Optimization with inexact gradient and function

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Outline for section 1

Why multiprecision?

Paraphrasing [Higham, 2017]:

- Variable precision is becoming more and more accessible in hardware and software.
- Using lower precision can drastically reduce computational running time (e.g. IEEE single up to 14 times faster than IEEE double).
- Our challenge is to better understand the accuracy of algorithms in low precision.

How does multiprecision arithmetic affect the convergence rate and final accuracy of minimization algorithms?

Why multiprecision?

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The (simple?) problem

We consider the unconstrained quadratic optimization (QO) problem:

$$\text{minimize} \quad q(x) = \tfrac{1}{2} x^T A x - b^T x$$

for $x, b \in \mathbb{R}^n$ and A an $n \times n$ symmetric positive-definite matrix.

A truly "core" problem in optimization (and linear algebra)

- the simplest nonlinear optimization problem
- subproblem in many methods for general nonlinear unconstrained optimization
- central in linear algebra (including solving elliptic PDEs)

Working assumptions

For what follows, we assume that

- the problem size n is large enough and A is dense enough to make factorization of A unavailable
- a Krylov iterative method (Conjugate Gradients, FOM) is used
- the cost of running this iterative method is dominated by the products Av

Focus on an $\boxed{\text{optimization point of view}}$: look at decrease in q rather than at decrease in the associated system's residual

ex: ensuring increase in the likelihood in statistics

Our aim, for x_* solution of QO,

$$\boxed{\text{Find } x_k \text{ such that } |q(x_k) - q(x_*)| \leq \epsilon |q(x_0) - q(x_*)|.}$$

A first motivating example: weather forecasting (1)

The $\boxed{\text{weakly-constrained 4D-Var}}$ formulation (See [Y. Tremolet 2006, 2007,..])

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|x_0 - x_b\|^2_{B^{-1}} + \frac{1}{2}\sum_{j=0}^{N} \|\mathcal{H}_j(x_j) - y_j\|^2_{R_j^{-1}} + \frac{1}{2}\sum_{j=1}^{N} \|\underbrace{x_j - \mathcal{M}_j(x_{j-1})}_{q_j}\|^2_{Q_j^{-1}}$$

- $x = (x_0, \ldots, x_N)^T$ is the state control variable (with $x_j = x(t_j)$)
- x_b is the background given at the initial time (t_0).
- $y_j \in \mathbb{R}^{m_j}$ is the observation vector over a given time interval
- $\mathcal{H}_j$ maps the state vector x_j from model space to observation space
- $\mathcal{M}_j$ is an integration of the numerical model from time t_{j-1} to t_j
- B, R_j and Q_j are the covariance matrices of background, observation and model error. B and Q_j impractical to "invert"

A first motivating example: weather forecasting (2)

Solve by a Gauss-Newton method whose subproblem (at iteration k) is

$$\min_{\delta x} \frac{1}{2}\|\delta x_0 - b^{(k)}\|^2_{\mathbf{B}^{-1}} + \frac{1}{2}\sum_{j=0}^{N}\left\|H_j^{(k)}\delta x_j - d_j^{(k)}\right\|^2_{\mathbf{R}_j^{-1}} + \frac{1}{2}\sum_{j=1}^{N}\|\underbrace{\delta x_j - M_j^{(k)}\delta x_{j-1}}_{\delta q_j} - c_j^{(k)}\|^2_{\mathbf{Q}_j^{-1}}$$

- δx is the increment in x.
- The vectors $b^{(k)}$, $c_j^{(k)}$ and $d_j^{(k)}$ are defined by

$$b^{(k)} = x_b - {x_0}^{(k)}, \quad c_j^{(k)} = q_j^{(k)}, \quad d_j^{(k)} = \mathcal{H}_j({x_j}^{(k)}) - y_j$$

and are calculated at the outer loop.

A first motivating example: weather forecasting (3)

Can be rewritten as

$$\min_{\delta x} \; q_{\rm st} = \frac{1}{2}\|L\delta x - b\|^2_{D^{-1}} + \frac{1}{2}\|H\delta x - d\|^2_{R^{-1}}$$

where

- $L = \begin{pmatrix} I & & & & \\ -M_1 & I & & & \\ & -M_2 & I & & \\ & & \ddots & \ddots & \\ & & & -M_N & I \end{pmatrix}$
- $d = (d_0, d_1, \ldots, d_N)^T$ and $b = (b, c_1, \ldots, c_N)^T$
- $H = \mathrm{diag}(H_0, H_1, \ldots, H_N)$
- $D = \mathrm{diag}(B, Q_1, \ldots, Q_N)$ and $R = \mathrm{diag}(R_0, R_1, \ldots, R_N)$

