^a$ in an irreducible representation $R_\mathcal{O}$ of $\mathrm{SO}(d)$, we should have

$$\langle \mathcal{O}_b|\mathcal{O}^a\rangle = c\delta^a_b \tag{6.42}$$

with $c>0$. Often, we normalize $\mathcal{O}$ so that $c=1$. We can compute norms of descendants using the conformal algebra, and demanding positivity gives interesting constraints on operator dimensions. These are called "unitarity bounds" because they follow from reflection-positivity, which is the Wick-rotated version of unitarity. For example,

$$\begin{aligned}(P_\mu|\mathcal{O}^a\rangle)^\dagger P_\nu|\mathcal{O}^b\rangle &= \langle\mathcal{O}_a|K_\mu P_\nu|\mathcal{O}^b\rangle \\ &= \langle\mathcal{O}_a|-2i(D\delta_{\mu\nu}-M_{\mu\nu})|\mathcal{O}^b\rangle \\ &= \langle\mathcal{O}_a|2\Delta\delta_{\mu\nu}+2i(\mathcal{S}_{\mu\nu})^b{}_c|\mathcal{O}^c\rangle \\ &= 2\Delta\delta_{\mu\nu}\delta^b_a+2i(S_{\mu\nu})^b{}_a &(6.43)\end{aligned}$$

If we think of this as a matrix with indices (μ,a) and (ν,b), then it must be positive semidefinite. This means that 2Δ must be greater than the maximum eigenvalue of $-2i(\mathcal{S}_{\mu\nu})^b{}_a$. For scalar operators, this gives $\Delta>0$. For non-scalars, let us write

$$\begin{aligned}-2i(\mathcal{S}_{\mu\nu})^b{}_a &= -(L^{\alpha\beta})_{\mu\nu}(\mathcal{S}_{\alpha\beta})^b{}_a \\ (L^{\alpha\beta})_{\mu\nu} &\equiv i(\delta^\alpha_\mu\delta^\beta_\nu-\delta^\alpha_\nu\delta^\beta_\mu) &(6.44)\end{aligned}$$

where $(L^{\alpha\beta})_{\mu\nu}$ is the generator of rotations in the vector representation V. Let us write this more abstractly as

$$\begin{aligned}-\mathbf{L}^A\mathbf{S}_A &= \frac{1}{2}\left(\mathbf{L}^2+\mathbf{S}^2-(\mathbf{L}+\mathbf{S})^2\right) \\ &= \frac{1}{2}\left(\mathrm{Casimir}(V)+\mathrm{Casimir}(R_\mathcal{O})-\mathrm{Casimir}(V\otimes R_\mathcal{O})\right) &(6.45)\end{aligned}$$

where $A = \alpha\beta$ indexes the adjoint representation of $\mathrm{SO}(d)$, and $\mathbf{L} = \mathbf{L} \otimes 1$ acts on the first factor in the tensor product $V \otimes R_{\mathcal{O}}$, while $\mathbf{S} = 1 \otimes \mathbf{S}$ acts on the second factor. This should look familiar from quantum mechanics when we compute the eigenvalues of angular momentum times spin.

Let's specialize to the case where $R_{\mathcal{O}}$ is V_ℓ, the spin-ℓ traceless symmetric tensor representation of $\mathrm{SO}(d)$. It has the Casimir $\ell(\ell+d-2)$. To get the maximal eigenvalue of $-\mathbf{L}\cdot\mathbf{S}$, we need the minimal eigenvalue of the Casimir acting on $V \otimes V_\ell = V_{\ell-1} \oplus \ldots$, where "$\ldots$" are representations with larger Casimirs. Thus,

$$\begin{aligned} \text{max-eigenvalue}(-\mathbf{L}\cdot\mathbf{S}) &= \frac{1}{2}\left((d-1) + \ell(\ell+d-2) - (\ell-1)(\ell-1+d-2)\right) \\ &= \ell + d - 2 \end{aligned} \tag{6.46}$$

This computation was valid for $\ell > 0$, since for scalars $V \otimes V_{\ell=0} = V$.

One can also consider more complicated descendants.

