RANDOM MATRIX THEORY – Spenta Wadia

Dyson: J. Math Phys $\underline{3}$ (1962)

Remarks – "... a new kind of statistical mechanics, in which we renounce exact knowledge of the system itself. We picture a complex nucleus as a "black box" in which a large number of particles are interacting according to unknown laws. The problem is then to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally probable."

$\langle i|H|j\rangle = M_{ij}$ is random, $M^{+} = M$

$$P(M)\,d\mu(M) = e^{-\frac{1}{2}t M^2} d\mu(M)$$

EXAMPLE

Many applications:

1. Spectra of chaotic systems
2. Statistical properties of S-matrix in quantum transport
3. Number theory: distribution of zeros of Riemann $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s \in \mathbb{C}$
4. Random surfaces and 2-dim quantum gravity 'Triangulations' ≡ Feynman graphs
5. Model SU(N) gauge theories
6. Statistics - free probability theory
7. Engineering - control theory
8. Statistics of critical points in a 'lanscape' Hessian is the random matrix

⋮

Wigner's semi-circle law

$$Z = \int [dM] \, e^{-S(M)}$$

$M = M^{\dagger}$ $N \times N$ $\quad S(U^{\dagger} M U) = S(M)$

$[dM] = ?$

space of hermitian matrices is euclidean (flat)

$ds^2 = \operatorname{tr} dM dM^{\dagger}$, $(dM)_{ij} = dM_{ij}$

'polar decomposition' : $M = U D U^{\dagger}$, $U^{\dagger} U = 1$

$D = [\lambda_1, \lambda_2, \ldots, \lambda_N]$

$dM = dD + [U^{\dagger} dU, D]$, $\quad U^{\dagger} dU \equiv d\Omega$ (polar coordinates)

$$dM_{ij} = d\lambda_i \delta_{ij} + (\lambda_i - \lambda_j) d\Omega_{ij}$$

$$ds^2 = dM_{ij} dM_{ji} = d\lambda_i d\lambda_i + \sum_{i,j} (\lambda_i - \lambda_j)^2 d\Omega_{ij} d\Omega_{ij}^{*}$$

Jacobian of the map: $M_{ij} \longrightarrow (\lambda_i, U_{ij})$

$$J \equiv \sqrt{G} = \prod_{i \neq j} (\lambda_i - \lambda_j) = \Delta(\lambda)^2$$

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det\left(\lambda_j^{i-1}\right)$$

Vandermonde determinant.

Since $S(U^\dagger M U) = S(M)$

$$[dM] \equiv \frac{\prod_{i=1}^{N} d\lambda_i \, \Delta(\lambda)^2 \prod_{i,j} d\Omega_{ij} \, d\Omega_{ij}^\dagger}{\mathrm{Vol}\, U(N)}$$

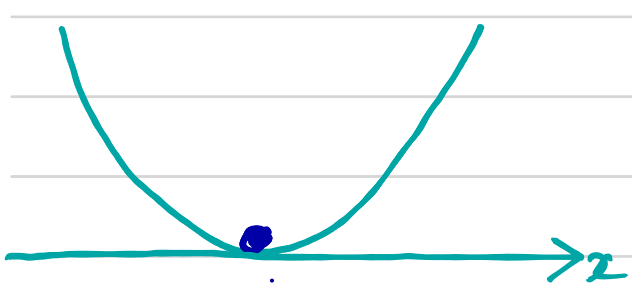
$$Z = \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \, e^{-S(\lambda_1, \ldots \lambda_N)}$$

$$S = S(\mathrm{tr} M^p, (\mathrm{tr} M^r)^s \ldots)$$

Wigner $S = \frac{1}{g} \mathrm{tr} M^2 = \frac{1}{g} \sum_i \lambda_i^2$

$$Z = \int \prod_{i=1}^{N} d\lambda_i \; e^{-\frac{1}{2g}\sum_i \lambda_i^2 + \sum_{i<j} \ln(\lambda_i - \lambda_j)^2}$$

One particle potential : $W(x) = \frac{1}{2} x^2$



harmonic well

log 'repulsion'

particles spread to a finite interval

General problem:

$$S(\lambda) = \frac{1}{g}\sum_j W(\lambda_j) - 2\sum_{i<j}\log(\lambda_i - \lambda_j)$$

excludes 'multitrace' terms

$$\frac{\partial S}{\partial \lambda_i} = 0 = \frac{1}{g}W'(\lambda_i) - 2\sum_{i\neq j}\frac{1}{(\lambda_i - \lambda_j)}$$