A first motivating example: weather forecasting (3)

$$\boxed{\min_{\delta x} \; q_{\rm st} = \frac{1}{2}\|L\delta x - b\|^2_{D^{-1}} + \frac{1}{2}\|H\delta x - d\|^2_{R^{-1}}}$$

This is a standard QO, but **HUGE!** Note that

$$\nabla^2 q_{\rm st} = L^T D^{-1} L + H^T R^{-1} H$$

In addition $\boxed{D^{-1} = \mathrm{diag}(B^{-1}, Q_1^{-1}, \ldots, Q_N^{-1}) \text{ is unavailable!}}$

Thus $\nabla^2 q_{st} v$ (a Hessian times vector product) must be computed by

- $w = Lv$,
- solve $Dz = w$ using some (preconditioned) Krylov method
- $v = L^T z + H^T R^{-1} Hv$

A second motivating example: variable precision arithmetic

Next barrier in hyper computing: energy dissipation!

Heat production is proportional to chip surface, hence

$$\text{energy output} \approx \left(\text{ number of digits used }\right)^2$$

Architectural trend: use multiprecision arithmetic

- graphical processing units (GPUs)
- hierarchy of specialized CPUs (double, single, half, . . .)

How to use this hierarchy optimally for fully accurate results?

Outline for section 2

Inaccuracy frameworks

Our proposal;

Make the Krylov methods for QO more efficient by allowing error on the matrix-vector product (the dominant computation)

Two frameworks of interest:

- Continuous accuracy levels
  ex: WC-4D-VAR, where accuracy in the inversion $Dz = w$ can be continuously chosen
- Discrete accuracy levels
  ex: double-single-half precision arithmetic

Considered here:

- Full orthonormalisation method (FOM)
- Conjugate Gradients (CG)

with (wlog) $x_0 = 0$ and $q(x_0) = 0$.

A central equality

Define $r(x) \overset{\text{def}}{=} Ax - b = \nabla q(x)$ and $Ax_* = b$.

$$\boxed{q(x) - q(x_*) = \tfrac{1}{2}\|r(x)\|^2_{A^{-1}}}$$

$$\begin{aligned}
\tfrac{1}{2}\|r(x)\|^2_{A^{-1}} &= \tfrac{1}{2}(Ax-b)^T A^{-1}(Ax-b) \\
&= \tfrac{1}{2}(x-x_*)^T A(x-x_*) \\
&= \tfrac{1}{2}(x^T Ax - 2x^T Ax_* + x_*^T Ax_*) \\
&= q(x) - q(x_*)
\end{aligned}$$

Hence

Decrease in q can be monitored by considering the A^{-1} norm of its gradient

The primal-dual norm

$\Rightarrow$ natural to consider the inaccuracy on the product Av by measuring the backward error

$$\|E\|_{A^{-1},A} \overset{\text{def}}{=} \sup_{x\neq 0} \frac{\|Ex\|_{A^{-1}}}{\|x\|_A} = \|A^{-1/2}EA^{-1/2}\|_2$$

(primal-dual norm)

Let A be a symmetric and positive definite matrix and E be any symmetric perturbation. Then, if $\|E\|_{A^{-1},A} < 1$, the matrix $A+E$ is symmetric positive definite.

The main idea

Krylov methods reduce the (internally recurred) residual r_k on successive nested Krylov spaces
$\Rightarrow$ can expect r_k to converge to zero
$\Rightarrow$ keep $r(x_k) - r_k$ small in the appropriate norm

From $q(x) - q(x_*) = \frac{1}{2}\|r(x)\|^2_{A^{-1}}$, $q(x_*) = -\frac{1}{2}\|b\|^2_{A^{-1}}$, and triangular inequality,

For FOM and CG, if

$$\max\left[\|r_k - r(x_k)\|_{A^{-1}}, \|r_k\|_{A^{-1}}\right] \le \frac{\sqrt{\epsilon}}{2}\|b\|_{A^{-1}}$$

then

$$|q(x_k) - q(x_*)| \le \epsilon |q(x_*)|$$

The inexact FOM algorithm

Theoretical inexact FOM algorithm

```
1.  Set $\beta = \|b\|_2$, and $v_1 = [b/\beta]$,
2.  For k=1, 2, . . . , do
3.    $w_k = (A + E_k)v_k$
4.    For $i = 1, \dots, k$ do
5.      $h_{i,k} = v_i^T w_k$
6.      $w_k = w_k - h_{i,k} v_i$
7.    EndFor
8.    $h_{k+1,k} = \|w_k\|_2$
9.    $y_k = H_k^{-1}(\beta e_1)$
10.   if $|h_{k+1,k} e_k^T y_k|$ is small enough then go to 13
11.   $v_{k+1} = w_k / h_{k+1,k}$
12. EndFor
13. $x_k = V_k y_k$
```