Exercise 16. *Compute the norm of $P_\mu P^\mu|\mathcal{O}\rangle$, where $\mathcal{O}$ is a scalar. Show that nonnegativity implies either $\Delta = 0$ or $\Delta \geq \frac{d-2}{2}$. This gives a slightly stronger condition than what we derived above ($\Delta \geq 0$) for scalars.*

It turns out that for a general conformal field theory, these inequalities are the best you can do (other descendants give no new information). In theories with more symmetry, like supersymmetric theories or 2d CFTs, unitarity bounds can be more interesting. In summary, we have

$$\begin{aligned} \Delta &= 0 \quad \text{(unit operator), or} \\ \Delta &\geq \begin{cases} \frac{d-2}{2} & \ell = 0 \\ \ell + d - 2 & \ell > 0 \end{cases} \end{aligned} \tag{6.47}$$

When Δ saturates these bounds, it means that the conformal multiplet has a null state. For the unit operator, all descendants are null. For a scalar, the null state is

$$P^2|\mathcal{O}\rangle = 0 \tag{6.48}$$

Translated into operator language, this says $\partial^2\mathcal{O}(x) = 0$, which means $\mathcal{O}$ satisfies the Klein-Gordon equation, and is thus a free scalar which decouples from the rest of the CFT. For a spin-ℓ operator, the null state is

$$P_\mu|\mathcal{O}^{\mu\mu_2\cdots\mu_\ell}\rangle = 0 \tag{6.49}$$

(this is related to the fact that we used $V_{\ell-1} \subset V \otimes V_\ell$ to compute the unitarity bound). In operator language, this is

$$\partial_\mu \mathcal{O}^{\mu\mu_2\cdots\mu_\ell}(x) = 0, \tag{6.50}$$

which is the equation for a conserved current. We can also run this logic backwards to conclude: an operator is a conserved current if and only if it satisfies

$$\Delta = \ell + d - 2 \qquad \text{(conserved current)} \tag{6.51}$$

Some important examples are global symmetry currents ($\ell = 1$, $\Delta = d - 1$) and the stress-tensor ($\ell = 2$, $\Delta = d$). It is expected that for CFTs in $d > 2$, higher spin currents cannot be present unless the theory is free. This has been proven in $d = 3$.

6.8 Only Primaries and Descendants

With a positive-definite inner product, we can now prove that all operators are linear combinations of primaries and descendants. We will use one additional physical assumption, which is that the partition function of the theory on $S^{d-1} \times S^1_\beta$ is finite,

$$\mathcal{Z}_{S^{d-1}\times S^1_\beta} = \mathrm{Tr}(e^{i\beta D}) < \infty. \tag{6.52}$$

In a unitary theory, this means that $e^{i\beta D}$ is Hermitian and trace-class, and hence diagonalizable. It follows that D is also diagonalizable with pure-imaginary eigenvalues $i\Delta$.

Now consider a local operator $\mathcal{O}$, and assume for simplicity it is an eigenvector of dilatation with dimension Δ. By finiteness of the partition function, there are a finite number of primary operators $\mathcal{O}_p$ with dimension less than or equal to Δ. Using the inner product, we may subtract off the projections of $\mathcal{O}$ onto the conformal multiplets of $\mathcal{O}_p$ to get $\mathcal{O}'$. Let us assume $\mathcal{O}' \neq 0$. Acting on it with K_μ's, we must eventually get zero (again by finiteness of the partition function), which means there is a new primary with dimension below Δ, a contradiction. Thus $\mathcal{O}' = 0$, and $\mathcal{O}$ is a linear combination of states in the multiplets $\mathcal{O}_p$.

7 The Operator Product Expansion

If we insert two operators $\mathcal{O}^a_i(x)\mathcal{O}^b_j(0)$ and perform the path integral over the interior of the sphere, then we get some state on the sphere. We can now decompose this state as

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle \;=\; \sum_k C_{ijk}(x, \partial_y)\mathcal{O}_k(y)|_{y=0}|0\rangle, \tag{7.1}$$

where k runs over primary operators and $C_{ijk}(x, \partial_y)$ is a differential operator that packages together primaries and descendants in the same conformal multiplet. This expansion is an exact equation that can be used in the path integral, as long as other operator insertions are outside the sphere with radius $|x|$. Thus, we often write

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \;=\; \sum_k C_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2) \tag{7.2}$$

where it is understood that this equation is valid inside any correlation function where the other operators $\mathcal{O}_m(x_m)$ have $|x_{2m}| \geq |x_{12}|$. This equation is called the OPE, or Operator Product Expansion.