(force balance)

$$\frac{1}{g}\sum_i \frac{W'(\lambda_i)}{(\lambda_i - z)} - 2\sum_{i\neq j}\frac{1}{(\lambda_i - z)(\lambda_i - \lambda_j)} = 0$$

$$\omega(z) = \frac{1}{N}\operatorname{tr}\frac{1}{(M-z)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{(\lambda_i - z)}$$

$$= \frac{1}{N}\int_0^\infty ds\, e^{-zs}\operatorname{tr}e^{-Ms} \quad \text{(loop operator)}$$

$$\boxed{\omega^2(z) - \frac{1}{N}\partial_z\omega(z) + \frac{1}{\mu}\omega(z)W'(z) + \frac{1}{4\mu^2}f(z) = 0}$$

$$f(z) = \frac{4\mu}{N}\sum_i\left[\frac{W'(z) - W'(\lambda_i)}{z - \lambda_i}\right], \quad gN = \mu$$

$$y(z) = 2\mu\omega(z) + W'(z) \Rightarrow \underbrace{y^2 - W'(z)^2 + f(z) = 0}_{\text{a curve in }(y,z)\text{ plane}}$$

Wigner's semi-circle law:

$N \to \infty$ limit $\quad \mu = gN$ fixed

$$\omega^2(z) + \frac{1}{\mu}\,\omega(z)\,z + \frac{1}{\mu}\,1 = 0$$

algebraic eqn.

$$\omega = -\frac{z}{2\mu} \pm \frac{1}{2\mu}\sqrt{z^2 - 4\mu} \qquad z \in \mathbb{C} - [\,,\,]$$

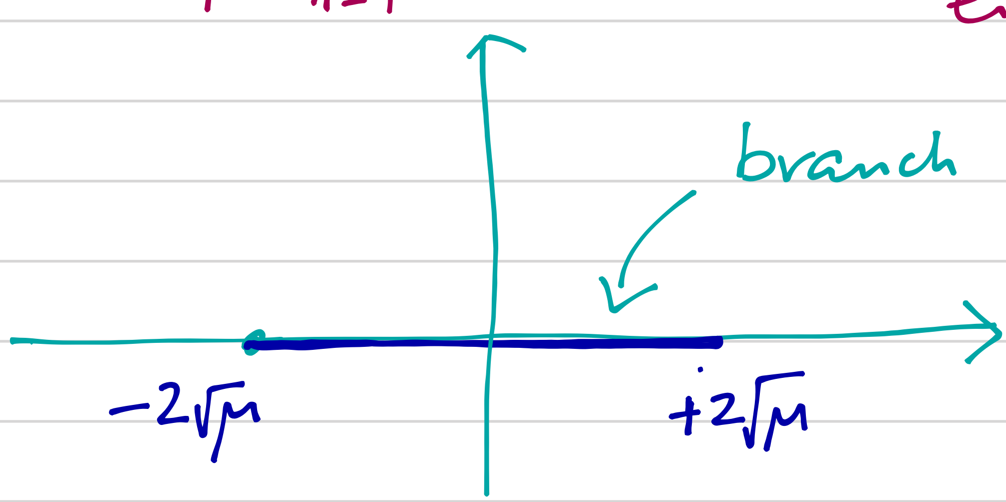
$$\omega(z \to \infty) \to -\frac{1}{z}$$

$$\Rightarrow \omega(z) = -\frac{z}{2\mu} + \frac{1}{2\mu}\sqrt{z^2 - 4\mu}$$

$$\left[\omega(z) = \frac{1}{N}\,\mathrm{tr}\left(\frac{1}{M - z}\right) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z}\right.$$

