LECTURE 4

NODAL LINES OF MAASS FORMS AND CRITICAL PERCOLATION

PETER SARNAK

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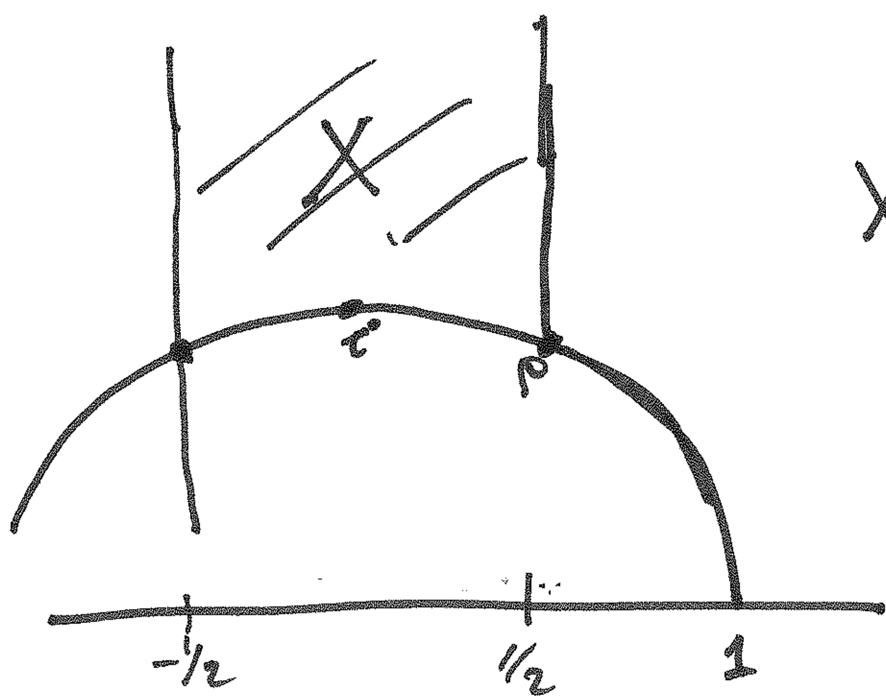
- AUTOMORPHIC CUSP FORMS ARE THE BUILDING BLOCKS OF MODERN AUTOMORPHIC FORM THEORY. THEIR EXISTENCE IS SUBTLE.

- EVERYWHERE UNRAMIFIED FORMS FOR $GL_2/\mathbb{Q}$ (ANALOGUES FOR GL_2 OF THE RIEMANN ZETA FUNCTION);

$$\Gamma = SL_2(\mathbb{Z})$$

ACTS ON $\mathbb{H}$ BY LINEAR FRACTIONAL TRANSF.



$$X = \Gamma \backslash \mathbb{H}$$

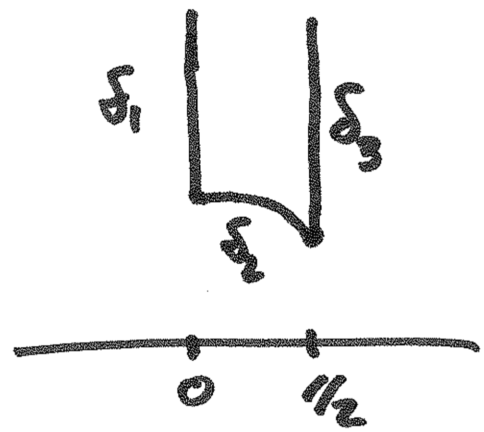
X is a hyperbolic surface, finite area

③

X has a symmetry σ induced by
$z \to -\bar{z}$ (orientation reversing)

$\delta := \mathrm{Fix}(\sigma) = \{ z \in X : \sigma(z) = z \}$

$\delta = \delta_1 \cup \delta_2 \cup \delta_3$



Maass cusp form (even unramified)

(i) $\phi : \mathbb{H} \to \mathbb{R}$

(ii) $\phi(\gamma z) = \phi(z)$ ϕ lives on X

(iii) $\phi(\sigma z) = \phi(z)$ ϕ is 'even'

(iv) $\int_X \phi^2(z)\, dA(z) = 1$

(v) $\Delta\phi + \lambda\phi = 0$, $\lambda > 0$.