We could have considered both operators away from the origin and written

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \;\;=\;\; \sum_k C'_{ijk}(x_1, x_2, \partial_y)\mathcal{O}_k(y)|_{y=0} \tag{7.3}$$

where $C'_{ijk}(x_1, x_2, \partial_y)$ is some other differential operator. It is more convenient for computations to use the first form of the OPE, but the existence of the second form is important. It shows that we can do the OPE between two operators, replacing them with an infinite sum over single operators, whenever it's possible to draw any sphere that separates the two operators from all the others.

We are being a bit schematic in writing the above equation. It's possible for all the operators to have spin. In this case, the OPE looks like

$$\mathcal{O}_i^a(x)\mathcal{O}_j^b(0) \;\;=\;\; \sum_k C_{ijkc}^{ab}(x, \partial_y)\mathcal{O}_k^c(y)|_{y=0} \tag{7.4}$$

where $C^{ab}{}_c$ is a differential operator with nontrivial SO(d) indices.

7.1 Consistency with Conformal Invariance

Conformal invariance strongly restricts the form of the OPE. For simplicity, let's focus on scalar operators. Acting with $e^{-iD\lambda}$ on (7.1), the left-hand side becomes

$$e^{\lambda(\Delta_i+\Delta_k)}\mathcal{O}_i(e^\lambda x)\mathcal{O}_j(0)|0\rangle \tag{7.5}$$

The right-hand side becomes

$$C(x, \partial_y)e^{\Delta_k\lambda}\mathcal{O}_k(e^\lambda y)|_{y=0}|0\rangle \;\;=\;\; C(x, e^\lambda\partial_y)e^{\Delta_k\lambda}\mathcal{O}_k(y)|_{y=0}|0\rangle. \tag{7.6}$$

Equating, we find

$$C(e^\lambda x, \partial_y) \;\;=\;\; e^{\lambda(\Delta_k-\Delta_i-\Delta_j)}C(x, e^\lambda\partial_y) \tag{7.7}$$

So that C has an expansion

$$C(x, \partial) \;\;\propto\;\; |x|^{\Delta_k-\Delta_i-\Delta_j}\,(1 + \#x^\mu\partial_\mu + \#x^\mu x^\nu\partial_\mu\partial_\nu + \ldots) \tag{7.8}$$

This is just a fancy way of saying we can do dimensional analysis: $\mathcal{O}_i$ has length-dimension $-\Delta_i$, so by matching dimensions, we see that $C(x, \partial)$ must have the expansion above. We're also implicitly using rotational invariance by contracting all the indices appropriately. We could have argued for this too by acting with $M_{\mu\nu}$.

We get a more interesting constraint by acting with K_μ. In fact, consistency with K_μ completely fixes C_{ijk} up to an overall coefficient. In particular, the coefficients in (7.8) can be determined in this way.

This computation is a little annoying (exercise!), so here's a simpler way to see why the form of the OPE is fixed, and to get the coefficients. Let us take correlation function of (7.2) with $\mathcal{O}_k(x_3)$ (we will assume $|x_{23}| \geq |x_{12}|$, so that the OPE is valid).

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \sum_{k'} C_{ijk'}(x_{12},\partial_2)\langle \mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle \tag{7.9}$$

The three-point function on the left-hand side is fixed by conformal invariance, and this will determine the right-hand side. Assume that primary operators are orthonormal, so that their 2-point functions are proportional to the identity $\langle \mathcal{O}_i\mathcal{O}_j\rangle = \delta_{ij}x^{-2\Delta_i}$. The sum collapses to a single term and we get

$$\frac{f_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k}x_{23}^{\Delta_j+\Delta_k-\Delta_i}x_{31}^{\Delta_k+\Delta_i-\Delta_j}} = C_{ijk}(x_{12},\partial_2)\frac{1}{x_{23}^{2\Delta_k}} \tag{7.10}$$

This determines C_{ijk} to be proportional to f_{ijk}, times a differential operator that depends only on the Δ_i's and can be obtained by matching the small $|x_{12}|/|x_{23}|$ expansion of both sides.