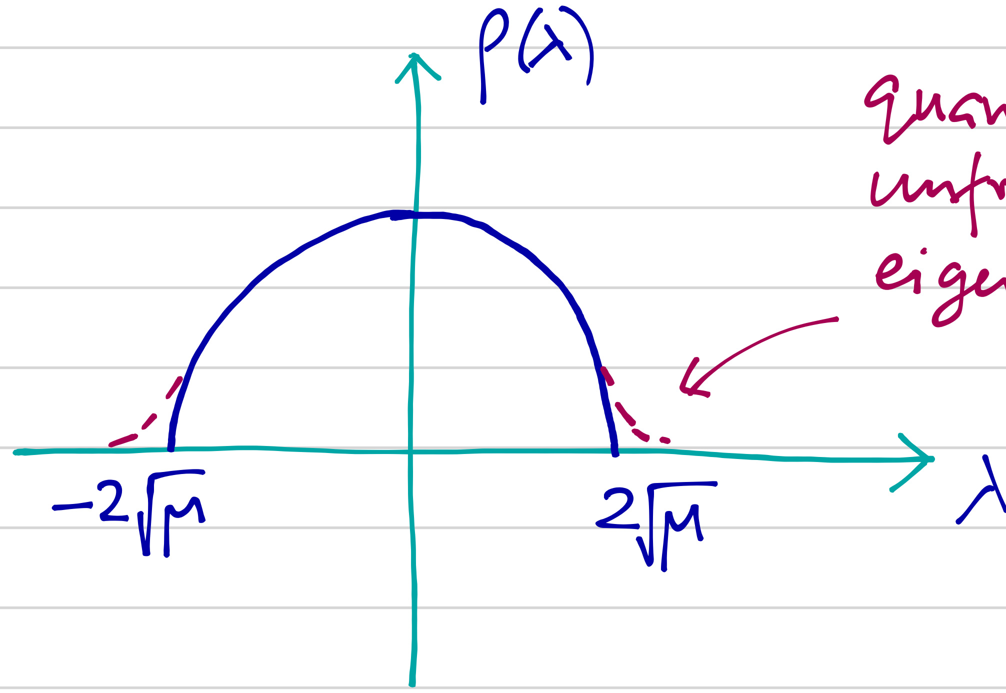
$$= \int \frac{\rho(\lambda)}{\lambda - z}$$

$$\left.\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i)\,, \quad \text{density of eigenvalues}\right]$$



branch cut (continuously spread eigenvalues)

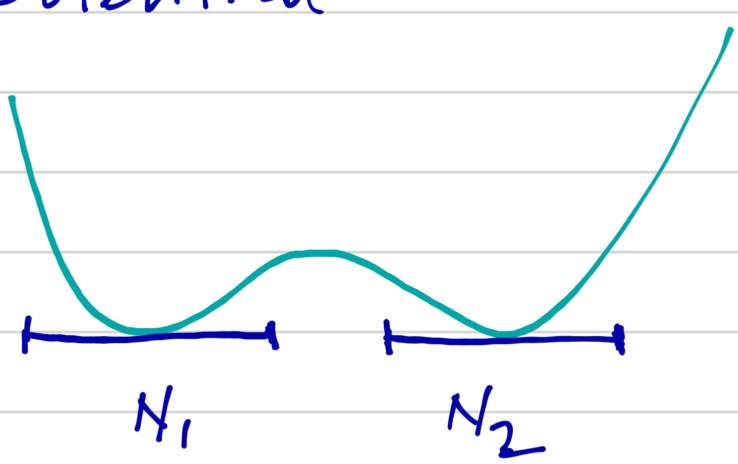
$$\rho(\lambda) = \frac{1}{2\pi i}\left(\omega(\lambda + i\epsilon) - \omega(\lambda - i\epsilon)\right) = \frac{1}{4\pi\mu}\sqrt{4\mu - \lambda^2}$$



quantum unfreezing of eigenvalues and criticality.

Comments:

1. For an arbitrary $W(\lambda)$ one can solve the algebraic eqn $W(z)$ in the cut plane with many branch cuts depending on the potential



multi-cut plane

$N_1 + N_2 = N$

Loop eqn for $W(z)$ can be solved beyond leading order in $\frac{1}{N}$.

