Random concave functions on an equilateral lattice with periodic hessians

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Connection to Machine learning

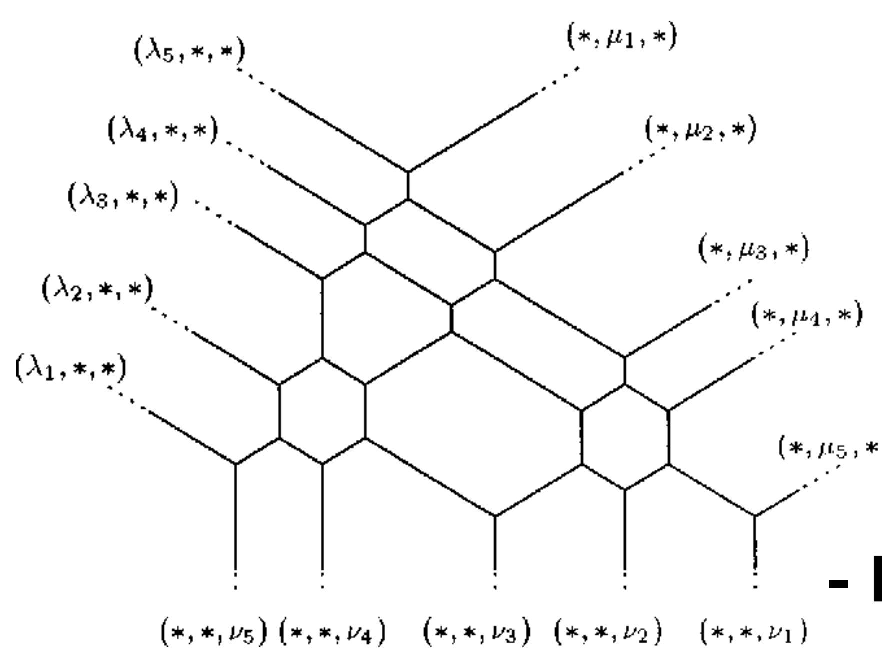
- The following questions are relevant to Machine learning.
- Let $\mathscr{F}$ be a class of real valued functions defined on a metric space. Given a prior distribution on $\mathscr{F}$, what can we say about typical functions from this distribution? Do they concentrate about their mean?

Random concave functions

- The prior distribution depends on the specific application.
- We focus on one particular example, namely, limits of uniform distributions applied to the set of concave functions defined on successively finer equilateral lattices in two dimensions, having periodic hessians of a certain average value.

Connection to statistical physics

- Height functions of random tilings.
- The gradients of a random concave function measured at the centers of the unit triangles form the vertices of a random hexagonal tiling of the plane.



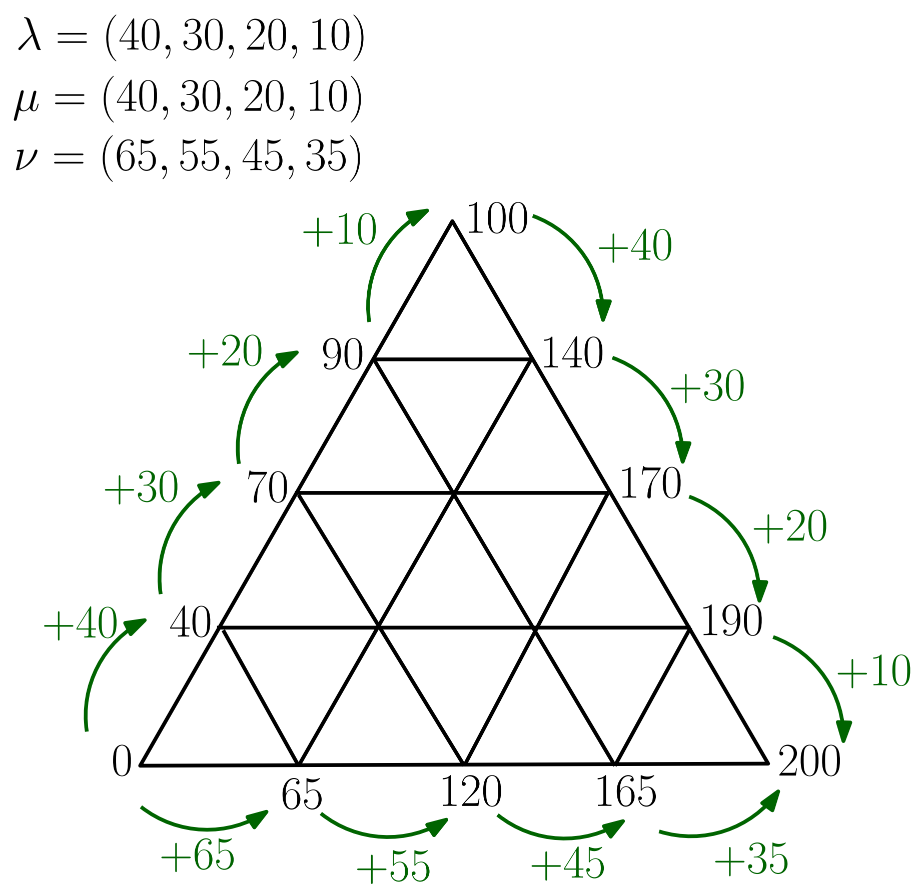
- Knutson-Tao 1999

Connection to representation theory

- The number of integer valued concave functions (also called hives) on a equilateral lattice on an equilateral triangle of side n with fixed boundary conditions count Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$, which are Clebsch-Gordon coefficients corresponding to the group $GL_n(\mathbb{C})$ (Knutson-Tao 1999).

