Exact Quantum Entropy

We would like to define an appropriate quantum generalization of the Bekenstein-Hawking area formula that captures the finite area corrections.

Even defining this presents several conceptual difficulties

I (1) What ensemble to choose?

(2) How to take into account 'nonlocal' effects from integrating out massless fields?

(3) Should we computing Index or degeneracy?

II (1) How to compute this quantum entropy

(2) How to compute the degeneracy?

It is convenient to embed this problem for defining exact quantum entropy in a broader context of AdS/CFT holography.

For an extremal (~~#~~ BPS) black hole the near horizon is $AdS_2 = AdS_{p+2}$ w/. $p=0$

The boundary is CFT_1.

$$A(\rho) = \left(1 - \frac{2GM}{\rho} + \frac{G\mathcal{Q}^2}{\rho^2}\right)$$ ②

$$= \frac{(\rho-\rho_+)(\rho-\rho_-)}{\rho^2}$$

Near Horizon Limit

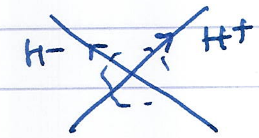
Consider the metric of a Reissner-Nordström black hole in 3+1 dimensions:

$$ds^2 = -A(\rho)\, d\tau^2 + \frac{d\rho^2}{A(r)} + \rho^2 \underbrace{(d\psi^2 + \sin^2\psi\, d\varphi^2)}_{S^2}$$

$$A(\rho) = \frac{(\rho-\rho_+)(\rho-\rho_-)}{\rho^2} \qquad \rho_+ = \rho_- \Rightarrow \text{extremal}$$

$\rho_\pm$ = outer/inner horizon.

$A(\rho)$ has a <u>single</u> zero and <u>not</u> a <u>double zero</u> if we took the extremal limit first. Thus it is a bifurcate horizon like for a Schwarzschild black hole



The near horizon limit is to be taken keeping the two horizons separate, i.e. for a near extremal black hole

$$r \doteq \frac{\rho - \frac{\rho_+ + \rho_-}{2}}{(\rho_+ - \rho_-)/2} \qquad t := \frac{\rho_+ - \rho_-}{2}\,\frac{\tau}{\rho_+^2} \qquad \rho_+ \to \rho_-$$

In this limit

$$ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2\left[d\psi^2 + \sin^2\psi\, d\varphi^2\right]$$

The two horizons are now at ~~the~~ $r = \pm 1$

AdS_2 spacetime

Let us focus on the AdS_2 factor

$$ds^2 = \left[-(r^2-1)\,dt^2 + \frac{dr^2}{(r^2-1)}\right] \rho_+^2$$

Euclidean continuation: $t = -i\theta$ $\quad$ $EAdS_2$

$$ds^2 = v\left[(r^2-1)\,d\theta^2 + \frac{dr^2}{(r^2-1)}\right]$$

$1 \leq r < \infty$
$0 \leq \theta < 2\pi$

The metric is invariant under

$SO(2,1) \simeq SL(2,\mathbb{R}) \simeq SU(1,1)$ symmetry

generated by L_0 $L_\pm$ virasoro-like generators

$\simeq$ Analytic continuation of S^2 w/ $SO(3)$ symmet

Maximally symmetric space $\quad S^2 = SO(3)/SO(2)$

$EAdS_2 = SO(2,1)/SO(2)$

We have kept v as a free parameter. All known extremal black holes have AdS_2 symmetry. We can take this to be the defn of an extrema black hole, with v to be determined by E.o.M. In addition to the metric we can have additional fields:

$U(1)$ gauge fields $=$ 1-form $\quad \{A^I\}$ $\quad I = 0 \cdots n_v$

complex scalar field $\quad \{z^a\}$

AdS_2 symmetry fixes not only the metric but also the form of these fields.