Exercise 17. *Consider the special case $\Delta_i = \Delta_j = \Delta_\phi$, $\Delta_k = \Delta$. Show*

$$\begin{aligned} C_{ijk}(x,\partial) &= f_{ijk}C(x,\partial) \\ C(x,\partial) &= x^{\Delta-2\Delta_\phi}\left(1+\frac{1}{2}x\cdot\partial+\alpha x^\mu x^\nu\partial_\mu\partial_\nu+\beta x^2\partial^2+\dots\right) \end{aligned} \tag{7.11}$$

where

$$\alpha = \frac{\Delta+2}{8(\Delta+1)}, \quad \textit{and} \quad \beta = -\frac{\Delta}{16(\Delta-\frac{d-2}{2})(\Delta+1)} \tag{7.12}$$

Equation (7.9) gives an example of using the OPE to reduce a three-point function to a sum of 2-point functions. In general, we can use the OPE to reduce any n-point function to a sum of $n-1$-point functions. Recursing, we can reduce everything to a sum of 1-point functions. In a CFT in flat space, all 1-point functions vanish by dimensional analysis, except for the unit operator which has $\langle 1\rangle = 1$.[14] This gives an algorithm for computing any flat-space correlation function using the OPE. It shows that all these correlators are determined by dimensions Δ_i and OPE coefficients f_{ijk}.

8 Conformal Blocks

8.1 Using the OPE

Let us use the OPE to compute a four-point function of identical scalars,

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{\Delta_\phi}x_{34}^{\Delta_\phi}}. \tag{8.1}$$

[14] The OPE is also valid on any manifold which is conformally flat. The difference is that on nontrivial manifolds, non-unit operators can have nonzero 1-point functions. An example is $\mathbb{R}^{d-1}\times S^1_\beta$, which has the interpretation as a CFT at finite temperature. By dimensional analysis, we have $\langle\mathcal{O}\rangle_{\mathbb{R}^{d-1}\times S^1_\beta} \propto \beta^{-\Delta_\mathcal{O}} \propto T^{\Delta_\mathcal{O}}$.

The OPE takes the form

$$\phi(x_1)\phi(x_2) \;=\; \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} C_a(x_{12}, \partial_2)\mathcal{O}^a(x_2), \tag{8.2}$$

where $\mathcal{O}^a$ can have nonzero spin in general. It turns out that in order for $\mathcal{O}^a$ to appear in the OPE of two scalars, it must transform in a spin-ℓ traceless symmetric tensor representation of SO(d).

Exercise 18. *By using the explicit form of the conformal 3-point function for two scalars and a spin-ℓ operator, show that $f_{\phi\phi\mathcal{O}}$ vanishes unless ℓ is even.*

Assuming the points are configured appropriately, we can pair up the operators (12) (34) and perform the OPE between them,[15]

$$\begin{aligned}
\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \sum_{\mathcal{O},\mathcal{O}'} f_{\phi\phi\mathcal{O}} f_{\phi\phi\mathcal{O}'} C_a(x_{12}, \partial_2) C_b(x_{34}, \partial_4) \langle\mathcal{O}^a(x_2)\mathcal{O}'^b(x_4)\rangle \\
&= \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} C_a(x_{12}, \partial_2) C_b(x_{34}, \partial_4) \frac{I^{ab}(x_{24})}{x_{24}^{2\Delta_{\mathcal{O}}}} \\
&= \frac{1}{x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi}} \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(x_i) && (8.3)\\
g_{\Delta,\ell}(x_i) &\equiv x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi} C_a(x_{12}, \partial_2) C_b(x_{34}, \partial_4) \frac{I^{ab}(x_{24})}{x_{24}^{2\Delta}}, && (8.4)
\end{aligned}$$

where we have chosen an orthonormal basis of operators, and used that the two-point function is fixed by conformal invariance to be

$$\begin{aligned}
\langle\mathcal{O}^a(x)\mathcal{O}'^b(0)\rangle &= \delta_{\mathcal{O}\mathcal{O}'} \frac{I^{ab}(x)}{x^{2\Delta_{\mathcal{O}}}} && (8.5)\\
I^{ab}(x) = I^{\mu_1\ldots\mu_\ell;\nu_1\ldots\nu_\ell}(x) &= \mathrm{sym}(I^{\mu_1\nu_1}(x)\cdots I^{\mu_\ell\nu_\ell}(x)) - \text{traces}. && (8.6)
\end{aligned}$$

The functions $g_{\Delta,\ell}(x_i)$ are called conformal blocks. Although it's not obvious from the way we've defined them, it turns out they are actually functions of the conformal cross-ratios u, v alone, $g_{\Delta,\ell}(u, v)$. We thus have the conformal block decomposition

$$g(u, v) \;=\; \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u, v) \tag{8.7}$$

Exercise 19. *Using the differential operator (7.11), show*

$$g_{\Delta,0}(u, v) \;=\; u^{\Delta/2}\,(1 + \ldots) \tag{8.8}$$

[15] Although our computation will make it look like we need $x_{3,4}$ to be sufficiently far from $x_{1,2}$, we will see shortly that the answer will be correct whenever we can draw any sphere separating x_1, x_2 from x_3, x_4.