$$V_\lambda \otimes V_\mu = \oplus_\nu \, c^{\nu}_{\lambda\mu} V_\nu$$

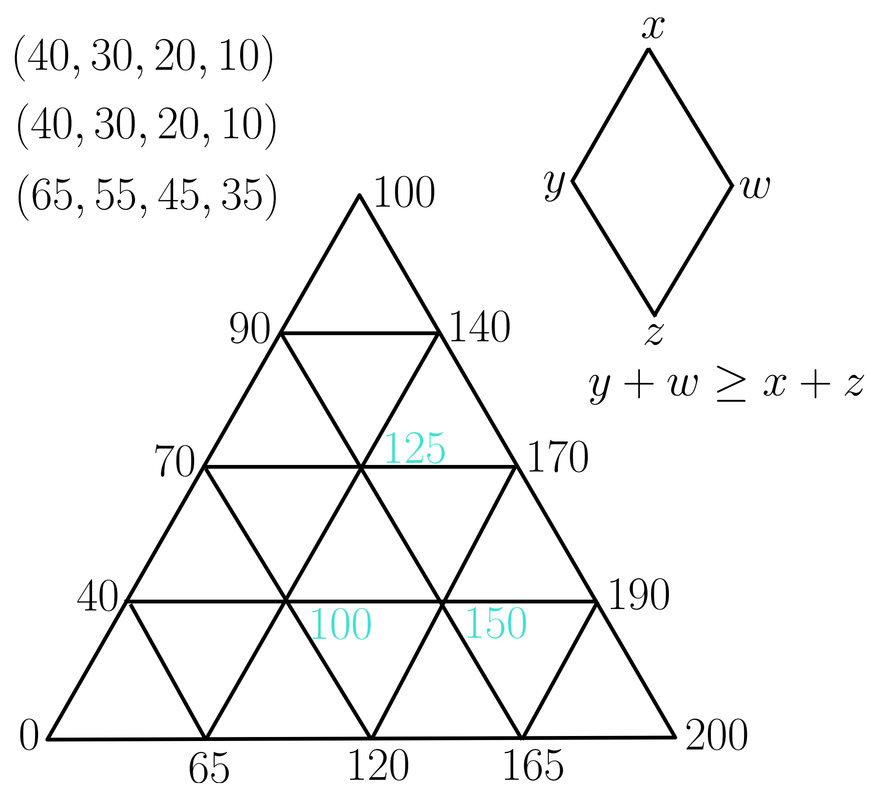
Connection to representation theory



$$V_\lambda \otimes V_\mu = \oplus_\nu \, c^\nu_{\lambda\mu} V_\nu$$

Connection to representation theory

$$\lambda = (40, 30, 20, 10)$$
$$\mu = (40, 30, 20, 10)$$
$$\nu = (65, 55, 45, 35)$$



$$V_\lambda \otimes V_\mu = \oplus_\nu \, c^\nu_{\lambda\mu} V_\nu$$

Connection to random matrices

- The volume of the polytope $P^{\nu}_{\lambda\mu}$ of all real hives is equal, up to known multiplicative factors involving Vandermonde determinants, to the probability density of obtaining a Hermitian matrix with spectrum ν when two Haar random Hermitian matrices with spectra λ, μ are added (Knutson-Tao 2003).

Scaling limits

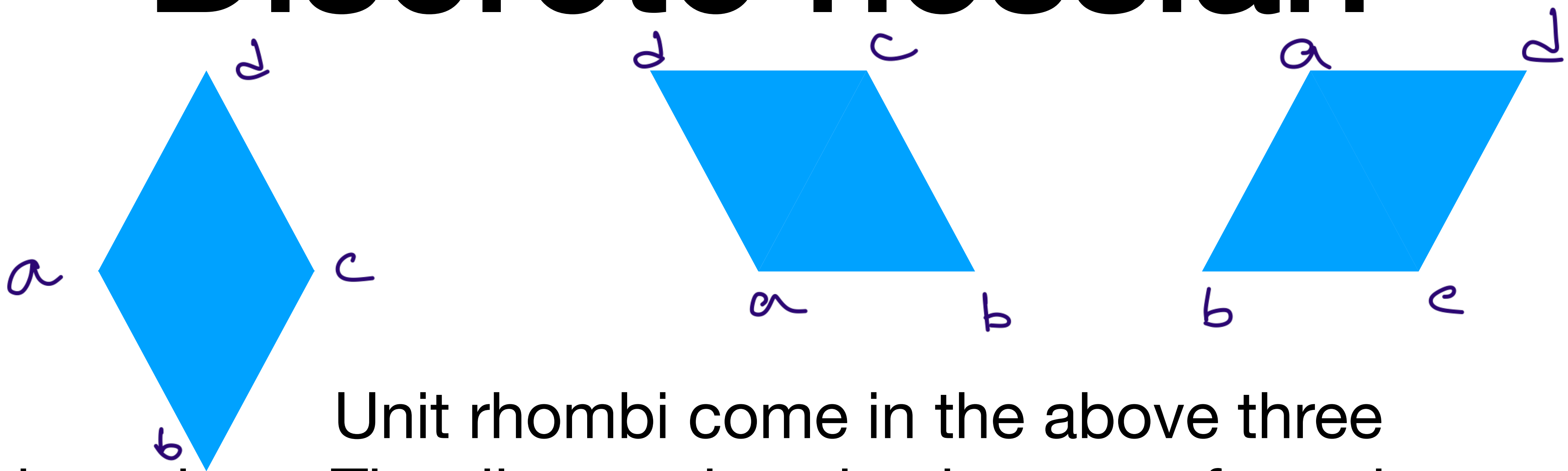
- We are interested in the shape of a random hive chosen uniformly from $P^{\nu}_{\lambda\mu}$, when the mesh of the equilateral lattice tends to zero, where λ, μ, ν are equally spaced samples from some one dimensional concave functions.