Δ = hyperbolic Laplacian.

write $\lambda_\phi = \frac{1}{4} + t_\phi^2$, $\lambda_\phi = \frac{1}{4} + t_\phi^2$

We also have that ϕ is a Hecke eigenform; $n \geqslant 1$

$$T_n \phi = \lambda_\phi(n) \phi$$

where
$$T_n \psi(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ b \bmod d}} \psi\left(\frac{az+b}{d}\right)$$

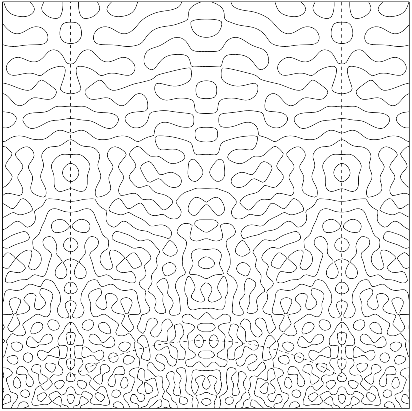
<u>Selberg</u> (existence) there are infinitely many such cusp forms in fact
$$\lambda_\phi \sim 24 n_\phi \quad \text{as } n_\phi \to \infty$$
where n_ϕ is its number ordered by eigenvalue.

Let $Z(\phi) \subset X = \{ z \in X : \phi(z) = 0 \}$, the nodal line.
$X \setminus Z(\phi)$ has $N(\phi)$ connected components called nodal domains.

Below is the nodal set $\{\phi = 0\}$ of a highly excited modular form for $\mathrm{SL}_2(\mathbb{Z})$.

$$\Delta\phi + \lambda\phi = 0, \quad \lambda = \frac{1}{4} + R^2.$$

$\phi(z)$ is $\mathrm{SL}_2(\mathbb{Z})$ periodic. Is the zero behaving randomly? How many components does it have?



Hejhal–Rackner nodal lines for $\lambda = 1/4 + R^2$, $R = 125.313840$

SOME GENERAL RESULTS FOR EIGEN-FUNCTIONS ON ANY COMPACT RIEMANNIAN SURFACE Y:

(i) COURANT'S NODAL DOMAIN THEOREM

$$N(\phi) \leq n_\phi .$$

(ii) (TOTH-ZELDITCH 2008): IF $\Omega \subset \mathbb{R}^2$ IS A COMPACT DOMAIN WITH REAL ANALYTIC BDRY AND ϕ SATISFIES NEUMANN BDRY CONDITIONS, THEN

$$|Z(\phi) \cap \partial\Omega| \ll t_\phi \quad (\lambda = t^2)$$

(iii) (DONNELLY-FEFFERMAN), X REAL ANALYTIC: $t \ll \text{length}(Z(\phi)) \ll t_\phi$

(iv). (HORMANDER) $\|\phi\|_\infty \ll t_\phi^{1/2}$ (SHARP ON S^2)