Results for the inexact FOM

Let $\epsilon_\pi > 0$ and let $\phi \in \mathbb{R}^k_+$ such that $\sum_{j=1}^k \phi_j^{-1} \le 1$. Suppose also that, for all $j \in \{1, \dots, k\}$,

$$\|E_j\|_{A^{-1},A} \le \omega_j^{\text{FOM}} \overset{\text{def}}{=} \min \left[1, \frac{\epsilon_\pi \, \|b\|_{A^{-1}}}{\phi_j \|v_j\|_A \|H_k^{-1}\|_2 \|r_{j-1}\|_2} \right] \quad (2.1)$$

Then
$$\|r(x_k) - r_k\|_{A^{-1}} \le \epsilon_\pi \, \|b\|_{A^{-1}}.$$

Let $\epsilon > 0$ and suppose that, at iteration $k > 0$ of the FOM algorithm,
$$\|r_k\|_{A^{-1}} \le \tfrac{1}{2}\sqrt{\epsilon}\, \|b\|_{A^{-1}}$$
and the product error matrices E_j satisfy (2.1) with $\epsilon_\pi = \frac{1}{2}\sqrt{\epsilon}$ for some $\phi \in \mathbb{R}^k$ (as above). Then
$$|q(x_k) - q(x_*)| \le \epsilon |q(x_*)|$$

The inexact Conjugate Gradients algorithm

Theoretical inexact CG algorithm

```
1.  Set $x_0 = 0$, $\beta_0 = \|b\|_2^2$, $r_0 = -b$ and $p_0 = r_0$
2.  For k=0, 1, . . . , do
3.     $c_k = (A + E_k)p_k$
4.     $\alpha_k = \beta_k / p_k^T c_k$
5.     $x_{k+1} = x_k + \alpha_k p_k$
6.     $r_{k+1} = r_k + \alpha_k c_k$
7.     if $r_{k+1}$ is small enough then stop
8.     $\beta_{k+1} = r_{k+1}^T r_{k+1}$
9.     $p_{k+1} = -r_{k+1} + (\beta_{k+1}/\beta_k)p_k$
10. EndFor
```

Results for the inexact CG

Let $\epsilon_\pi > 0$ and let $\phi \in \mathbb{R}_+^k$ such that $\sum_{j=1}^k \phi_j^{-1} \leq 1$. Suppose also that, for all $j \in \{0, \ldots, k-1\}$,

$$\|E_j\|_{A^{-1},A} \leq \omega_j^{\mathrm{CG}} \overset{\mathrm{def}}{=} \frac{\epsilon_\pi \, \|b\|_{A^{-1}} \|p_j\|_A}{\phi_{j+1}\|r_j\|_2^2 + \epsilon_\pi \, \|b\|_{A^{-1}} \|p_j\|_A} \qquad (2.2)$$

Then

$$\|r(x_k) - r_k\|_{A^{-1}} \leq \epsilon_\pi \, \|b\|_{A^{-1}}.$$

Let $\epsilon > 0$ and suppose that, at iteration $k > 0$ of the CG algorithm,

$$\|r_k\|_{A^{-1}} \leq \tfrac{1}{2}\sqrt{\epsilon} \, \|b\|_{A^{-1}}$$

and the product error matrices E_j satisfy (2.2) with $\epsilon_\pi = \frac{1}{2}\sqrt{\epsilon}$ for some $\phi \in \mathbb{R}^k$ (as above). Then

$$|q(x_k) - q(x_*)| \leq \epsilon |q(x_*)|$$

Using the true quantities (1)

Would this work at all if using the true $\|b\|_{A^{-1}}$, $\|v_j\|_A$ and $\|p_j\|_A$?

Consider 6 algorithms:

FOM:	the standard full-accuracy FOM
iFOM:	the inexact FOM (with exact bounds, for now)
CG:	the standard full-accuracy CG
CGR:	the full-accuracy $\boxed{\text{CG } \textit{with reorthogonalization}}$
iCG:	the inexact CG (with exact bounds, for now)
iCGR:	the inexact CGR (with exact bounds, for now)

Continuous accuracy levels (1)

Comparing equivalent numbers of full accuracy products:

- Assume obtaining full accuracy is a linearly convergent process of rate ρ
  (realistic for our weather prediction data assimilation example)
- Cost of an ϵ-accurate solution:

$$\frac{\log(\epsilon)}{\log(\rho)}$$

$\Rightarrow$ sum these values during computing and compare them.