2. Multi-trace operators

$$S(M) = \frac{1}{2}\mathrm{tr}\, M^2 + \frac{c_1}{4}\mathrm{tr}\, M^4 + \frac{c_2}{N}(\mathrm{tr}\, M^2)^2$$

$N \to \infty$ $\quad \langle (\mathrm{tr}\, M^2)^2 \rangle = \langle \mathrm{tr}\, M^2 \rangle^2$ (discuss later)

no fluctuation

$$\to \quad S_{eff} = \frac{1}{2}\mathrm{tr}\, M^2 + \frac{c_1}{4}\mathrm{tr}\, M^4 + \frac{c_2}{N}\langle \mathrm{tr} M^2 \rangle \mathrm{tr} M^2$$

Same method as before but

$\langle \mathrm{tr} M^2 \rangle$ is determined self consistently

3. Traditional way to solve for $\rho(\lambda)$:

$$W'(\lambda) = 2 \fint \frac{\rho(\lambda')}{\lambda' - \lambda}, \quad \int \rho(\lambda) d\lambda = 1$$

Singular integral eqn. $\Rightarrow$

Riemann - Hilbert problem for

$$\omega(z) = \int \frac{\rho(\lambda')}{\lambda' - z}, \qquad \omega(z \to \infty) = -\frac{1}{z}$$

$$\omega_+ - \omega_- = \rho(\lambda), \quad \omega_+ + \omega_- = W'(\lambda)$$

$$\frac{\omega_+ \circlearrowright}{\omega_- \circlearrowleft}$$

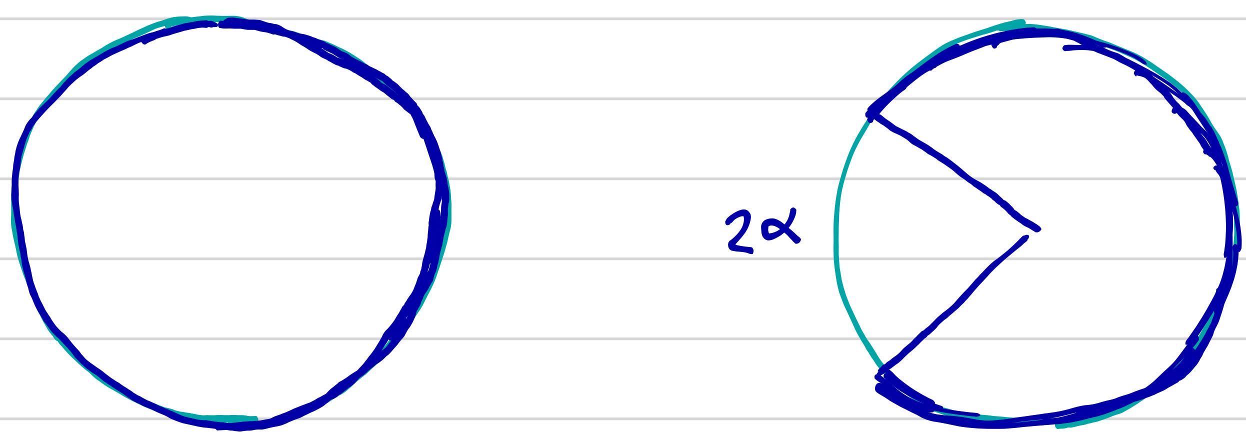
4. Unitary matrix model

$$S(U) = \frac{NE}{2}\left(\mathrm{tr}\,U + \mathrm{tr}\,U^{\dagger}\right) + \cdots, \quad U^{\dagger}U = 1$$

eigenvalues of U : $e^{i\theta_i}$, $i = 1, \ldots, N$

$$\rho(\theta) = \frac{1}{N}\sum_i \delta(\theta - \theta_i)$$

Charges in an electric field E



$\rho(\theta) = 1 + E\cos\theta$

$0 \le E \le 1$

$\rho(\theta) = 2E\cos\frac{\theta}{2}\sqrt{\frac{1}{E} - \sin^2\theta/2}$

$1 \le E < \infty$, $\frac{1}{E} = \sin^2\frac{\alpha}{2}$

As the electric field E increases there appears a gap in the distribution of eigenvalues at $E_{\text{critical}} = 1$.