1.
$$A^I = e^I (r-1)\, dt = -ie^I (r-1)\, d\theta$$

so that
$$F^I = dA^I = -ie^I\, dr \wedge d\theta$$

- Note that $\sqrt{g} = 1$ so $dr \wedge d\theta$ is the $SO(2,1)$ invariant volume form. So a 2-form like F^I must be proportional to the volume form
- The -1 in $A^I = -ie^I (r-1)\, d\theta$ is required so that A^I is well-defined at the origin $r=1$

To see this note that

$$ds^2 = (r^2-1)\, d\theta^2 + \frac{dr^2}{(r^2-1)} \qquad r = \cosh R.$$
$$= \sinh^2 R\, d\theta^2 + dR^2 \qquad r-1 \sim R^2 \quad \text{at } r \sim 1$$
$$\simeq R^2 d\theta^2 + dR^2 \quad \text{at } R \to 0 \text{ or } r \to 1$$

Therefore $e^\theta = R\, d\theta$ is the orthonormal 1-form.

$$\Rightarrow \quad A^I \simeq -ie^I R^2\, d\theta = -ie^I R\, e^\theta$$

Without the -1, $A^I \simeq -ie^I \frac{e^\theta}{R}$ would be singular the orthonormal frame at $R \to 0$

2. Similarly,

$$z^a = c^a \quad \text{by symmetry.}$$

We regard $\{e^I\}$, $\{c^a\}$, v as undetermined. Using the classical action, they can be determined by extremization. Note that by using the AdS_2 near horizon symmetries, we are led to solving extremization eqns for these constants, instead of functions of r & θ. These are algebraic eqns and not differential eqns. Enormous simplification. Related to the attractor mechanism :

Electrodynamics in EAdS$_2$

Consider the Euclidean action for the gauge fields $\{A^I\}$

$$I = \frac{v}{2}\int_M N_{IJ}(c^a)\, F^I_{r\theta} F^J_{r\theta}\, dr\, d\theta + i q_I \int_{\partial M} A^I_\theta\, d\theta$$

The second term is needed to make the variational problem well defined. N_{IJ} in general are functions of the values of the scalar fields $\mathcal{I}^a = c^a$.

$$\delta I = v\int N_{IJ}(c)\left[F^J_{r\theta}\, \partial_r(\delta A^I_\theta) + F^I_{\theta r}\, \partial_\theta(\delta A^J_r)\right] dr\, d\theta$$
$$+ i q_I \int \delta A^I_\theta\, d\theta = 0$$

$\Rightarrow$ $v N_{IJ} F^J{}_{r\theta} = -i q_I$ ① from bdry term

$\partial_r(F_{r\theta}) = \partial_\theta(F_{r\theta}) = 0$ ②

Let $F^I_{r\theta} = -i e^I$, then ① $\Rightarrow$ $q_I = v N_{IJ} e^J$

Hence the electric field is determined in terms of the charges. We can now choose a gauge $A_r = 0$

$\Rightarrow F_{r\theta} = \partial_r A_\theta$ ② $\Rightarrow$ $\partial_r \partial_\theta A_\theta = 0$ ⓐ

and $\partial_r^2 A_\theta = 0$ ⓑ

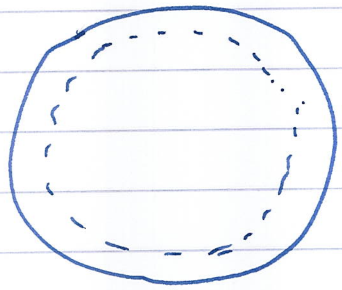
$\Rightarrow A_\theta = a^I r + b^I$ (ⓐ $\Rightarrow$ $A'_\theta = f(\theta) + g(r)$ then ⓑ $\Rightarrow$ $g(r) = ar + b$; $f(\theta)$ can be removed by the residual gauge symmetry $\varepsilon(\theta)$ that leaves $A_r = 0$ gauge unchanged.

$\Rightarrow$ $A^I_\theta = -ie^I(r-1)$ with e^I determined in terms of q_I by $v N_{IJ}(c)\, e^J = q_I$

Since $q_I = -\dfrac{\partial I}{\partial F^I_{r\theta}}$ it can be identified with the asymptotic electric charge of the black hole = integer

Wald Entropy:

Euclidean AdS_2 is the Poincaré disk.



$$1 \leq r < r_0 \to \infty$$
$$0 \leq \theta < 2\pi$$
$$\text{Vol} = \int_1^{r_0} \sqrt{g}\, dr\, d\theta = 2\pi(r_0 - 1) \to \infty$$