Continuous accuracy levels (2)

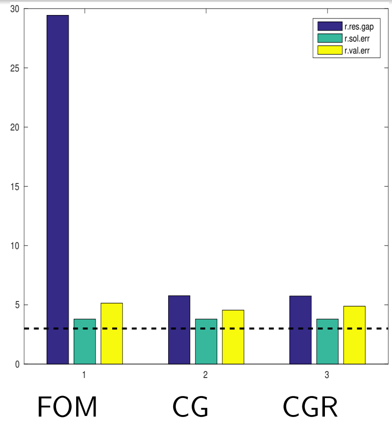
Compare on:

- synthetic matrices of size 1000×1000 with varying conditioning (from 10^1 to 10^8) and log-linearly spaced eigenvalues
- "real" matrices from the NIST Matrix Market
- use different levels of final accuracy ($\epsilon = 10^{-3},\ 10^{-5}$)

Note that

Continuous accuracy levels $\Rightarrow$ no room for inaccuracy budget management!

Continuous accuracy levels (3)



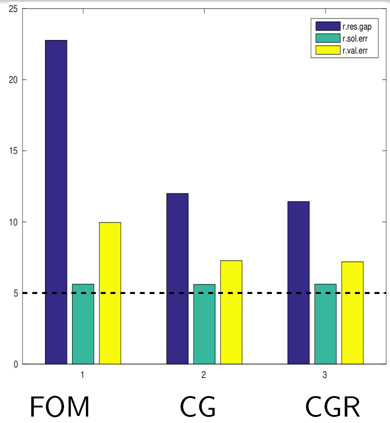


Figure: Exact bounds, $\kappa(A) = 10^1$, $\epsilon = 10^{-3}$ (left), $\kappa(A) = 10^5$, $\epsilon = 10^{-5}$ (right); continuous case

Want green (gap) and blue (stopping criterion error on the quadratic) close to epsilon, and yellow (approximate error on the quadratic) close to green

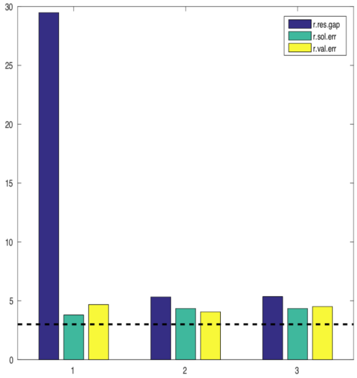
Multiprecision (1)

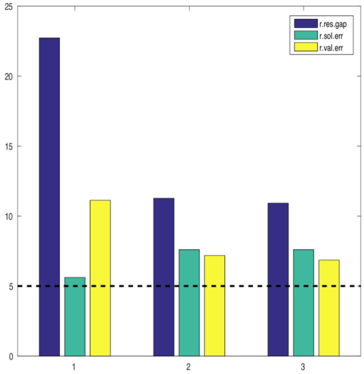
Focus on $\boxed{\text{multiprecision arithmetic}}$. Assume

- 3 levels of accuracy (double, single, half)
- a ratio of 4 in efficiency when moving from one level to the next

Use the sames matrices and final accuracies as above.

Multiprecision (2)

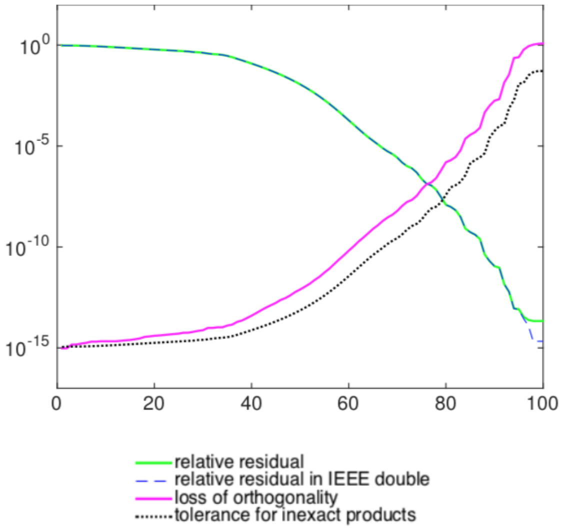




FOM CG CGR FOM CG CGR

Figure: Exact bounds, $\kappa(A) = 10^1$, $\epsilon = 10^{-3}$ (left), $\kappa(A) = 10^5$, $\epsilon = 10^{-5}$ (right); discontinuous case

An beyond : inexact scalar products



Just relax !

Perfect in theory but...