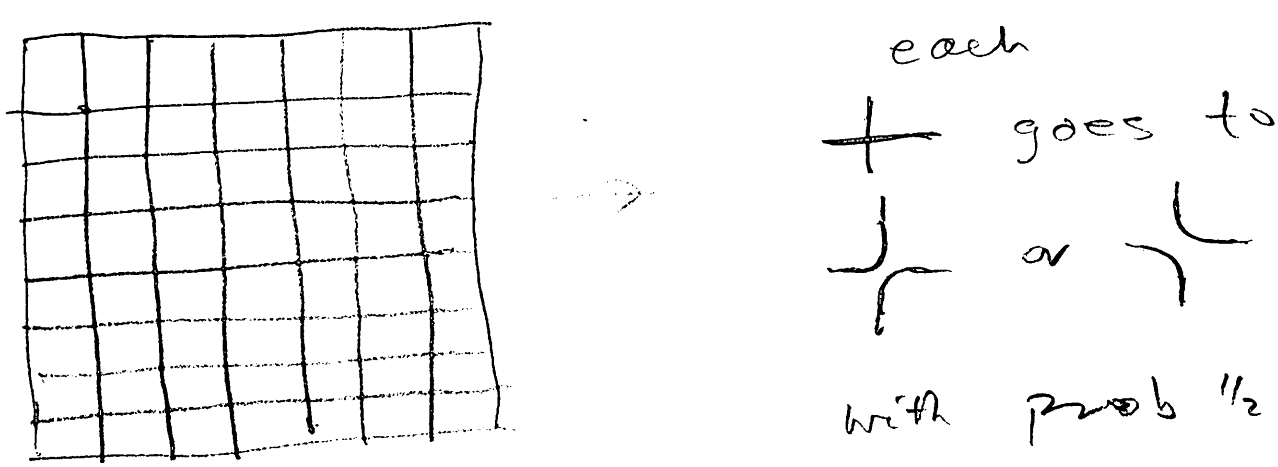
(v). (BURQ-GERARD-TZVETKOV) LET γ BE A GEODESIC SEGMENT IN Y, THEN

$$\|\phi|_\gamma\|_2 \ll t_\phi^{1/4}, \quad \text{SHARP ON } S^2.$$

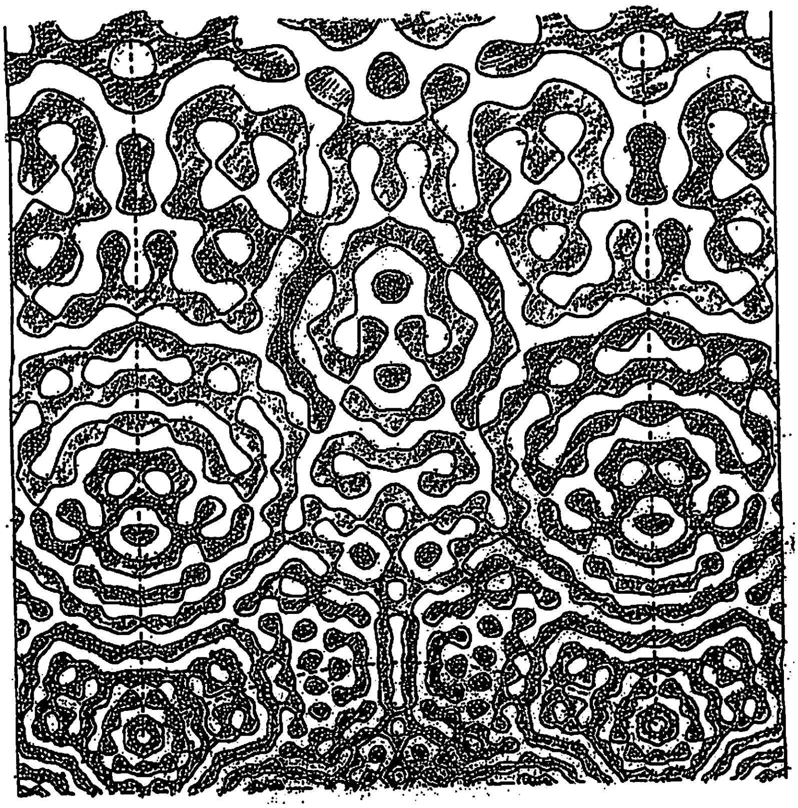
Based on heuristics asserting that eigenfunctions of quantizations of classicaly chaotic Hamiltonians behave like random band limited functions and that the latter have their zero sets behave like an exactly soluble critical bond percolation model, Bogomolny and Schmit (2002) have made a general conjecture which specialized to our X:

<u>BOGOMOLNY-SCHMIT CONJ FOR X:</u>

$$N(\phi) \sim \frac{2}{\pi}(3\sqrt{3}-5)\,n_\phi \quad \text{as } n_\phi \to \infty.$$