Scaling limits

- We are interested in the shape of a random hive chosen uniformly from $P_{\lambda\mu}^{\nu}$, when the mesh of the equilateral lattice tends to zero, where λ, μ, ν are equally spaced samples from some one dimensional concave functions.

- In order to prove the existence of a scaling limit, the strategy followed by Cohn, Kenyon and Propp 2001, Sheffield 2005, etc, is to first consider the case with periodic boundary conditions.

Discrete hessian



- Unit rhombi come in the above three orientations. The discrete hessian is a map from the set of unit rhombi to the reals.

- For a given unit rhombus, with the above labels (denoting values that the hive takes), the hessian is b+d-a-c in each case.

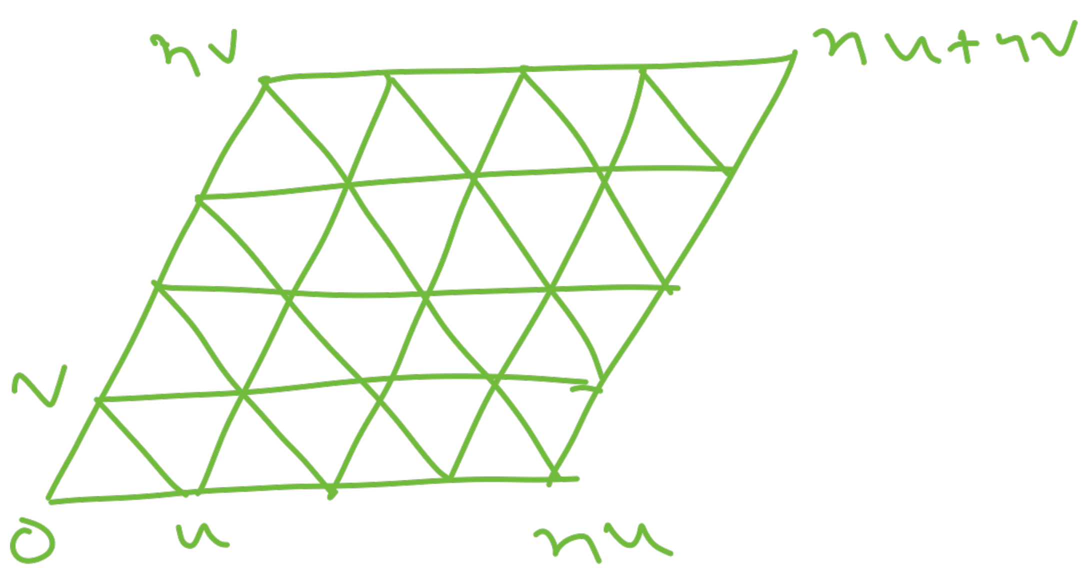
Average hessian

- Unit rhombi come in the above three orientations. The discrete hessian is a map from the set of unit rhombi to the reals.

- For a given unit rhombus, the average hessian is a triple of numbers representing the average of the hessians of the rhombi of each of the three orientations.

Scaling limits with periodic hessians

- Consider the polytope of infinite hives P_s on an equilateral lattice generated by unit vectors u and v in two dimensional euclidean space, whose discrete hessian has an average equal to s, is invariant under translations by nu and nv, modulo the addition of an affine function. This is a polytope of dimension $n^2 - 1$.



Scaling limits with periodic hessians

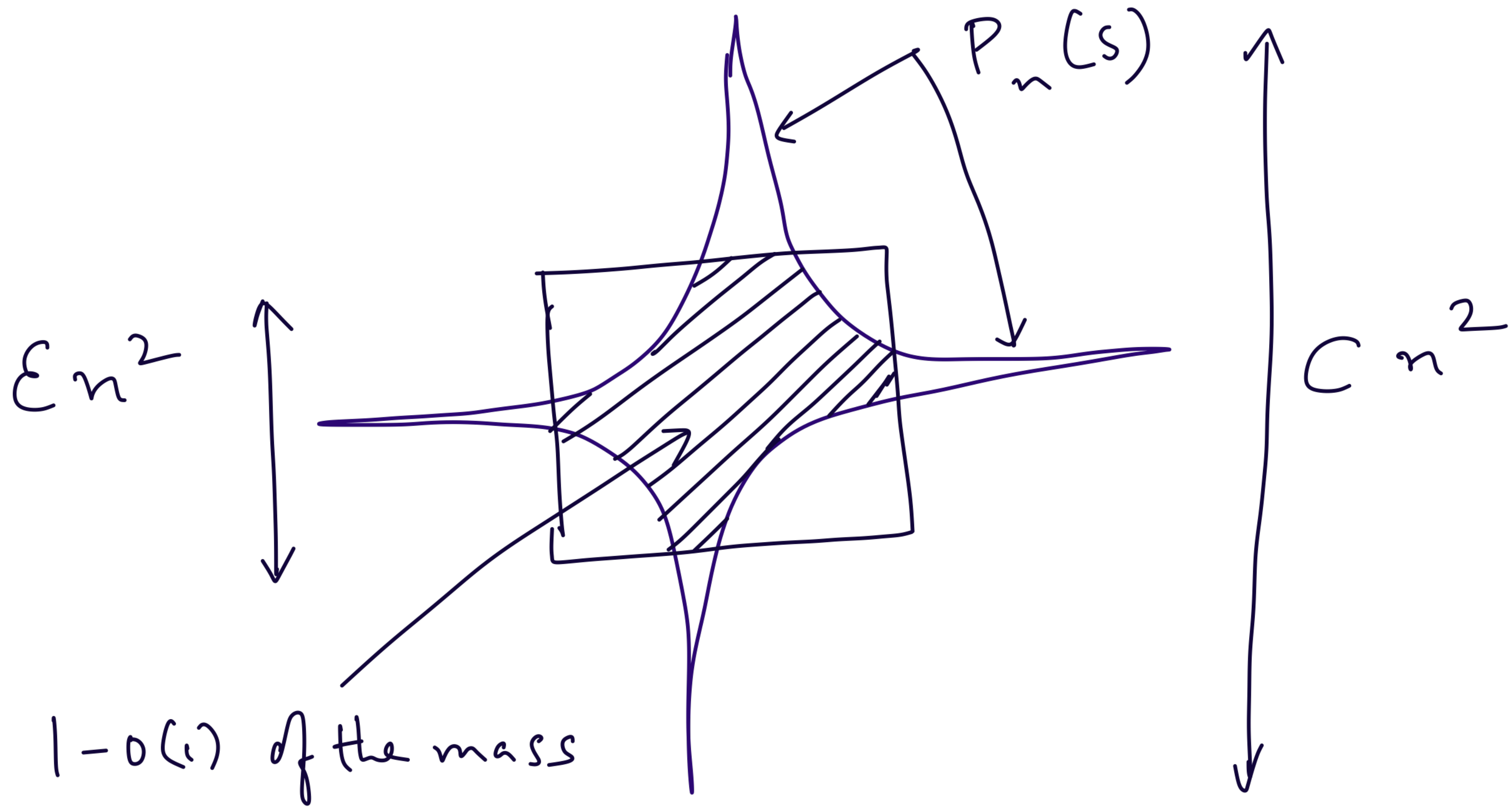
- The diameter of P_s in l_∞ is at least cn^2 for some positive constant c depending on s, as n tends to infinity. This can be seen as follows.