- The primal-dual norm $\|E_j\|_{A^{-1},A}$ is sometimes difficult to evaluate
- The error bounds remain unfortunately hard to estimate (they involve $\|b\|_{A^{-1}}$, $\|v_j\|_A$ or $\|p_j\|_A$, which cannot be computed readily in the course of the FOM or CG algorithm).
- The termination test $\|r_k\|_{A^{-1}} \leq \frac{1}{2}\sqrt{\epsilon}\,\|b\|_{A^{-1}}$ also involves the unavailable $\|r_k\|_{A^{-1}}$

Give up? Not quite. . .

- the FOM error bound allows a growth of the error in $\|r_j\|^{-1}$ while CG allows a growth of the order of $\|r_j\|^{-2}\|p_j\|_A$ instead.

Adhoc approximations

Abandon theoretical but unavailable quantities $\rightarrow$ approximate them:

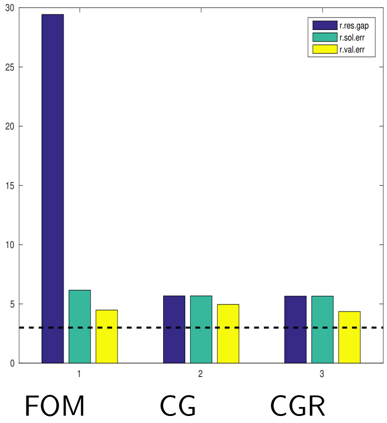
- $\|E\|_{A^{-1},A} \geq \lambda_{\min}(A)^{-1}\|E\|_2$
- $\|p\|_A \approx \sqrt{\frac{1}{n}\mathrm{Tr}(A)}\|p\|_2$
  (ok for p with random independent components)
- $\|b\|_{A^{-1}} = \sqrt{2|q(x_*)|} \approx \sqrt{2|q_k|} \approx \sqrt{|b^T x_k|}$
- $\|H_k^{-1}\| = \frac{1}{\lambda_{\min}(H_k)} \leq \frac{1}{\lambda_{\min}(A)}$ (FOM only)
- $k_{\max} \approx \frac{\log(\epsilon)}{\log(\rho)}$ with $\rho \overset{\text{def}}{=} \frac{\sqrt{\lambda_{\max}/\lambda_{\min}}-1}{\sqrt{\lambda_{\max}/\lambda_{\min}}+1}$

Termination test (Arioli & Gratton):

$$q_{k-d} - q_k \leq \tfrac{1}{4}\epsilon|q_k|$$

for some stabilization delay d (e.g. 10)

Does it still work (continuous accuracy levels)?



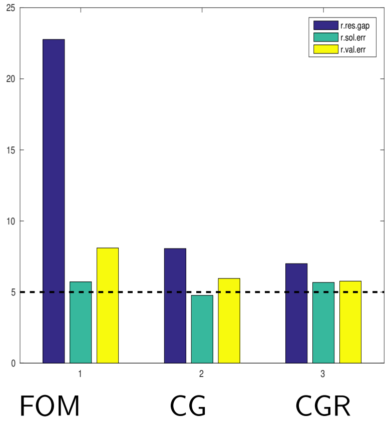
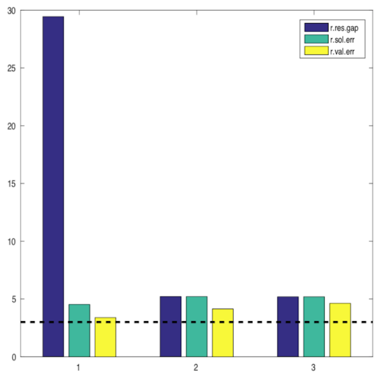
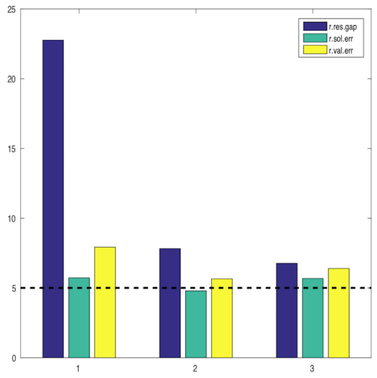


Figure: Exact bounds, $\kappa(A) = 10^1$, $\epsilon = 10^{-3}$ (left), $\kappa(A) = 10^5$, $\epsilon = 10^{-5}$ (right); continuous case

Does it still work (multiprecision)?