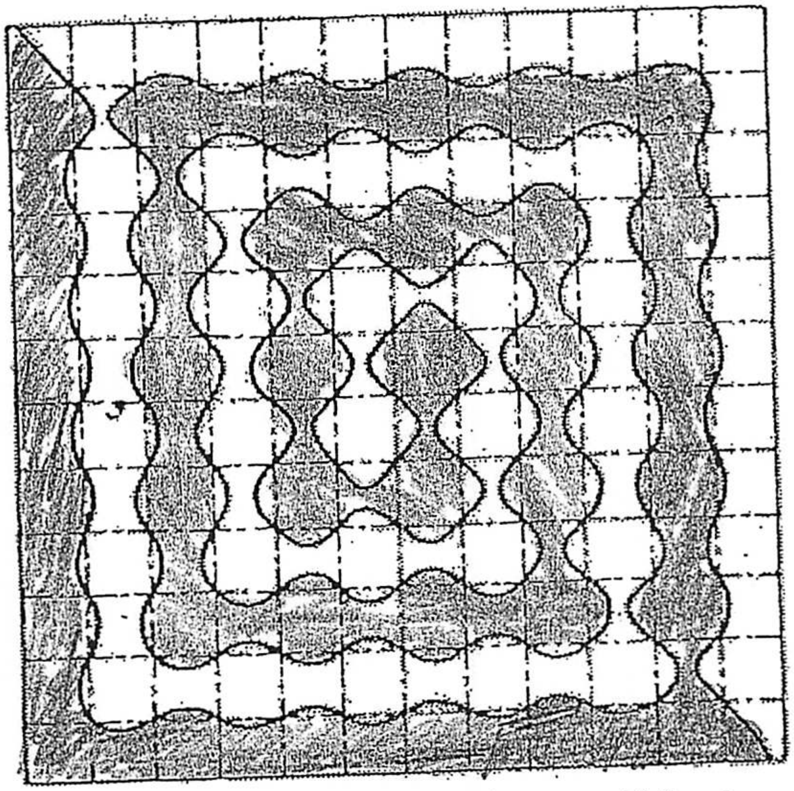
Below a picture of the zero set $\phi = 0$ of such a "Maass form" for $\mathrm{SL}_2(\mathbb{Z})$, $\lambda = \frac{1}{4} + t^2$, $t = 125.34\ldots$ (Hejhal–Rackner).
Is the zero set behaving randomly? How many components does it have?



58 nodal domains in A

TWO COLORABLE !

The trouble is giving a lower bound for $N(\phi)$. In general it need not grow!



Nodal domain of an eigenfunction on the square, $N(\phi) = 2$.
From Courant–Hilbert; Vol I. Thesis A. Stern, Gottingen, 1925.

Symmetry for X:
The isometry $z \mapsto -\overline{z}$ of $\mathbb{H}$ induces an isometry $\sigma : X \to X$.

$$\delta = \mathrm{Fix}(\sigma) = \{z : \sigma(z) = z\} = \delta_1 \cup \delta_2 \cup \delta_3.$$

ϕ is either even or odd with respect to σ. We stick to the even ones.

- If Ω is a nodal domain for ϕ, then so is $\sigma(\Omega)$.
  $\Longrightarrow$ either $\sigma(\Omega) = \Omega$, we call Ω inert (or real), or $\sigma(\Omega) \cap \Omega = \emptyset$, we call Ω split.
- Ω is inert iff Ω meets δ nontrivially.