- 3rd derivative of $\ln Z$ is discontinuous at $E = 1$ (Gross-Witten-Wadia 3rd order phase transition)

5. Level spacing statistics

$\lambda_1 < \lambda_2 \ldots < \lambda_N$

$s = (\lambda_{n+1} - \lambda_n)/\langle s \rangle$, $\langle s \rangle = \langle \lambda_{n+1} - \lambda_n \rangle$

$P_1(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2}$ ⟨ orthogonal symmetric matrix $\prod_{i<j}(\lambda_i - \lambda_j)$

$P_2(s) = \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi}s^2}$ ⟨ hermitian matrix $\prod_{i<j}(\lambda_i - \lambda_j)^2$

$P_4(s) = \frac{2^{18}}{3^6\pi^3} s^4 e^{-\frac{64}{9\pi}s^2}$ ⟨ hermitian quaternionic matrices $\prod_{i<j}(\lambda_i - \lambda_j)^4$

$P_i(s=0) = 0 \qquad i = 1, 2, 4$

(level repulsion)

↓

a property that is relevant for modelling quantum chaotic systems.

Large N factorization

1- matrix model

$$\left\langle \frac{tr}{N} M^{2p} \frac{tr}{N} M^{2q} \right\rangle = \int d\lambda\, \lambda^{2p} \bar{\rho}(\lambda) \cdot \int d\lambda'\, \lambda'^{2q} \bar{\rho}(\lambda')$$

$$= \left\langle \frac{tr}{N} M^{2p} \right\rangle \left\langle \frac{tr}{N} M^{2q} \right\rangle + o\left(\frac{1}{N^2}\right)$$

All correlations are evaluated by a single function $\bar{\rho}(\lambda)$ the eigenvalue density at the saddle point.

This 'method' does not work for more than one matrix

e.g M_1 and M_2, $[M_1, M_2] \neq 0$

Question:

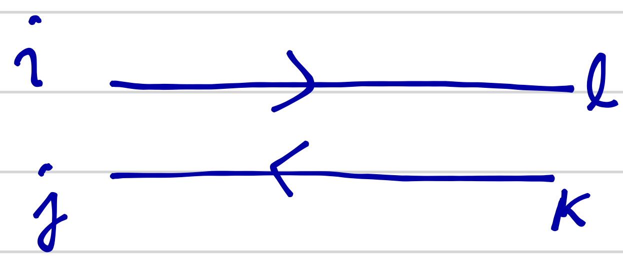
Is it possible to formulate the large N limit of an arbitrarily complicate matrix / field theory?

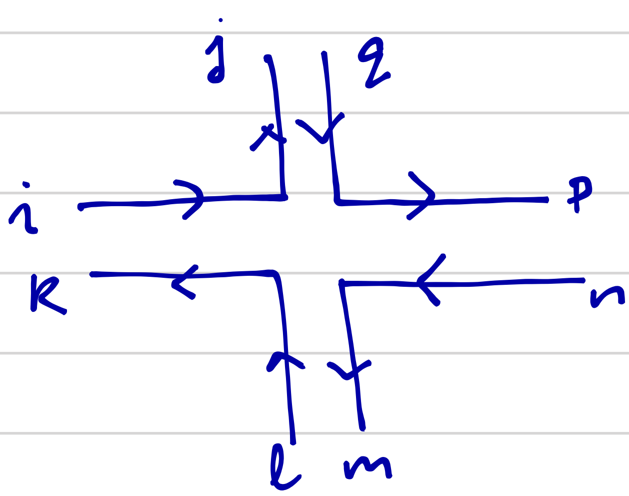
e.g multi-matrix models; non-abelian gauge theories ....

Factorization in a simple model:

$$\int [dM]\; e^{-\frac{N}{\lambda}\left(\mathrm{tr}\,M^2 + \mathrm{tr}\,M^3 + \mathrm{tr}\,M^4\right)} \cdots$$

$$\langle M_{ij}\, M_{kl} \rangle = \frac{\lambda}{N}\, \delta_{il}\, \delta_{jk} \quad , \quad \delta_{ii} = N$$