FOM CG CGR FOM CG CGR

Figure: Approximate bounds, $\kappa(A) = 10^1$, $\epsilon = 10^{-3}$ (left), $\kappa(A) = 10^5$, $\epsilon = 10^{-5}$ (right); multiprecision

Outline for section 3

Consider

$$\boxed{\min_{x\in\mathbf{R}^n} f(x).}$$

The dynamic accuracy setting of trust-region methods [CGT 2000], it is assumed that

- The value of the objective can be approximated with a prespecified level of accuracy ω_f :

$$\boxed{|\overline{f}(x,\omega_f) - \overline{f}(x,0)| \le \omega_f \quad \text{and} \quad \overline{f}(x,0) = f(x)}$$

- Following [Carter 1993; G., L.N Vicente and Z. Zhang 2018], the case where the gradient is inexact can be handled:

$$\boxed{\|\overline{g}(x,\omega_g) - \overline{g}(x,0)\| \le \omega_g \|\overline{g}(x,\omega_g)\| \;\text{ and }\; \overline{g}(x,0) = \nabla_x^1 f(x)}$$

We recall that the convergence at step k

$$\boxed{\|\nabla_x^1 f(x_k)\| \le \|\overline{g}(x_k,\omega_{g,k})\| + \|\overline{g}(x_k,\omega_{g,k}) - \overline{g}(x,0)\| \le \epsilon.}$$

is gained provided, for some constant κ_g, $\omega_{g,k} \le \kappa_g$ and $\|\overline{g}(x_k,\omega_{g,k})\| \le \frac{\epsilon}{1+\kappa_g}$.

TR with dynamic accuracy on f and g (algo TR**1**DA) (Step computation)

Step 1 Check for termination. If $k = 0$ or $x_k \neq x_{k-1}$, choose $\omega_{g,k} \in (0, \kappa_g]$ and compute $\overline{g}_k = \overline{g}(x_k, \omega_{g,k})$ such that $\|\overline{g}(x_k, \omega_{g,k}) - \overline{g}(x_k, 0)\| \leq \omega_{g,k}\|\overline{g}(x, \omega_{g,k})\|$. Terminate if $\|\overline{g}(x_k, \omega_{g,k})\| \leq \frac{\epsilon}{1+\kappa_g}$.

Step 2 Step calculation. Sufficiently reduce the model $m(x_k, s) = f_k + \overline{g}_k^T s + \frac{1}{2} s^T H_k s$ in the Trust-Region $\{s_k, \|s_k\| \leq \Delta_k\}$ in the sense that

$$m(x_k, 0) - m(x_k, s_k) \geq \tfrac{1}{2}\|\overline{g}_k\| \min\left[\frac{\|\overline{g}_k\|}{\|H_k\|}, \Delta_k\right]$$

Step 3 Evaluate the objective function. Select $\omega_{f,k}^+ \in \Big(0, \eta_0[m(x_k, 0) - m(x_k, s_k)]\Big]$ and compute $f_k^+ = \overline{f}(x_k + s_k, \omega_{f,k}^+)$. If $\omega_{f,k}^+ < \omega_{f,k}$, recompute $f_k = \overline{f}(x_k, \omega_{f,k}^+)$.

TR with dynamic accuracy on f and g (TR**1**DA) (Step acceptance)

Step 4 Acceptance of the trial point. Define the ratio

$$\rho_k = \frac{f_k - f_k^+}{m(x_k, 0) - m(x_k, s_k)}.$$

If $\rho_k \geq \eta_1$, then define $x_{k+1} = x_k + s_k$ and set $\omega_{f,k+1} = \omega_{f,k}^+$.
Otherwise set $x_{k+1} = x_k$, $\omega_{f,k+1} = \omega_{f,k}$ and $\omega_{g,k+1} = \omega_{g,k}$.

Step 5 Standard trust-radius update.

Set

$$\Delta_{k+1} \in \begin{cases} [\Delta_k, \infty) & \text{if } \rho_k \geq \eta_2, \quad \nearrow \\ [\gamma_2 \Delta_k, \Delta_k) & \text{if } \rho_k \in [\eta_1, \eta_2), \quad \searrow \\ [\gamma_1 \Delta_k, \gamma_2 \Delta_k] & \text{if } \rho_k < \eta_1. \quad \downarrow \end{cases}$$

Increment k by 1 and go to Step 2.

Assumptions

AS.1: The objective function f is twice continuously differentiable in $\mathbb{R}^n$ and there exist a constant $\kappa_\nabla \geq 0$ such that $\|\nabla_x^2 f(x)\| \leq \kappa_\nabla$ for all $x \in \mathbb{R}^n$.

AS.2: There exists a constant $\kappa_H \geq 0$ such that $\|H_k\| \leq \kappa_H$ for all $k \geq 0$.

AS.3 There exists a constant $\kappa_{\rm low}$ such that $f(x) \geq \kappa_{\rm low}$ for all $x \in \mathbb{R}^n$.