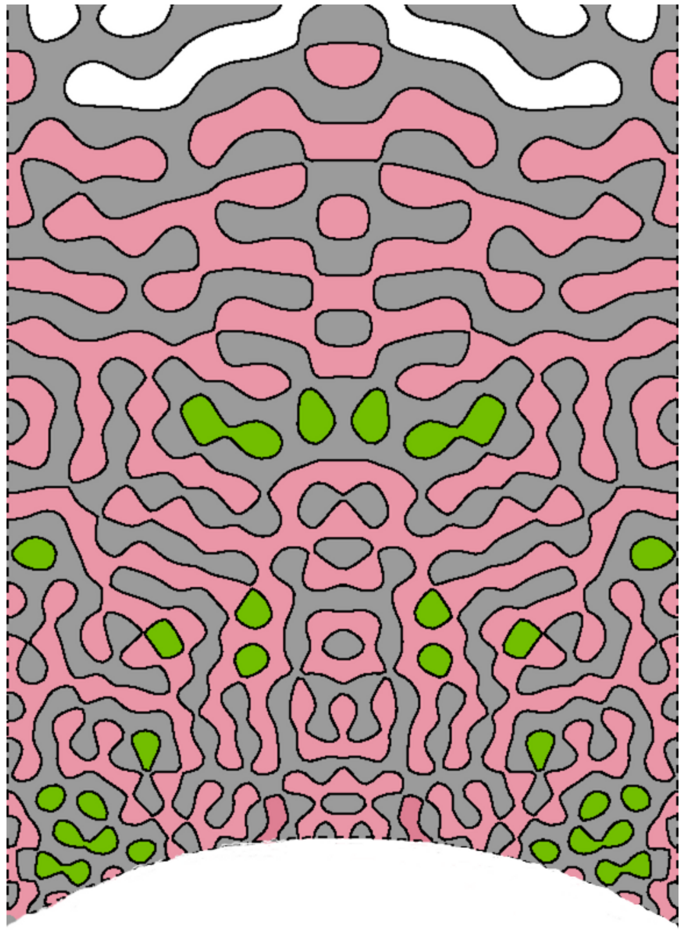


FIGURE 2

Denote by $\mathrm{K}^{\beta}(\phi)$ the number of sign changes of ϕ when traversing β. A key role is played by the arc $\delta = \{z \in \mathbb{X} : \sigma(z) = z\}$ which is fixed by σ. This arc is composed of three piecewise analytic geodesics, denoted by δ_1, δ_2 and δ_3 in Figure 3. A simple topological analysis (see Section 1) shows that

$$1 + \frac{1}{2}\mathrm{K}^{\delta}(\phi) \leq \mathrm{N}_{\mathrm{in}}(\phi) \leq |Z_\phi \cap \delta|, \tag{5}$$

and more generally for β a nonempty segment of δ

$$1 + \frac{1}{2}\mathrm{K}^{\beta}(\phi) \leq \mathrm{N}^{\beta}_{\mathrm{in}}(\phi) \leq |Z_\phi \cap \beta| + 1, \tag{6}$$

The complexification technique of Toth and Zelditch (using a form of Jensen's Lemma) may be applied to our ϕ's whose normal derivative vanishes along δ, to

Let

$$N_i(\phi) = \# \text{ of inert nodal domains}$$

$$N_s(\phi) = \# \text{ of split nodal domains}$$
(which is even)

$$N(\phi) = N_i(\phi) + N_s(\phi)$$

<u>Topological Proposition:</u>

Let k_ϕ be the number of sign changes of ϕ going around δ then

$$\frac{k_\phi}{2} + 1 \le N_i(\phi) \le |\, Z(\phi) \cap \delta \,|$$

There is also a local version of this proposition for $\beta \subset \delta$ a compact segment.

We are lead to study the number of intersections of $Z(\phi)$ with a given arc.

RESTRICTION RESULTS FOR X:

TO BIG SETS:

- QUE (QUANTUM UNIQUE ERGODICITY)
  E. LINDENSTRAUSS, K. SOUNDARARAJAN

$B \subset X$, area$(B) > 0$ (nice)

$$\|\phi|_B\|_2^2 \longrightarrow \frac{\text{area}(B)}{\text{area}(X)} \quad \text{as } t_\phi \to \infty.$$

