Lecture 3

MÖBIUS RANDOMNESS AND HOROCYCLE FLOWS

PETER SARNAK

TIFR 2012.

$n \geq 1$,

$$\mu(n) = \begin{cases} (-1)^t & \text{if } n = p_1 p_2 \cdots p_t \text{ distinct,} \\ 0 & \text{if } n \text{ has a square factor.} \end{cases}$$

$$1, -1, -1, 0, -1, 1, -1, 1, -1, 0, 0, 1, \ldots.$$

Is this a "random" sequence?

$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

so the zeros of $\zeta(s)$ are closely connected to

$$\sum_{n \leq N} \mu(n).$$

Prime Number Theorem

$$\overset{\text{elementarily}}{\Longleftrightarrow}$$

$$\sum_{n \leq N} \mu(n) = \sum_{n \leq N} \mu(n) \cdot 1 = o(N).$$

Riemann Hypothesis $\Longleftrightarrow$ For $\varepsilon > 0$,

$$\sum_{n \leq N} \mu(n) = O_{\varepsilon}(N^{1/2+\varepsilon}).$$

- Usual randomness of $\mu(n)$, square-root cancellation.

(Old Heurestic) "Möbius Randomness Law" (EG, I–K)

$$\sum_{n \leq N} \mu(n)\xi(n) = o(N)$$

for any "reasonable" independently defined bounded $\xi(n)$.

This is often used to guess the behaviour for sums on primes using

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^e, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

What is "reasonable"?
Computational Complexity (?): $\xi \in P$ if $\xi(n)$ can be computed in $\mathrm{polylog}(n)$ steps.
Perhaps $\xi \in P \implies \mu$ is orthogonal to ξ?

I don't believe so since I believe factoring and μ itself is in P.

Problem: Construct $\xi \in P$ bounded such that

$$\frac{1}{N}\sum_{n\le N}\mu(n)\xi(n)\to\alpha\neq 0.$$

Dynamical view of complexity of a sequence (Furstenberg disjointness paper 1967)
Flow: $F=(X,T)$, X a compact metric space, $T:X\to X$ continuous. If $x\in X$ and $f\in C(X)$, the sequence ("return times")

$$\xi(n)=f(T^n x)$$

is realized in F.

Idea is to measure the complexity of $\xi(n)$ by realizing $\xi(n)$ in a flow F of low complexity.

Every bounded sequence can be realized; say $\xi(n) \in \{0,1\}$, $\Omega = \{0,1\}^{\mathbb{N}}$, $T : \Omega \to \Omega$,

$$T((x_1, x_2, \ldots)) = (x_2, x_3, \ldots)$$

i.e. shift.
If $\xi = (\xi(1), \xi(2), \ldots) \in \Omega$ and $f(x) = x_1$, $x = \xi$ realizes $\xi(n)$.

In fact, $\xi(n)$ is already realized in the potentially much simpler flow $F_\xi = (X_\xi, T)$, $X_\xi = \overline{\{T^j \xi\}_{j=1}^{\infty}} \subset \Omega$.

The crudest measure of the complexity of a flow is its Topological Entropy $h(F)$. This measures the exponential growth rate of distinct orbits of length m, $m \to \infty$.

Definition

F is deterministic if $h(F) = 0$. $\xi(n)$ is deterministic if it can be realized in a deterministic flow.

A Process: is a flow together with an invariant probability measure

$$F_\nu = (X, T, \nu),$$
$$\nu(T^{-1}A) = \nu(A) \quad \text{for all (Borel) sets } A \subset X.$$

$h(F_\nu) =$ Kolmogorov–Sinai entropy.
$h(F_\nu) = 0$, F_ν is deterministic, and it means that with ν-probability one, $\xi(1)$ is determined from $\xi(2), \xi(3), \ldots$.

Theorem

$\mu(n)$ is not deterministic.

A much stronger form of this should be that $\mu(n)$ cannot be approximated by a deterministic sequence.

Definition

$\mu(n)$ is disjoint (or orthogonal) from F if

$$\sum_{n \le N} \mu(n)\xi(n) = o(N)$$

for every ξ belonging to F.

Main Conjecture (Möbius Randomness Law)

μ is disjoint from any deterministic F. In particular, μ is orthogonal to any deterministic sequence.

NB We don't ask for rates in $o(N)$.

Why believe this conjecture?

There is an old conjecture.

Conjecture (Chowla: self correlations)

$0 \leq a_1 < a_2 < \ldots < a_t$,

$$\sum_{n \leq N} \mu(n+a_1)\mu(n+a_2)\cdots\mu(n+a_t) = o(N).$$

The trouble with this is no techniques are known to attack it and nothing is known towards it.