- Corresponding to hessian s, for every n, there is a discretisation of the unique quadratic hive whose hessian is everywhere s, but there are also hives obtained by discretising a polyhedral hive. The l_∞ norm of the difference between two such hives is at least cn^2.

Scaling limits with periodic hessians

- Our main theorem is the following: Let $s = (s_1, s_2, s_3)$ be the negative of the average hessian where each component is non-negative. Further suppose that $s_1 = s_2 \leq s_3$. For any positive ϵ a random hive sampled from the uniform distribution on $P_n(s)$ is within ϵn^2 in the l_∞ norm of the unique quadratic hive having average hessian $-s$ with probability tending to 1, as n tends to infinity.

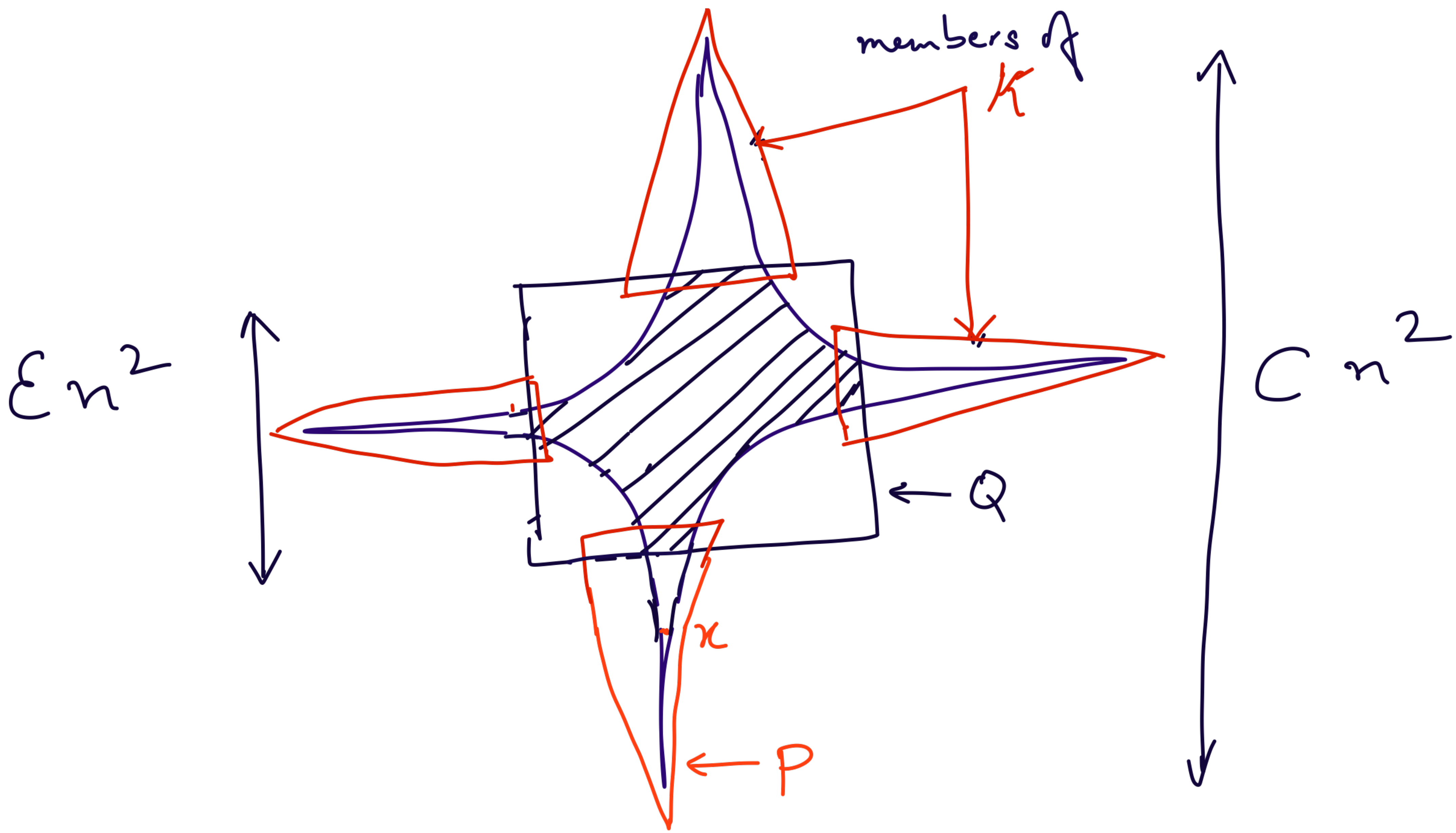
Main theorem



Proof strategy

- Define a family $\mathscr{K}$ of polytopes whose total volume is at most $o(|P_n(s)|)$ such that for every point $x \in P_n(s) \setminus Q$, where Q is the cube of side ϵn^2 centered at the quadratic hive q_s, there is a polytope $P^o \in \mathscr{K}$ such that $x \in P^o$.

Proof strategy



Preliminaries

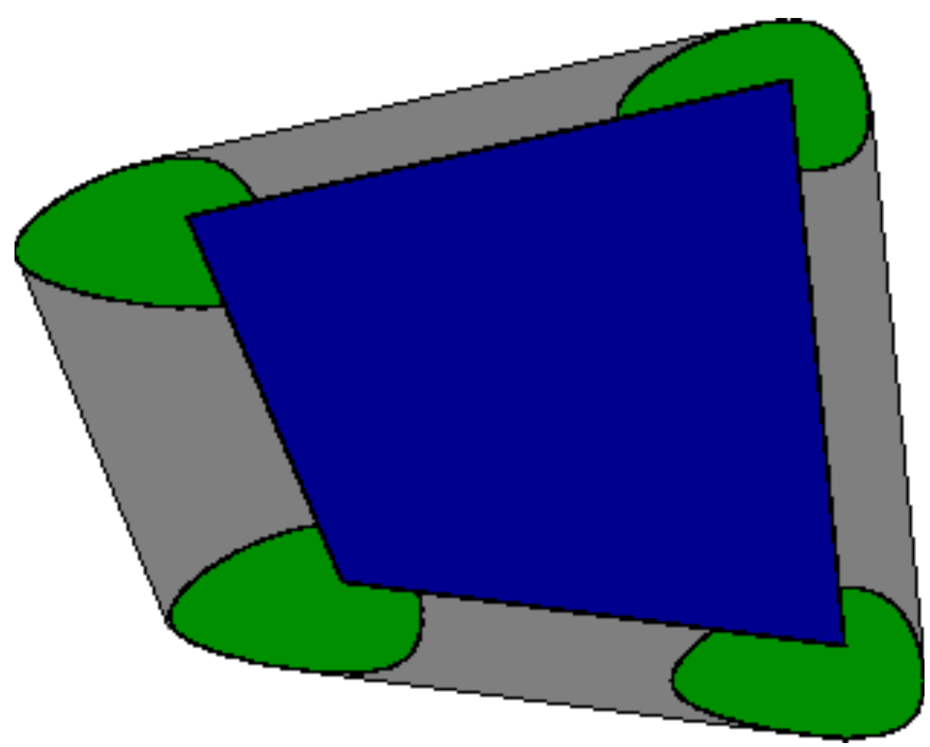
- We will outline the proof in the case when $s = (2,2,2)$. We translate $P_n(s)$ by subtracting, from each hive in it, the quadratic hive q_s.