TO POINTS, IE L^∞ BOUNDS

- (IWANIEC - S) SUBCONVEX BOUND

$$\|\phi\|_\infty \ll_\varepsilon t_\phi^{5/12+\varepsilon} .$$

NOTE: THE CONJECTURE HERE IS THAT FOR $K \subset X$ COMPACT FIXED

$$\|\phi|_K\|_\infty \ll_\varepsilon t_\phi^{\varepsilon} .$$

IF THIS IS TRUE IT IS VERY STRONG AS IT IMPLIES THE LINDELOF HYPOTHESIS FOR THE RIEMANN ZETA FUNCTION

$$\zeta(\tfrac{1}{2}+it) \ll_\varepsilon (1+|t|)^\varepsilon .$$

• (MARSHALL 2012): Let $\gamma \subset X$ be a geodesic segment then

$$\| \phi|_{\gamma} \|_2 \ll_{\varepsilon} t_{\phi}^{3/14+\varepsilon} .$$

FOR OUR ANALYSIS OF NODAL LINES WE ALSO NEED LOWER BOUNDS FOR RESTRICTION:

<u>ELEMENTARY FACT:</u>

IF h AND γ ARE horocycle or geodesic segments the $\phi|_h$ and $\phi|_{\gamma}$ cannot vanish identically.

<u>PROOF:</u> USES THE DISTRIBUTION OF $\tilde{h}$ IN X (horocycle dynamics), $\tilde{h}$ the continuation of h, and ϕ is odd about $\tilde{\gamma}$, WHILE THE REFLECTION GROUP $\delta_1, \delta_2, \delta_3$ IS MAXIMAL DISCRETE.

SHARP RESTRICTION THEOREMS:

THEOREM 1:

(a) Let C be a closed horocycle in X then for $\varepsilon > 0$;

$$t_\phi^{-\varepsilon} \ll \|\phi|_C\|_2 \ll t_\phi^{\varepsilon}.$$

(b) Let $\beta \subset \delta$ be a sufficiently long compact subsegment of δ then for $\varepsilon > 0$

$$1 \ll \|\phi|_\beta\|_2 \ll_\varepsilon t_\phi^{\varepsilon}.$$

COR 1: LET C be a closed horocycle then

$$t_\phi^{1/12} \ll |Z(\phi) \cap C| \ll t_\phi$$

REMARK: THE LAST UPPER BOUND CAN BE PROVEN WITHOUT ANY ARITHMETIC INPUT, IE FOR ANY HYPERBOLIC SURFACE Y (JUNG 2011).

COR 2: ASSUME THE LINDELOF HYPOTHESIS FOR THE STANDARD L-FUNCTIONS $L(s,\phi)$ THEN FOR β AS IN THEOREM 1

$$t^{1/12} \ll |Z(\phi) \cap \beta| \ll t$$

PROOF OF COR 2: $\Lambda(s,\phi) = \int_0^\infty \phi(iy) \cdot y^s \frac{dy}{y}$ (Hecke)

Lindelof $\Rightarrow$ for $\alpha < \alpha' < \beta$

$$\int_\alpha^{\alpha'} \phi(iy) \frac{dy}{y} \ll_\varepsilon t_\phi^{-1/2+\varepsilon}$$

Let $\alpha_1 < \alpha_2 \dots < \alpha_\ell$ be the sign change pts of ϕ on β

$$\int_\beta |\phi(iy)| \frac{dy}{y} = \sum_{j=1}^{\ell} \operatorname{sgn}(\phi_{[\alpha_j,\alpha_{j+1}]}) \int_{\alpha_j}^{\alpha_{j+1}} \phi(iy) \frac{dy}{y} \ll \ell . t^{-1/2+\varepsilon} .$$

By theorem 1 (b) lower bound and L^∞ subconvexity

$$1 \ll \int_\beta |\phi(iy)|^2 \frac{dy}{y} \le (\max_{y \in \beta} |\phi(iy)|) \int_\beta |\phi(iy)| \frac{dy}{y} \ll \ell t^{-1/2+\varepsilon} . t^{5/12}$$

$$\Rightarrow \quad \ell \geq t_\phi^{1/12-\varepsilon} .$$

COMBINING COR 2 WITH THE TOPOLOGICAL LEMMA WE OBTAIN OUR MAIN RESULT

<u>THEOREM 2:</u> ASSUME THE LINDELOF HYPOTHESIS FOR $L(s,\phi)$ THEN FOR β A FIXED LONG COMPACT SEGMENT OF δ

$$t^{1/12} \ll N_i^{(\beta)}(\phi) \ll t$$

IN PARTICULAR THE NUMBER OF NODAL DOMAINS IN A COMPACT PART OF X GOES TO INFINITY.