Proposition

Chowla $\Longrightarrow$ *Main Conjecture.*

The proof is purely combinatorial and applies to any uncorrelated sequence.

The point is that progress on the main conjecture can be made, and these hard-earned results have far-reaching applications.
The key tool is the bilinear method of Vinogradov — we explain it in dynamical terms at the end.

Cases of Main Conjecture Known:

(i) F is a point $\Longleftrightarrow$ Prime Number Theorem.

(ii) F finite $\Longleftrightarrow$ Dirichlet's theorem on primes in progressions.

(iii) $F = (\mathbb{R}/\mathbb{Z}, T_\alpha)$, $T_\alpha(x) = x + \alpha$, rotation of circle; Vinogradov/Davenport 1937.

(IV) $F = (\Gamma\backslash N, T_\alpha)$, where N is a nilpotent Lie group and Γ a lattice in N, $T_\alpha(\Gamma x) = \Gamma x \alpha$, $\alpha \in N$ fixed (GREEN-TAO 2009)

(EG: $N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $N(\mathbb{R}) = N$
$\Gamma = N(\mathbb{Z})$)

(V) If $F = (X, T)$ is the dynamical flow corresponding to the Morse sequence (connected to the parity of sums of dyadic digits of n); Mauduit-Rivat (2005).

• The last is related to a proof that $\mu(n)$ is orthogonal to bounded depth polynomial size circuit function (GREEN 2011)

MONOTONE CIRCUIT (BOURGAIN 2011)

SEE GIL KALAI'S BLOG (2011)

- IN ALL OF THE ABOVE THE DYNAMICS F IS VERY RIGID. FOR EXAMPLE IT IS NOT WEAK MIXING.

(VI) A SOURCE OF MUCH MORE COMPLEX DYNAMICS BUT STILL DETERMINISTIC IN THE HOMOGENEOUS SETTING IS TO REPLACE THE ABELIAN OR NILPOTENT GROUPS BY SEMI SIMPLE ONES.

THE SIMPLEST EXAMPLE IS THE "HOROCYCLE FLOW"

$G = SL_2(\mathbb{R})$, Γ A LATTICE IN G (eg $SL_2(\mathbb{Z})$)

$$F = (X, T), \quad X = \Gamma\backslash G, \quad T(\Gamma x) = \Gamma x \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- F IS MIXING OF ALL ORDERS (MARCUS/MOSES)
- F IS RIGID (DANI, IN GENERAL RATNER)

<u>THEOREM</u> (BOURGAIN-ZIEGLER-S 2011):
THE MAIN DISJOINTNESS CONJECTURE IS TRUE FOR HOROCYCLE FLOWS.

Dynamical System associated with μ

Simplest realization of μ:

$$\{-1,0,1\}^{\mathbb{N}} = X, \qquad T \text{ shift}$$

$$\omega = (\mu(1), \mu(2), \ldots) \in X$$

$$X_M = \overline{\{T^j\omega\}_{j=1}^{\infty}} \subset X$$

$$M = (X_M, T_M) \text{ is the } \underline{\text{Möbius flow}}.$$

Look for factors and extensions:

$$\eta = (\mu^2(1), \mu^2(2), \ldots) \in Y = \{0,1\}^{\mathbb{N}}$$

$$Y_S = \text{closure in } Y \text{ of } T^j\eta$$

$$S := (Y_S, T_S) \text{ is the } \underline{\text{square-free flow}}.$$

$$\pi : X_M \to Y_M$$

$$(x_1, x_2, \ldots) \mapsto (x_1^2, x_2^2, \ldots)$$

$$\begin{array}{ccc} X_M & \xrightarrow{T_M} & X_M \\ \pi \downarrow & & \downarrow \pi \\ Y_S & \xrightarrow{T_S} & Y_S \end{array}$$

S is a factor of M.

Using an elementary square-free sieve, one can study S!

Definition

$A \subset \mathbb{N}$ is admissible if the reduction $\overline{A}$ of $A \pmod{p^2}$ is not all of the residue classes $\pmod{p^2}$ for every prime p.

Theorem

(i) *Y_S consists of all points $y \in Y$ whose support is admissible.*

(ii) *The flow S is not deterministic; in fact,*

$$h(S) = \frac{6}{\pi^2} \log 2.$$

(iii) *S is proximal;*

$$\inf_{n \geq 1} d(T^n x, T^n y) = 0 \quad \textit{for all } x, y.$$

(iv) *S has a nontrivial joining with the Kronecker flow $K = (G, T)$, $G = \prod_p (\mathbb{Z}/p^2\mathbb{Z})$, $Tx = x + (1, 1, \ldots)$.*

(v) *S is not weak mixing.*

AT THE MEASURE THEORETIC LEVEL THERE ARE TWO IMPORTANT INVARIANT MEASURES FOR S.