We can bound the accuracy on the model w.r.t the exact function:

Suppose AS.1 and AS.2 hold. Then, for each $k \geq 0$,

$$|f(x_k + s_k) - m(x_k, s_k)| \leq |f_k - f(x_k)| + \kappa_g \|\overline{g}(x_k, \omega_{g,k})\| \Delta_k + \kappa_{H\nabla} \Delta_k^2$$

for $\kappa_{H\nabla} = 1 + \max[\kappa_H, \kappa_\nabla]$.

The observed ρ can be interpreted as a true function versus model reduction

We have that, for all $k \geq 0$,

$$\max\left[|f_k - f(x_k)|, |f_k^+ - f(x_k+s_k)|\right] \leq \eta_0 \left[m(x_k,0) - m(x_k,s_k)\right]$$

and

$$\rho_k \geq \eta_1 \quad \text{implies that} \quad \frac{f(x_k) - f(x_k+s_k)}{m(x_k,0) - m(x_k,s_k)} \geq \eta_1 - 2\eta_0 > 0.$$

Proof. This follows from the accuracy management and from

$$\rho_k = \frac{f_k - f_k^+}{m(x_k,0) - m(x_k,s_k)} = \frac{f(x_k) - f(x_k+s_k)}{m(x_k,0) - m(x_k,s_k)} + \frac{[f_k - f(x_k)] + \left[|f_k^+ - f(x_k+s_k)\right]}{m(x_k,0) - m(x_k,s_k)}$$

□

Suppose AS.1 and AS.2 hold, and that $\overline{g}(x_k, \omega_{g,k}) \neq 0$. Then

$$\Delta_k \leq \frac{\|\overline{g}(x_k,\omega_{g,k})\|}{2\kappa_{H\nabla}}\left[\tfrac{1}{2}(1-\eta_1)-\eta_0-\kappa_g\right] \text{ implies that } \Delta_{k+1} \geq \Delta_k.$$

Proof.

$$\begin{aligned}
|\rho_k - 1| &\leq \frac{|f_k^+ - f(x_k+s_k)| + |f(x_k+s_k) - m(x_k,s_k)|}{m(x_k,0) - m(x_k,s_k)} \\
&\leq 2\eta_0 + \frac{\kappa_g\|\overline{g}(x_k,\omega_{g,k})\|\Delta_k + \kappa_{H\nabla}\Delta_k^2}{\frac{1}{2}\|\overline{g}(x_k,\omega_{g,k})\|\,\Delta_k} \\
&\leq 2\eta_0 + 2\kappa_g + 2\kappa_{H\nabla}\frac{\Delta_k}{\|\overline{g}(x_k,\omega_{g,k})\|} \\
&\leq 1 - \eta_2
\end{aligned}$$

where we used $\eta_0 + \kappa_g < \frac{1}{2}(1-\eta_2)$. □

Suppose $\Delta_0 \geq \theta\epsilon$. The TR1DA algorithm produces an iterate x_k such that $\|\nabla_x^1 f(x_k)\| \leq \epsilon$ in at most $\tau_S \overset{\text{def}}{=} \frac{2(f(x_0)-\kappa_{\text{low}})(1+\kappa_g)}{(\eta_1-2\eta_0)\theta} \cdot \frac{1}{\epsilon^2}$ successful iterations, and at most

$$\tau_{\text{tot}} \overset{\text{def}}{=} \tau_S \left(1 - \frac{\log \gamma_3}{\log \gamma_2}\right) + \frac{1}{|\log \gamma_2|} \log\left(\frac{\Delta_0}{\theta\epsilon}\right) \tag{3.3}$$

iterations in total.

Proof.

$$\begin{aligned}
f(x_0) - \kappa_{\text{low}} &\geq \sum_{j\in\mathcal{S}_k} [f(x_j) - f(x_{j+1})] \\
&\geq \tfrac{1}{2}(\eta_1 - 2\eta_0) \sum_{j\in\mathcal{S}_k} \|\overline{g}(x_j,\omega_{g,j})\| \min\left[\frac{\|\overline{g}(x_j,\omega_{g,j})\|}{1+\|H_j\|}, \Delta_j\right] \\
&\geq \tfrac{1}{2}|\mathcal{S}_k|(\eta_1 - 2\eta_0)\frac{\epsilon}{1+\kappa_g} \min\left[\frac{\epsilon}{\kappa_{H\nabla}(1+\kappa_g)}, \min\left[\Delta_0, \theta\epsilon\right.\right. \\
&= |\mathcal{S}_k| \frac{(\eta_1 - 2\eta_0)}{2(1+\kappa_g)} \min\left[\frac{1}{\kappa_{H\nabla}(1+\kappa_g)}, \theta\right] \epsilon^2
\end{aligned}$$