<u>NOTE:</u> 1) WE DON'T KNOW HOW TO PRODUCE ANY SPLIT NODAL DOMAINS, WHICH ACCORDING TO BOGOMOLNY-SCHMIT SHOULD BE MOST OF THEM ($\approx t^2$).

2) IN THE REGION $y \gg t_\phi$, IE THE CUSP OF X, THE PROFILE OF ϕ SEPARATES INTO A PRODUCT FORM AND THE NODAL LINES CAN BE ANALYZED MORE EASILY. THERE ARE ABOUT t_ϕ NODAL DOMAINS INERT.

Something about the proof of Theorem 1(b).

FOURIER EXPASION

$$\phi(z) = \sum_{n=1}^{\infty}{}' \rho_\phi(n)\, y^{1/2} K_{it_\phi}(2\pi n y) \cos(2\pi n x)$$

and on δ_1

$$\phi(iy) = \sum_{n=1}^{\infty}{}' \rho_\phi(n)\, y^{1/2} K_{it_\phi}(2\pi n y)$$

- $\rho_\phi(n) = \rho_\phi(1)\lambda_\phi(n)$ and the Hecke eigenvalues $\lambda_\phi(n)$ carry the arithmetic
- $K_{it_\phi}(\xi)$ the Bessel function carries the analysis and its behavior as $t_\phi \to \infty$ has 3-ranges for ξ, the transitional range $t_\phi - t_\phi^{1/3} < \xi < t_\phi + t_\phi^{1/3}$ is the most subtle — Airy function.

$$\int_0^\infty (\phi(iy))^2 \beta(y) \frac{dy}{y} = \sum_{n,m}{}' \rho_\phi(n)\rho_\phi(m)\, \Delta_\phi(n,m)$$

$$\Delta_\phi(n,m) = \int_0^\infty \beta(y)\, K_{it_\phi}(2\pi n y) K_{it_\phi}(2\pi m y)\, dy$$

Using classical identities and uniform asymptotics for such special functions (Olver, Dunster ...).

$\Longrightarrow$

Off diag factor $\Delta_\phi(n,m)$ is small if $|n-m|$ is ~~large~~ growing. and one of n or m is $\geq \varepsilon_0 t_\phi$.

Modify: $\quad F(y) := \sum_{m \geq \varepsilon_0 t_\phi} \rho_\phi(m)\, y^{1/2} K_{it_\phi}(2\pi m y)$

then

$$\int_0^\infty F(y)\,\phi(iy)\,\beta(y)\,\frac{dy}{y} \leq \left(\int_0^\infty F(y)^2 \beta(y)\frac{dy}{y}\right)^{1/2}\left(\int_0^\infty \phi(iy)^2 \beta(y)\frac{dy}{y}\right)^{1/2}$$

So if l.h.s. has a correct order lower bound ✓

$$\text{LHS} = \sum_{\substack{m \geq \varepsilon_0 t_\phi \\ n}} \rho_\phi(m)\,\rho_\phi(n)\,\Delta(m,n)$$

$\Delta(m,n)$ is small if $|m-n|$ is large

For $m=n$, Diagonal we get a lower bound from QUE and this is the main term (arithmetic sum)

For $0 < m - n = h$ small

QUE again gives cancellation in the sum ($\Delta(m, m+h)$ is smoothly varying).

In terms of proving anything about the zero sets of random band limited functions (one of the two heuristic steps in B-S) there is very nice progress by Nazorov and Sodin. Their methods can be used to prove the following (as part of a universal such result):

Let
$$\psi = \sum_{T \le t_j \le T+1} c_j \, \phi_j \, ,$$
There are about T such t_j's, with c_j i.i.d standard Gaussians.

Let $N(\psi)$ be the no' of nodal sets for ψ, then

$$N(\psi) \sim c\,T^2 \, ,$$ for a universal positive constant c (it is perhaps the B-G explicit c but no connection to percolation is known)

Some references:

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