8.2 In Radial Quantization

A conformal block represents the contribution of a single conformal multiplet to a four-point function. It is instructive to think about these contributions in radial quantization. Let us pick an origin such that $|x_{3,4}| \geq |x_{1,2}|$, so that

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle \tag{8.9}$$

For a primary operator $\mathcal{O}$, let $|\mathcal{O}|$ be the projector onto the conformal multiplet of $\mathcal{O}$,

$$|\mathcal{O}| \equiv \sum_{\alpha,\beta=\mathcal{O},P\mathcal{O},PP\mathcal{O},\ldots} |\alpha\rangle \mathcal{N}^{-1}_{\alpha\beta}\langle\beta|, \qquad \mathcal{N}_{\alpha\beta} \equiv \langle\alpha|\beta\rangle \tag{8.10}$$

The identity is the sum of these projectors over all primary operators.

$$\mathbf{1} = \sum_{\mathcal{O}} |\mathcal{O}| \tag{8.11}$$

Inserting this into our four-point function gives

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O}} \langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle \tag{8.12}$$

Each term in the sum is essentially a conformal block,

$$\langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle = \frac{f^2_{\phi\phi\mathcal{O}}}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} g_{\Delta_\mathcal{O},\ell_\mathcal{O}}(u,v) \tag{8.13}$$

One can verify the equivalence between this expression and the one in the previous section by performing the OPE between $\phi(x_3)\phi(x_4)$ and $\phi(x_1)\phi(x_2)$ above.

This expression makes it clear why $g_{\Delta,\ell}(u,v)$ is a function of u and v: the projector $|\mathcal{O}|$ commutes with all conformal generators (by construction). Thus, the object above satisfies all the same Ward identities as a four-point function of primaries, and it must take the form (5.9). In path integral language, we can think of $|\mathcal{O}|$ as a new type of topological surface operator. Here, we've inserted it on a sphere separating $x_{1,2}$ from $x_{3,4}$.

8.3 From the Conformal Casimir

Recall that the conformal algebra is isomorphic to the $\mathrm{SO}(d+1,1)$, with generators L_{ab} given by (3.15). The Casimir $C = \frac{1}{2}L^{ab}L_{ab}$ acts with the same eigenvalue on every state in a conformal multiplet. For example, for an operator $\mathcal{O}(x)$ with dimension Δ and spin ℓ, we have

$$C\mathcal{O}(x)|0\rangle = \lambda_{\Delta,\ell}\mathcal{O}(x)|0\rangle \tag{8.14}$$
$$\lambda_{\Delta,\ell} \equiv -\Delta(\Delta-d) - \ell(\ell+d-2) \tag{8.15}$$

It follows that C gives this same eigenvalue when acting on the projection operator $|\mathcal{O}|$ from either the right or left,

$$C|\mathcal{O}| = |\mathcal{O}|C = \lambda_{\Delta,\ell}|\mathcal{O}| \tag{8.16}$$

Let $\mathcal{L}_{ab,i}$ be the differential operator giving the action of L_{ab} on the operator $\phi(x_i)$. Note that

$$\begin{aligned}(\mathcal{L}_{ab,1}+\mathcal{L}_{ab,2})\phi(x_1)\phi(x_2)|0\rangle &= ([L_{ab},\phi(x_1)]\phi(x_2)+\phi(x_1)[L_{ab},\phi(x_2)])\,|0\rangle \\ &= L_{ab}\phi(x_1)\phi(x_2)|0\rangle \end{aligned}\tag{8.17}$$

Thus, we can generate the Casimir acting on $\phi(x_1)\phi(x_2)$ with a differential operator,

$$\mathcal{D}_{1,2}\phi(x_1)\phi(x_2)|0\rangle = C\phi(x_1)\phi(x_2)|0\rangle \tag{8.18}$$

$$\mathcal{D}_{1,2} \equiv \frac{1}{2}(\mathcal{L}_1^{ab}+\mathcal{L}_2^{ab})(\mathcal{L}_{ab,1}+\mathcal{L}_{ab,2}) \tag{8.19}$$