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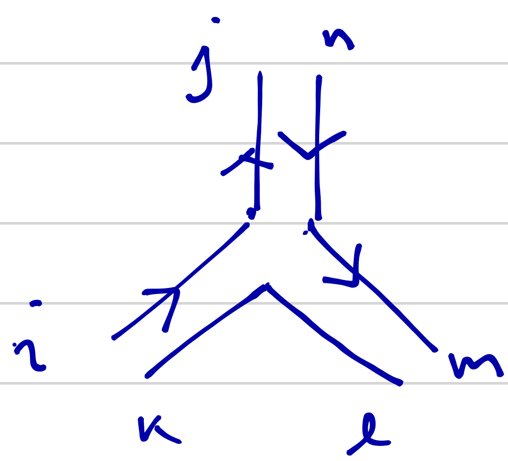
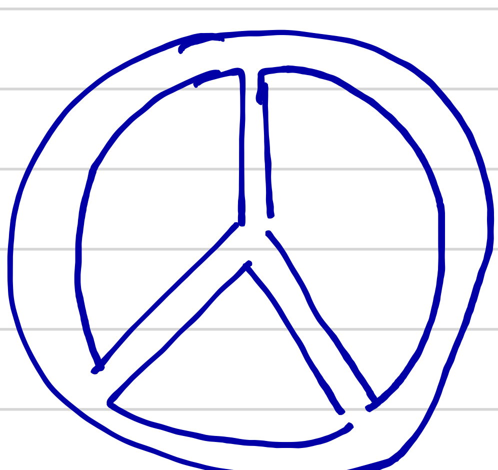


. . .

$\propto \frac{N}{\lambda}$ $\qquad\qquad$ $\propto \frac{N}{\lambda}$

Typical planar feynman - tHooft diagram



4 3-pt vertices = V
6 propagators = E
4 index loops = F

$F - E + V = 2$

$\propto N^2 \lambda^2$

Topological expansion:

In general a diagram will have

E propagators, V interaction vertices and

F index loops

Diagram in pert theory $\propto$

$$\left(\frac{\lambda}{N}\right)^{E} \left(\frac{N}{\lambda}\right)^{V} N^{F} = N^{\chi} \lambda^{E-V}$$

$\chi = F - E + V$ is the Euler character.

Dominant contribution at large N comes from $\chi = 2$ (diagrams that can be put on a sphere)

 $\simeq$ 2 discs fitted on S^2.

$$\ln Z = \sum_{h=0}^{\infty} N^{2-2h} F_h(\lambda) \quad , \chi = 2-2h$$

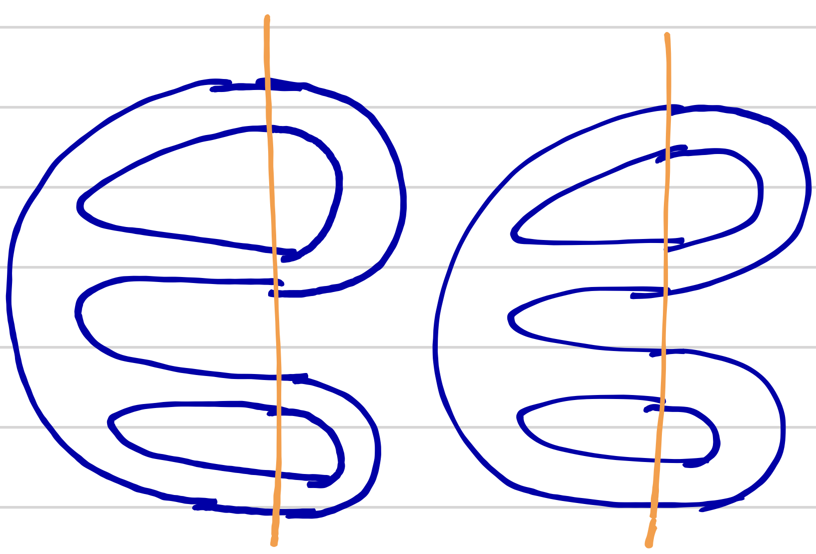
(topological expansion)

handle on a sphere.

Factorization :

$\langle \underline{tr}\, M^4 \, \underline{tr}\, M^4 \rangle$, $S = \frac{N}{\lambda} \mathrm{tr}\, M^2$