Practical setting

In our numerical experiments with TR1DA

- We perfom 20 runs on 86 Cuter problems
- We assume that the objective function's value $\overline{f}(x_k, \omega_k)$ and the gradient $\overline{g}(x_k, \omega_k)$ can be computed with corresponding accuracy level equal to machine precision, half machine precision or quarter machine precision
- The computational cost of an operation is devided by 4 when passing from one level to the immediate next one: half precision corresponds to double-precision costs divided by 16
- Hessian approximation are obtained with a limited-memory symmetric rank-one (SR1) quasi-Newton update

Practical setting

To set the stage, our first experiment starts by comparing three variants of the TR1DA algorithm:

- LMQN: a version using $\omega_f = \omega_g = 0$ for all k (i.e. using the full double precision arithmetic throughout),
- LMQN-s: a version using single precision evaluation of the objective function and gradient for all k,
- LMQN-h: a version using half precision evaluation of the objective function and gradient for all k.

Simple minded approach: expensive parts of the optimization calculation conducted in reduced precision no further adaptive accuracy management.

Simple approach

						relative to LMQN		
ϵ	Variant	nsucc	its.	costf	costg	its.	costf	costg
1e-03	LMQN	82	41.05	42.04	42.04			
	LMQN-s	78	41.40	42.60	42.60	1.03	1.04	1.04
	LMQN-h	22	16.95	1.12	1.12	0.97	0.06	0.06
1e-05	LMQN	80	46.34	47.38	47.38			
	LMQN-s	48	47.79	48.96	48.96	1.08	1.08	1.08
	LMQN-h	10	17.80	1.18	1.18	1.38	0.08	0.08
1e-07	LMQN	67	62.76	63.85	63.85			
	LMQN-s	25	28.28	28.96	28.96	0.82	0.81	0.81
	LMQN-h	6	15.83	1.05	1.05	0.97	0.06	0.06

Table: Results for LMQN-s and LMQN-h compared to LMQN

- Quickly decreasing robustness when a tight accuracy is demanded
- In most cases, $\boxed{\text{no improvement}}$, in costf and costg
- When LMQN-h happens to succeed its cost is very low

Two variant of TR1DA

- LMQN: as above,
- iLMQN-a: a variant of the TR1DA algorithm where

$$\omega_{f,k} = \min[\tfrac{1}{10}, \tfrac{4}{100}\eta_1\left(m_k(0) - m_k(s_k)\right)] \quad \text{and} \quad \omega_{g,k} = \tfrac{1}{2}\kappa_g.$$

- iLMQN-b: a variant of the TR1DA algorithm where,

$$\omega_{f,k} = \min[\tfrac{1}{10}, \tfrac{4}{100}\eta_1\left(m_k(0) - m_k(s_k)\right)] \quad \text{and} \quad \omega_{g,k} = \min[\kappa_g, \omega_{f,k}].$$

Variable precision approach

						relative to LMQN		
ϵ	Variant	nsucc	its.	costf	costg	its.	costf	costg
1e-03	LMQN	82	41.05	42.04	42.04			
	iLMQN-a	80	50.05	9.88	6.11	1.23	0.24	0.15
	iLMQN-b	76	52.67	13.85	3.34	1.36	0.35	0.08
1e-05	LMQN	80	46.34	47.38	47.38			
	iLMQN-a	75	75.92	36.21	24.77	1.40	0.63	0.42
	iLMQN-b	63	72.57	39.85	4.60	1.78	0.95	0.11
1e-07	LMQN	67	62.76	63.85	63.85			
	iLMQN-a	47	65.83	58.97	37.50	1.18	1.03	0.65
	iLMQN-b	40	87.35	95.09	5.52	1.39	1.45	0.09

Table: Results for the variable-precision variants

Summary of the experiments

- For $\epsilon = 10^{-3}$ or 10^{-5}, inexact variants iLMQN-a and iLMQN-b perform well in cost for gradient and function
- iLMQN-a appears to dominate iLMQN-b in the evaluation of the objective function
- iLMQN-b shows significantly larger savings in the gradient evaluation costs
- When the final $\boxed{\text{accuracy is tigther}}$ inexact methods appear to loose their edge in robustness. Gains in function evaluation costs disappear
- Comparison of iLMQN-a and even iLMQN-b with LMQN-s and LMQN-h clearly $\boxed{\text{favours the new methods}}$

Outline for section 4

Conclusions and perspectives

Summary:

- Optimization-focused theory for with inexact functon/gradient evaluation
- Theoretical gains substantial
- Translates well to practice after approximations

Perspectives:

- More general (controlable) inexactness in constrained optimization
- Probabilistic error specification

Thank your for your attention!

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