Now, acting with $\mathcal{D}_{1,2}$ on our conformal block, we have

$$\begin{aligned}\mathcal{D}_{1,2}\langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle &= \langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|C\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle \\ &= \lambda_{\Delta,\ell}\langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle\end{aligned}\tag{8.20}$$

It follows that $g_{\Delta,\ell}$ satisfies a differential equation of the form

$$\mathcal{D}g_{\Delta,\ell}(u,v) = \lambda_{\Delta,\ell}g_{\Delta,\ell}(u,v). \tag{8.21}$$

where $\mathcal{D}$ is a second-order differential operator in u,v. This equation can be solved to determine $g_{\Delta,\ell}(u,v)$. For example,

$$g_{\Delta,\ell}^{(2d)}(u,v) = k_{\Delta+\ell}(z)k_{\Delta-\ell}(\overline{z})+k_{\Delta-\ell}(z)k_{\Delta+\ell}(\overline{z}) \tag{8.22}$$

$$g_{\Delta,\ell}^{(4d)}(u,v) = \frac{z\overline{z}}{z-\overline{z}}\left(k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\overline{z})-k_{\Delta-\ell-2}(z)k_{\Delta+\ell}(\overline{z})\right) \tag{8.23}$$

$$k_\beta(x) \equiv x^{\beta/2}{}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},\beta,x\right) \tag{8.24}$$

There are similar explicit formulae in any even dimension. In odd dimensions, no explicit formula in terms of elementary functions is known. However the blocks can still be computed from the Casimir equation, and alternative techniques like recursion relations.

8.4 Series Expansion

We can understand the series expansion of the blocks in terms of the exchange of states in radial quantization. For these purposes, it's useful to introduce a new cross-ratio. Using conformal transformations, we can place all four operators on a 2-plane. In complex coordinates on this 2-plane, let us set

$$x_4 = -1 \tag{8.25}$$
$$x_3 = 1 \tag{8.26}$$
$$x_2 = \rho = re^{i\theta} \tag{8.27}$$
$$x_1 = -\rho = -e^{i\theta} \tag{8.28}$$

with $r = e^{-\tau}$. This corresponds to placing operators $x_{1,2}$ and $x_{3,4}$ at diametrically opposite points on the cylinder, separated by time τ. The coordinate ρ is related to z via

$$\rho = \frac{z}{(1+\sqrt{1-z})^2}, \qquad z = \frac{4\rho}{(1+\rho)^2}. \tag{8.29}$$

In terms of cylinder operators

$$\phi_{\text{cyl.}}(\tau, \mathbf{n}) \equiv \Phi(\tau, \mathbf{n}) = e^{2\Delta_\phi \tau}\phi(e^\tau \mathbf{n}) \tag{8.30}$$

we can write the block as

$$\begin{aligned} 4^{-\Delta} g_{\Delta,\ell}(u,v) &= \langle 0|\Phi(0,\mathbf{n}_1)\Phi(0,-\mathbf{n}_1)|\mathcal{O}|\Phi(-\tau,\mathbf{n}_2)\Phi(-\tau,-\mathbf{n}_2)|0\rangle \\ &= \langle 0|\Phi(0,\mathbf{n}_1)\Phi(0,-\mathbf{n}_1)|\mathcal{O}|e^{iD\tau}\Phi(0,\mathbf{n}_2)\Phi(0,-\mathbf{n}_2)|0\rangle \end{aligned} \tag{8.31}$$

Let us write the projector $|\mathcal{O}|$ as a sum over states with definite energy $E = \Delta + n$ and angular momentum j

$$|\mathcal{O}|e^{iD\tau} = \sum_{E,j}\sum_a |E,j\rangle^a{}_a\langle E,j|e^{-E\tau} \tag{8.32}$$

where a is an SO(d) index. The block becomes

$$= \sum_{E,j} e^{-\tau E}\langle 0|\Phi(0,\mathbf{n}_1)\Phi(0,-\mathbf{n}_1)|E,j\rangle^a{}_a\langle E,j|\Phi(0,\mathbf{n}_2)\Phi(0,-\mathbf{n}_2)|0\rangle \tag{8.33}$$