Dominant contribution from disconnected diagrams

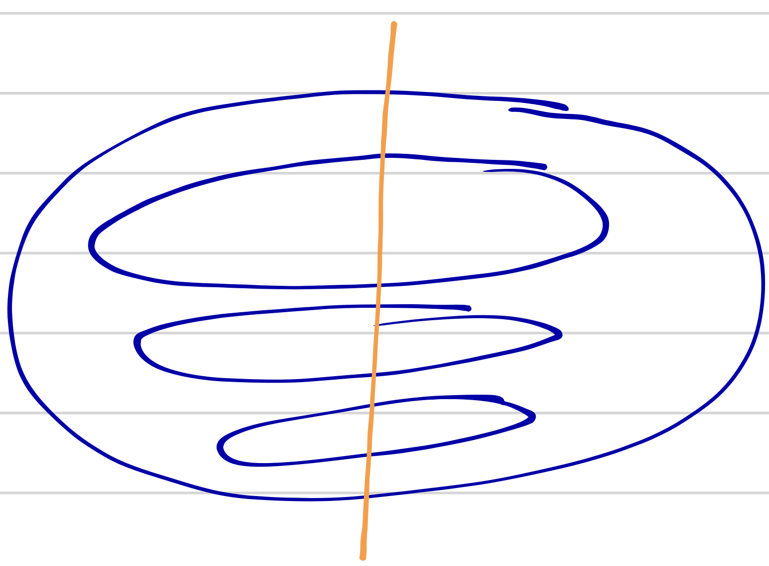


4-propagators = E
6-index loops = F

$\propto N^2$

Disconnected

+



4 propagators = E

4 index loops = F

$\propto N^0$

connected

$$\left\langle \frac{tr}{N} M^4 \frac{tr}{N} M^4 \right\rangle = \left\langle \frac{tr}{N} M^4 \right\rangle \left\langle \frac{tr}{N} M^4 \right\rangle + O(1) + O\left(\frac{1}{N^2}\right).$$

One can convince oneself that disconnected diagrams dominate at large N, in general.

Factorization at large N
$\Rightarrow$ closed set of Schwinger-Dyson equations

2-examples:

1-matrix model

SD :
$$\int [dM] \frac{\partial}{\partial M_{ij}} \left(M^n_{ij} \, e^{-S} \right) = 0$$

using
$$\frac{\partial}{\partial M_{ij}} (M^n)_{ij} = \sum_{k=0}^{n-1} (M^k)_{jj} (M^{n-1-k})_{ii}$$
$$n \geq 1$$

and
$$\langle \operatorname{tr} M^r \operatorname{tr} M^s \rangle = \langle \operatorname{tr} M^r \rangle \langle \operatorname{tr} M^s \rangle$$

one can reproduce :
$$\omega^2(z) + \frac{1}{\mu} \omega(z) S'(z) + \frac{1}{4\mu^2} f(z) = 0$$

$$\omega(z) = \operatorname{tr} \frac{1}{M-z}$$

2-matrix model

(Deo, Jain, Shastry)

$$Z = \int [dA][dB]\, e^{-S(A,B)}$$

$$S = N\,\mathrm{tr}\left(\frac{A^2}{2} + \frac{B^2}{2} - c\,AB\right)$$

Closed SD eqns at large N for:

$$\left\langle \mathrm{tr}\left(\frac{1}{z-A}\,\frac{1}{w-B}\right)\right\rangle_c$$

$$\left\langle \mathrm{tr}\,\frac{1}{z-A}\right\rangle, \quad \left\langle \mathrm{tr}\,\frac{1}{z-B}\right\rangle$$

$$\left\langle \mathrm{tr}\,\frac{1}{z-A}B\right\rangle, \quad \left\langle \mathrm{tr}\,\frac{1}{z-B}A\right\rangle$$

$$\left\langle \mathrm{tr}\,\frac{1}{z-A}\,\mathrm{tr}\,\frac{1}{w-B}\right\rangle$$

Answer:

$$\left\langle \mathrm{tr}\,\frac{1}{z-A}\,\mathrm{tr}\,\frac{1}{w-B}\right\rangle = \frac{c_1}{[1 - c_2 W(z) W(w)]^2}$$

$$\times \left[\frac{W(z)^2}{1 - c_3 W^2(z)}\right]\left[z \to w\right]$$

c_1, c_2, c_3 are known constants.

Comments :

1. SD + factorization $\Rightarrow$ exact equations for Wilson loops in a non-abelian lattice gauge theory

2. In models with vector degrees of freedom (eg $S^a(\vec{x})$ spin, $\psi^a(\vec{x})$ fermion)

Bilocal variables act like random metrics

e.g. $G(\vec{x}, \vec{y}) = \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}) \equiv M_{\vec{x}, \vec{y}}$ ....

Applications to Chern-Simon's theory, SYK model

(Sachdev-Ye-Kitaev)

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