- We are now dealing with zero mean functions on a discrete $n \times n$ torus whose hessian on each unit rhombus is bounded above by 2.

- We wish to show that the l_∞ norm of a random function in the translated polytope $P_n(s)$ is less than ϵn^2 with probability at least $1 - o(1)$.

Convex geometry

- Let K and L be compact convex subsets of $\mathbb{R}^m$, where $m \geq 1$. The Brunn-Minkowski inequality states that $|K+L|^{\frac{1}{m}} \geq |K|^{\frac{1}{m}} + |L|^{\frac{1}{m}}$.



Minkowski Sum

Convex geometry

- Define the anisotropic surface area $S_K(L)$ of L w.r.t K by $\lim_{\epsilon \to 0^+} \frac{|L + \epsilon K| - |L|}{\epsilon}$. Then the following anisotropic isoperimetric inequality follows from Brunn-Minkowski: $S_K(L) \geq m|K|^{\frac{1}{m}}|L|^{\frac{m-1}{m}}$. We rewrite the inequality as
- $\frac{|K|}{|L|} \leq \left(\frac{S_K(L)}{m|L|} \right)^m$. This is how we control the volumes of polytopes in $\mathscr{K}$.

Volume of $P_n(s)$

- We see that any mean zero function on the discrete torus taking values in $[-1/2, 1/2]$ belongs to $P_n(s)$, which therefore contains a central section of the unit cube. By a result of J. Vaaler, any central section of the unit cube has volume at least 1. Therefore $|P_n(s)| \geq 1$.

- A more involved argument involving differential entropy shows that $\limsup_{n \to \infty} |P_n(s)|^{\frac{1}{n^2-1}} \leq 2e$.