By rotational invariance, we must have

$${}_{\mu_1\cdots\mu_j}\langle E,j|\Phi(0,\mathbf{n})\Phi(0,-\mathbf{n})|0\rangle = a_{E,j}(\mathbf{n}_{\mu_1}\cdots\mathbf{n}_{\mu_j} - \text{traces.}) \tag{8.34}$$

A contraction of two of these objects is a Gegenbauer polynomial

$$C_j^{(d-2)/2}(\mathbf{n}_1\cdot\mathbf{n}_2) = (\mathbf{n}_{1\mu_1}\cdots\mathbf{n}_{1\mu_j} - \text{traces.})(\mathbf{n}_2^{\mu_1}\cdots\mathbf{n}_2^{\mu_j} - \text{traces.}) \tag{8.35}$$

Thus,

$$g_{\Delta,\ell}(u,v) = \sum_{E,j} b_{E,j} r^E C_j^{(d-2)/2}(\cos\theta) \tag{8.36}$$

where we've defined $b_{E,j} = 4^\Delta a_{E,j}^2$.

We notice a couple things

- $g_{\Delta,\ell}(u,v)$ has an expansion in r with terms that go like $r^{\Delta+n}$.
- The coefficients are linear combinations of Gegenbauer polynomials $C_j^{(d-2)/2}(\cos\theta)$ with positive coefficients (in a unitary theory).
- The coefficients $b_{E,j}$ are rational functions of Δ because they are squares of overlaps of normalized descendant states with $\Phi\Phi|0\rangle$. This is also clear from the construction using the differential operators $C(x,\partial)$, since their coefficients are also rational functions of Δ, as in (7.12).

Exercise 20. *Expand $g^{(2d)}_{\Delta,\ell}(u,v)$ and $g^{(4d)}_{\Delta,\ell}(u,v)$ to the first few orders in $\rho,\overline{\rho}$, and verify the above properties. Verify that some of the coefficients $b_{E,j}$ become negative when Δ violates the unitarity bound.*

9 The Conformal Bootstrap

Let us summarize the consequences of conformal symmetry and unitarity so far

- All operators are primaries or descendants.
- Primary operators satisfy the unitarity bounds

$$\begin{aligned} \Delta &= 0 \qquad \text{(unit operator)} \\ \Delta &\geq \begin{cases} \frac{d-2}{2} & (\ell=0) \\ \ell+d-2 & (\ell>0) \end{cases} \end{aligned} \tag{9.1}$$

- Correlation functions can be computed using the OPE

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k f_{ijk} C(x,\partial)\mathcal{O}_k(0) \tag{9.2}$$

  where the $C(x,\partial)$ are fixed by conformal invariance.
- For four-point functions, this implies

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} \tag{9.3}$$

$$g(u,v) = \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v) \tag{9.4}$$

  where $g_{\Delta,\ell}(u,v)$ are conformal blocks. Only even spins $\ell_{\mathcal{O}}$ appear in a four-point function of identical scalars.

Recall that permutation symmetry of the x_i led to the constraint

$$g(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} g(v,u). \tag{9.5}$$

This is not obvious from the conformal block expansion. It imposes an infinite set of consistency conditions on the OPE coefficients $f_{\phi\phi\mathcal{O}}$, dimensions $\Delta_{\mathcal{O}}$ and spins $\ell_{\mathcal{O}}$.

Exercise 21. *The other constraint*

$$g(u,v) \quad = \quad g\left(\frac{u}{v},\frac{1}{v}\right) \tag{9.6}$$

follows from the fact that only even spins appear in a four-point function of scalars. Show that $g_{\Delta,\ell}(-\rho,-\overline{\rho}) = (-1)^{\ell} g_{\Delta,\ell}(\rho,\overline{\rho})$, *and show that when rewritten in terms of* u, v *this implies the above holds block-by-block.*

More generally, the OPE should be associative

$$\overbracket{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_3 \quad = \quad \mathcal{O}_1\overbracket{\mathcal{O}_2\mathcal{O}_3} \tag{9.7}$$

If this holds, then we will get the same answer in an n-point function no matter what order we do the OPE in. Associativity of the OPE is equivalent to crossing symmetry of *all* four-point functions in the theory. There is a simple graphical argument: crossing of a four-point function means gives us a move that we can use in any network of OPEs to turn it into any other network of OPEs, see figure **??**.