Work with bosonic string theory for simplicity (formal due to tachyons in loops).

1. off-shell amplitude:

Relax $Q_B|A\rangle = 0$ condition.

Still impose $(L_0 - \bar{L}_0)|A\rangle = 0$, $(b_0 - \bar{b}_0)|A\rangle = 0$

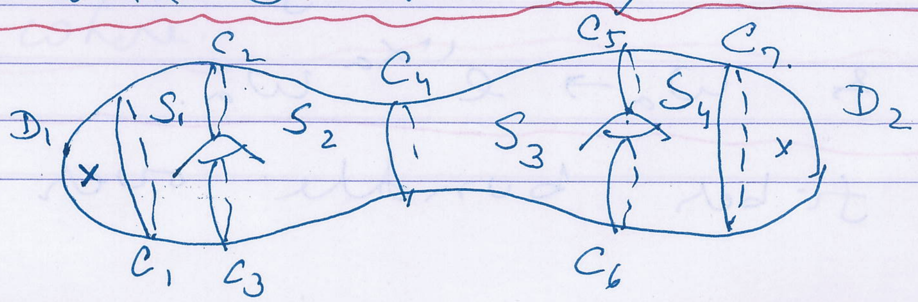
zero modes of b-ghost

Vertex ops. $A_i$ can no longer be taken to be dimension 0 primaries.

$\langle \prod_i A_i \times \text{ghost} \rangle_{\Sigma_{g,n}}$ depends on the choice of world-sheet metric near the punctures.

Need to pick a metric.

Take $\Sigma_{g,n}$ & divide it into ~~a union of~~ $n$ disks $D_1, \ldots D_n$ and $2g-2+n$ spheres with 3 holes by $3g-3+2n$ circles



$w_a$: Coordinate on $D_a$, $z_i$: Coordinate on $S_i$.

On common boundaries:

$z_i = f_{ia}(w_a)$, $z_i = F_{ij}(z_j)$.

$\{f_{ai}\}$, $\{F_{ij}\}$: Capture full information about $\Sigma_{g,n}$.

$\mathcal{M}_{g,n}$: Space of $\{f_{ai}\}$, $\{F_{ij}\}$ modulo equivalence due to reparametrization of $z_i$ & $w_a$.

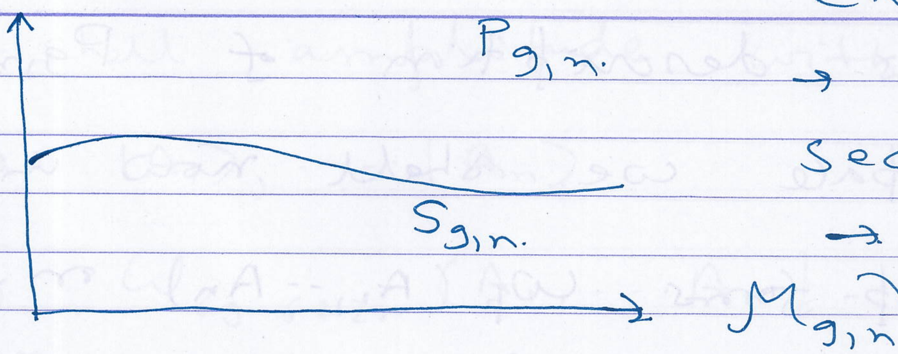
Choose metric on $D_a$: $|dw_a|^2$.

Off-shell amplitudes are not invariant under $w_a$ reparametrization except for $w_a \to e^{i\alpha_a} w_a$.

$P_{g,n}$: Space of $\{f_{ai}\}$, $\{F_{ij}\}$ modulo equivalence due to reparametrization of $z_i$ & $w_a \to e^{i\alpha_a} w_a$.

$\Rightarrow$ a fiber bundle over $\mathcal{M}_{g,n}$.

$\{w_a\}$ reparametrization.




Choice of $w_a$ → choice of section $S_{g,n}$ → needed to define off-shell amplit…

Tangent space of $P_{g,n}$:

Infinitesimal deformation:

e.g. $f_{ia} \to f_{ia} - \delta f_{ia}$

For fixed $w_a$, this changes $z_i$.

e.g. $z_i = f_{ia}(w_a) \Rightarrow z_i = f_{ia}(w_a) - \delta f_{ia}(w_a)$

$$= z_i^{old} - \delta f_{ia}\left(f_{ia}^{-1}(z_i^{old})\right)$$

$v(z_i) = \delta f_{ia}\left(f_{ia}^{-1}(z_i)\right)$ is an infinitesimal holomorphic vector field on $\Sigma$ defined on the overlap circle of $D_a$ and $S_i$

$v(z_i)$ can have ~~poles~~ singularities elsewhere.