Volume of $P_n(s)$

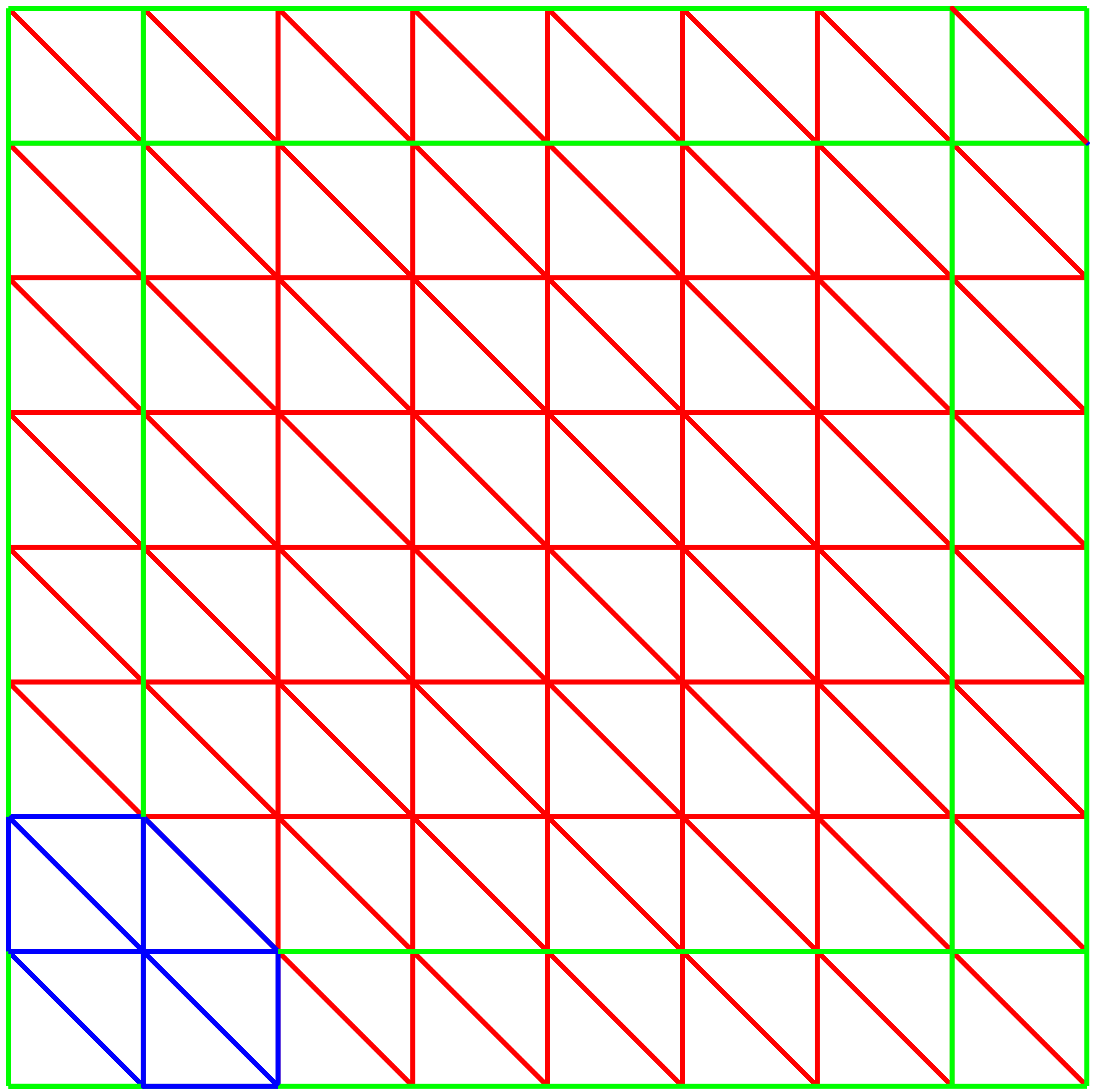
- We thus have $1 \leq \liminf_{n\to\infty} |P_n(s)|^{\frac{1}{n^2-1}} \leq \limsup_{n\to\infty} |P_n(s)|^{\frac{1}{n^2-1}} \leq 2e$.

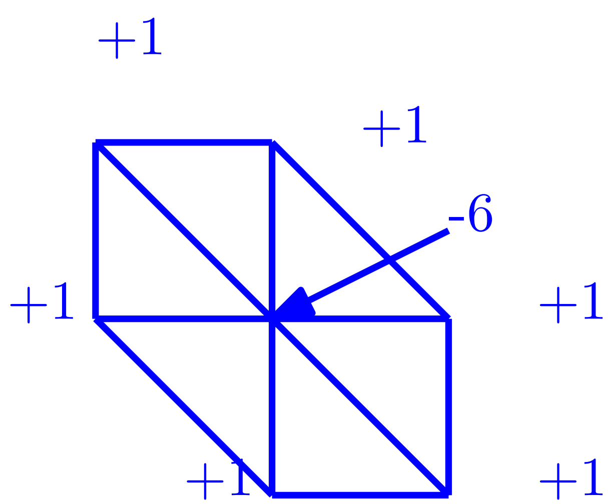
- We show that as a function of n, $P_n(s)^{\frac{1}{n^2-1}}$ is "approximately" monotonically increasing, and this tells us that $\lim_{n\to\infty} |P_n(s)|^{\frac{1}{n^2-1}} \in [1,2e]$.

- We wish to show that the l_∞ norm of a random function in the translated polytope $P_n(s)$ is less than ϵn^2 with probability at least $1 - o(1)$.

- Fix point x in $P_n(s)$. Let Δ be the laplacian on the discrete torus. Let $\square$ be a rhombus of side length $n_2 = \epsilon_1 n$. Let Φ be n_1^{-2} times the indicator of $\square$. Let $\star$ denote the convolution operator on the discrete torus.

- Lemma: If $\|x\|_\infty \geq \epsilon n^2$, then $\|\Phi \star \Delta \star x\|_2 \geq \Omega(\epsilon^2 n)$.