The volume is infinite which can be regularized by putting a cutoff at $r = r_0$

Because of the boundary term for the maxwell fields, we are really computing the expectation value of the Wilson line inserted at the boundary

$$W = \left\langle e^{-i q_I \int A_\theta^I \, d\theta} \right\rangle \quad \& \text{ not } Z = \langle 1 \rangle$$

$$\simeq e^{-I_{bulk} + I_{bdry}} \quad \text{in the classical limit}$$

$$I_{bulk} = -v \int_1^{r_0} \mathcal{L}^{(2)} \, dr\, d\theta = -2\pi(r_0 - 1)\mathcal{L}^{(2)} v$$

$$I_{bdry} = 2\pi(r_0 - 1) q_I e^I .$$

$\mathcal{L}^{(2)}$ includes the Einstein-Hilbert term. Magnetic charges of the black hole $\{p^I\}$ are fluxes on the S^2 which appear as parameters of "S^2 compactification

$$W = e^{-I} = e^{-2\pi r_0 c}\, e^{\mathcal{E}}$$

$$\mathcal{E} = 2\pi q_I e^I - v \mathcal{L}^{(2)}(c, e^I) = 2\pi(q \cdot e - f)$$

Extremization $\partial \mathcal{E} / \partial e^I = \frac{\partial \mathcal{E}}{\partial v} = \frac{\partial \mathcal{E}}{\partial c^a} = 0$ at e_*^I, v_*, c_*^a

$$\Rightarrow \boxed{\mathcal{E}_* = S_{Wald}}$$

We have dropped $-2\pi r_0 c$ term by an appropriate boundary counterterm

Quantum Entropy

This suggests a natural definition for the quantum entropy as the path integral over all ~~qua~~ quantum fields of string theory.

$$W(\vec{q}) = \langle e^{-iq_I \int A_t^I \, d\theta} \rangle = d(\vec{q})$$

AdS_2 = CFT$_1$.

The near horizon limit corresponds to taking the lowest energy states. These are precisely the microstates of the black hole. with vanishing energy $H = 0$.

$$d(\vec{q}) = \mathrm{Tr}_{CFT_1}\, e^{-\beta H} = \#\text{ of ground states.}$$

There is an ambiguity in the origin of ~~energy~~ energy. The trace is at the boundary $\beta = 2\pi r_0$

$$\Rightarrow \; d(\vec{q}) \to e^{-2\pi r_0 E_0}\, d(\vec{q})$$

We choose the origin of H such that the r_0 dependent term can be removed, to obtain a finite answer. This "counterterm" in the boundary is precisely the piece $-2\pi C r_0$ we dropped in the bulk: Holographic renormalization.

$$\boxed{W(\vec{q}) \overset{?}{=} d(\vec{q})}$$

AdS_2 CFT$_1$.

Comments:

- Wald Entropy formula is valid for any local action — obtained for example by integrating out massive fields like massive string states. Quantum entropy enables us to take into account systematically effects from integrating out massless fields.

Microcanonical Ensemble

The AdS_2 boundary condition naturally lead us to consider the microcanonical ensemble

~~for~~ Microcanonical	(grand)canonical
fix charges $\vec{q}$	fix chemical potentials $\vec{\mu}$

AdS_2 ~~Asymptotic~~

$$A \sim qr + \mu$$

growing mode as $r \to r_0$

fix q

AdS_4 ~~Asymptotic~~

$$A \sim \mu + \frac{q}{r}$$

dominant mode (nonnormalizable) as $r \to r_0$.

Index vs Degeneracy

The entropy of any thermodynamic ensemble must equal the log degeneracy by Boltzmann.

$$S = \log d \qquad d = n_B + n_F.$$

But this is usually very difficult to compute.

Almost all cases studied in string theory one computes ~~I~~ $I_{asymptotic} = Tr(-1)^F = n_B - n_F$.

There is no a-priori reason why I should equal d

$$I \neq d \quad \text{in general}$$

AdS_2 + 4 supersymmetries give a rationale why $I = d$.

Index ~~=~~ = Degeneracy.

a) AdS_2 has $SO(2,1) \simeq SU(1,1)$ symmetry
Suppose the black hole near horizon geometry has ~~4~~ 4 supersymmetries.
Then closure of the superalgebra implies that we have $SU(1,1|2)$ symmetry

$$\psi^a_\alpha \qquad SU(1,1) \times SU(2)$$

$\Rightarrow$ Black hole horizon must be spherically symmetric: $\Rightarrow J=0$ or $M=0$

b) Using AdS_2 quantum ~~entropy~~ boundary condition we conclude it is J that must be zero. because we have a microcanonical ensemble.

Surprising ~~prediction~~ result that gives an IR prediction for the UV bound states of the theory.