1) ν DEFINED ON CYLINDER SETS C_A, (A FINITE) BY,

If $$C_A = \{ y \in Y : y_a = 1 \text{ for } a \in A \}$$

then
$$\nu(C_A) = \prod_p \left(1 - \frac{t(\bar{A}, p^2)}{p^2}\right)$$

where $t(\bar{A}, p^2)$ is the number of reduced classes of A mod p^2.

<u>THEOREM:</u> $S_\nu = (Y_S, T_S, \nu)$ satisfies

(i) η is generic for ν, that is $T^n \eta \in Y$ is ν-equidistd.

(ii) S_ν is ergodic and deterministic.

(iii) S_ν has $K_\nu = (K, T, dg)$ as a Kronecker factor.

• RECENTLY CELLAROSI AND SINAI HAVE SHOWN THAT S_ν IS ISOMORPHIC (MEASURE THEORETICALLY) WITH THE KRONECKER PROCESS. IN PARTICULAR THE QUOTIENT MAP IN (iii) IS AN ISOMORPHISM.

2) m – measure of maximal entropy for S (realizing top entropy).

R. Peckner has shown that there is a unique such measure for S and moreover the process

$(Y_S, T_S, m) = S_m$ is measure theoretically isomorphic to $B \times K$ where B is a Bernoulli process of entropy $\frac{6}{\pi^2}\log 2$ and K is the Kronecker process above.

- Since S is a factor of M, $h(M) \geq h(S) > 0 \Longrightarrow \mu(n)$ is not deterministic!
- Once can form a process N_ν which is a completely positive extension of S and which conjecturally describes M and hence the precise randomness of $\mu(n)$. In this way, the Main Conjecture can be seen as a consequence of a disjointness statement in Furstenberg's general theory.
- We don't know how to establish any more randomness in M than the factor S provides.
- The best we know are the cases of disjointness proved.

Vinogradov (Vaughan) “Sieve” expresses $\sum_{n\leq N}\mu(n)F(n)$ in terms of Type I and Type II sums:
In dynamical terms:

$$I)\quad \sum_{n\leq N} f(T^{nd_1}x).$$

Individual Birkhoff sums associated with (X, T^{d_1}), i.e. sums of f on arithmetic progressions.

$$II)\quad \sum_{n\leq N} f(T^{d_1 n}x)f(T^{d_2 n}x) \qquad \text{(Bilinear sums).}$$

Individual Birkhoff sums associated with the joinings (X, T^{d_1}) with (X, T^{d_2}).

In Bourgain – Ziegler – S we give a finite version of this process. Allows for having <u>no rates</u> (only main terms) in the type II sums.

With this and $X = (\Gamma \backslash SL_2(\mathbb{R}), T_\alpha)$, $\alpha = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, one can then appeal to Ratner's joinings of horocycle theory (1983) to compute and handle the type II sums. $\Rightarrow$ disjointness of $\mu(n)$ with such horocycle flows.

The method applies to some other zero entropy systems:

minimal self joining flows (Veech, Rudolf ...)

Bourgain has dealt with some rank 1 systems and substitution dynamics.

Without rates we aren't able to produce primes for such horocycle flows.

What one can show is

THEOREM (Ubis-S 2011):

- $X = (SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}), T_\alpha)$, $x \in X$ not periodic for T then the orbit $T^p x$ at prime times meets every set $\Omega \subset X$ with $Vol(\Omega) > 9/10$.
- The orbit $T^n x \in X$ with n varying over numbers[*] with at most 100 primes ("almost prime times") is dense in X.

The proof involves among other things effectivizing Ratner (which in this case is a Theorem of Dani) for X.

and T. Ziegler

Some references:

- J. Bourgain and P. Sarnak, "Disjointness of Möbius from horocycle flows", preprint, 2011.
- S. Chowla, *The Riemann Hypothesis and Hilbert's Tenth Problem*, Gordon and Breach, New York, 1965.
- H. Davenport, Quat. J. Math. **8** (1937), 313–320.
- H. Furstenberg, Math. Syst. Th. **1** (1961), 1–49.
- B. Green and T. Tao, "The Möbius function is orthogonal to nilsequences", to appear in Ann. Math.
- H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS, 2004.
- Gil Kalai, Blog, gilkalai.wordpress.com/2011/02/21.
- M. Ratner, Ann. Math. **118** (1983), 277–313.
- M. Ratner, Ann. Math. **134** (1991), 545–607.

P. Sarnak, "Three lectures on the Möbius function randomness and dynamics", publications.ias.edu/sarnak.

I. M. Vinogradov, Recueiv Math. **8** (1937), 313–320.