The side length of the above “rhombus” is n_1

- Fix point x in $P_n(s)$. Let Δ be the laplacian on the discrete torus. Let $\square$ be a rhombus of side length $n_2 = \epsilon_1 n$ on the torus. Let Φ be n_1^{-2} times the indicator of $\square$. Let $\star$ denote the convolution operator on the discrete torus.

- Lemma: If $\|x\|_\infty \geq \epsilon n^2$, then $\|\Phi \star \Delta \star x\|_2 \geq \Omega(\epsilon^2 n)$.

- Proof sketch: Due to the one sided bound on the hessian of x, $\|\Phi \star x\|_2$ is bounded below by $\Omega(\epsilon^2 n^3)$. The smallest eigenvalue of $-\Delta$ on the discrete torus is at least $\frac{1}{n^2}$. The lemma follows.

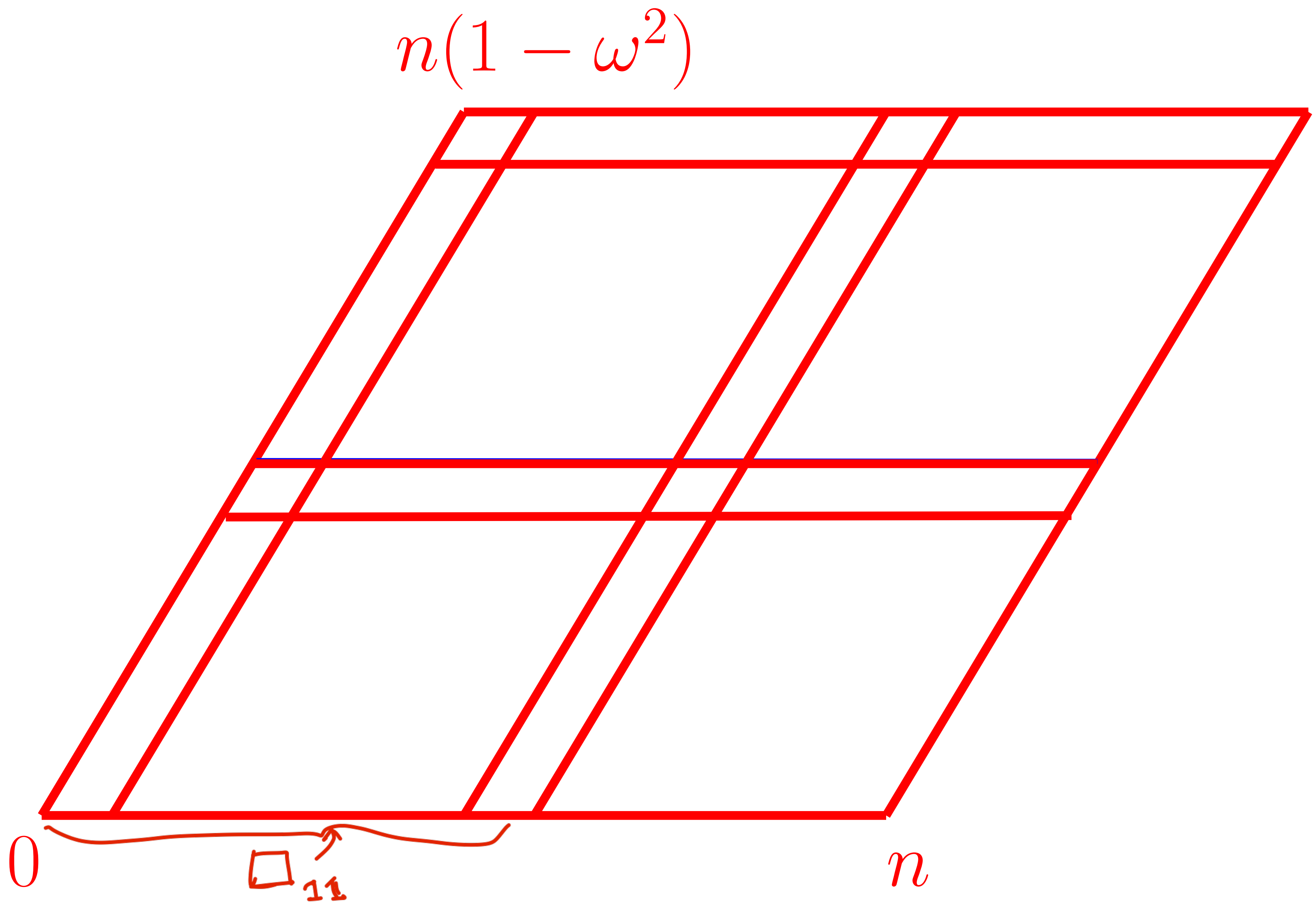
Convex geometry

- Let $L = P_{n_1}(s)$. Recall that $\frac{|K|}{|L|} \leq \left(\frac{S_K(L)}{m|L|}\right)^m$. Further, by the symmetries of the discrete torus and $s = (2,2,2)$, all the codimension 1 facets of $P_{n_1}(s)$ are congruent, and hence have the same $n_1^2 - 2$ dimensional volume.

Convex geometry

- Let $\partial \square$ denote a double layer boundary of $\square$. Conditional on the values that a hive takes on $\partial \square$, the values that x takes on the interior of $\square$, are independent of the values outside $\square$.

- Roughly speaking, we choose K to be the projection of $P_n(s)$ to $\mathbb{R}^{\square}$ intersected with the subspace $\{y \in \mathbb{R}^{\square}$ such that $y|_{\partial\square} = x|_{\partial\square}\}$. This set represents local (to the interior of $\square$) perturbations of x, that continue to lie in $P_n(s)$.