Using this description of $P_{g,n}$ & its tangent space we shall now introduce a set of $p$-forms $\omega_p(A_1, \cdots A_n)$ on $\mathcal{M}_{g,n}$.

$$\omega_0(A_1, \cdots A_n) = (2\pi i)^{-3g+3-n} \langle \prod_{i=1}^{n} A_i \rangle_{\Sigma_{g,n}}$$

vertex op. for off-shell external states.

To specify ~~$\omega_p$~~ $\omega_p$, it is enough to specify its contraction with $p$ arbitrary tangent vectors ~~fields~~ $V^{(1)}, \cdots V^{(p)}$ of $P_{g,n}$ / vector fields $v^{(1)}, \cdots v^{(p)}$ on $\Sigma_{g,n}$ defined on overlap circles $C^{(1)}, \cdots C^{(p)}$.

$$B[V^{(i)}] \equiv \oint_{C^{(i)}} b(z)\, v^{(i)}(z)\, dz + \oint_{C^{(i)}} \bar{b}(\bar{z})\, \overline{v^{(i)}(z)}\, d\bar{z}$$

$$\omega_p[V^{(1)}, \cdots V^{(p)}] = (2\pi i)^{-3g+3-n} \langle B[V^{(1)}] \cdots \cdots B[V^{(p)}] \prod_{i=1}^{n} A_i \rangle_{\Sigma_{g,n}}$$

Off-shell amplitude with external states $|A_1\rangle, \ldots |A_n\rangle$

$$\int_{S_{g,n}} \omega_{6g-6+2n}(A_1, \ldots A_n)$$

② IPI amplitudes:

Plumbing fixture: Family of maps from

$$P_{g_1,n_1} \times P_{g_2,n_2} \to P_{g_1+g_2, n_1+n_2-2}.$$

(a) Take a puncture each from $\Sigma_{g_1,n_1}$ & $\Sigma_{g_2,n_2}$ ~~& ...~~:

Let $w_1, w_2$ be local coordinates around punctures.

(b) Glue $\Sigma_{g_1,n_1}$ to $\Sigma_{g_2,n_2}$ via

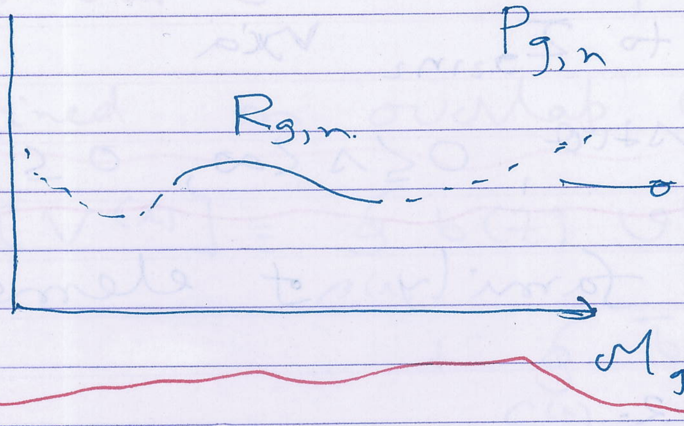
$$w_1 w_2 = e^{-s+i\theta}, \quad 0 \leq s < \infty, \quad 0 \leq \theta < 2\pi$$

$\to$ 2 parameter family of elements of $P_{g_1+g_2, n_1+n_2-2}$.

Note: $\Sigma_{g_1+g_2, n_1+n_2-2}$ inherits local coordinates ~~...~~ around punctures from $\Sigma_{g_1,n_1}$ & $\Sigma_{g_2,n_2}$

③ Choose $S_{g,n}$ for different $g,n$ respecting this map, e.g. if $p_1 \in S_{g_1,n_1}$, $p_2 \in S_{g_2,n_2}$ then the plumbing fixture of $p_1$ & $p_2$ $\in S_{g_1+g_2,\, n_1+n_2-2}$ for $0 \le s < \infty$, $0 \le \theta < 2\pi$.

This allows us to identify a subspace $R_{g,n} \subset S_{g,n}$ $\forall\, g,n$ such that elements of $\{R_{g,n}\}$ & their plumbing fixture generate the whole $\{S_{g,n}\}$.




Generated from plumbing fixture

1PI amplitude $= \int_{R_{g,n}} \omega_{6g-6+2n}(A_1, \dots A_n)$

Note: $R_{g,n}$ does not include any separating type degenerations & hence is free from tadpole & mass renormalization divergences.