$n(1-\omega^2)$
0
n
$\square_{11}$

- We prove that if $\square_{ij}^{o}$ is a random translate of $\square$, and K_{ij}^{o} is the corresponding convex set, then

$$\mathbb{E}\left|\frac{S_{K_{ij}^{o}}(L)}{m|L|} - 1 - (\Phi \star \Delta \star x)(\text{center of } \square_{ij}^{o})\right| \leq \frac{1}{n_1}.$$

- Thus,

$$\left(\frac{|K_{ij}^{o}|}{|L|}\right)^{\frac{1}{m}} \leq \frac{S_{K_{ij}^{o}}(L)}{m|L|} \lesssim 1 + (\Phi \star \Delta \star x)(\text{center of } \square_{ij}^{o}).$$

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- $$\ln a \leq a - 1 - \frac{(a-1)^2}{2\max(a,1)}.$$

- Therefore, setting $y_{ij}^o =$ center of $\square_{ij}^o$, it turns out that
$$\ln\left(\frac{|K_{ij}^o|}{|L|}\right) \lesssim m(\Phi \star \Delta \star x)(y_{ij}^o) - \frac{(\Phi \star \Delta \star x)^2(y_{ij}^o)}{6/\epsilon_1^2}$$

- $$\ln\left(\frac{|K_{ij}^o|}{|L|}\right) \lesssim m(\Phi \star \Delta \star x)(y_{ij}^o) - \frac{m(\Phi \star \Delta \star x)^2(y_{ij}^o)}{6/\epsilon_1^2}$$

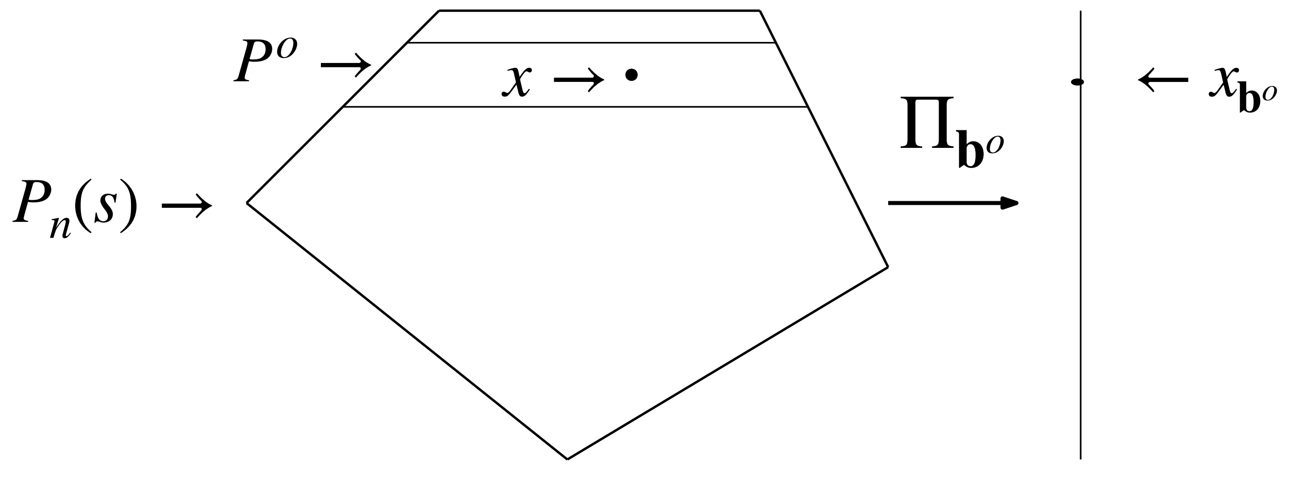
  Summing over o, i, j, the linear term effectively vanishes and so:

- $$\sum_{o,i,j} \ln\left(\frac{|K_{ij}^o|}{|L|}\right) \lesssim -\left(\frac{1}{6/\epsilon_1^2}\right)\sum_{o,i,j} m\|\Phi \star \Delta \star x\|_2^2$$

- $$\leq -\Omega(mn\epsilon^4)\,.$$

- For a given offset o, let the interiors of disjoint $\square_{ij}^{o}$ cover most of the torus leaving out a small subset of the lattice points, which we denote $\mathbf{b}^o$. Let $\Pi_{\mathbf{b}^o}$ denote the projection that restricts $x' \in \text{span } P_n(s)$ to its coordinates $x'_{\mathbf{b}^o}$ corresponding to $\mathbf{b}^o$. Then, define

$$P^o := \left(\Pi_{\mathbf{b}^o}^{-1}(x_{\mathbf{b}^o} + [-n^{-6}, n^{-6}]^{\mathbf{b}^o})\right) \cap P_n(s)$$



- We show that $|P^o| \lesssim \prod_{i,j} |K_{ij}^o|$. Recall that
- $$\sum_{o,i,j} \ln\left(\frac{|K_{ij}^o|}{|L|}\right) \lesssim -\Omega(mn\epsilon^4).$$
- This leads to the existence of an offset o, such that $|P^o| \lesssim \exp(-\Omega(m\epsilon^4))|L|^{\frac{n^2}{n_1^2}}$.

- This leads to the existence of an offset o, such that $|P^o| \lesssim \exp(-\Omega(m\epsilon^4))|P_{n_1}(s)|^{\frac{n^2}{n_1^2}}$.

- But the total number of polytopes of the form P^o in $\mathscr{K}$ for some $x \in P_n(s)$ is $\lesssim n^{\Theta(n/\epsilon_1)}$. Since $m = n_1^2 - 1 \approx \left(\frac{n}{\epsilon_1}\right)^2$, we see that the total volume of all polytopes in $\mathscr{K}$ is $o(|P_n(s)|)$, which is what we sought to prove.

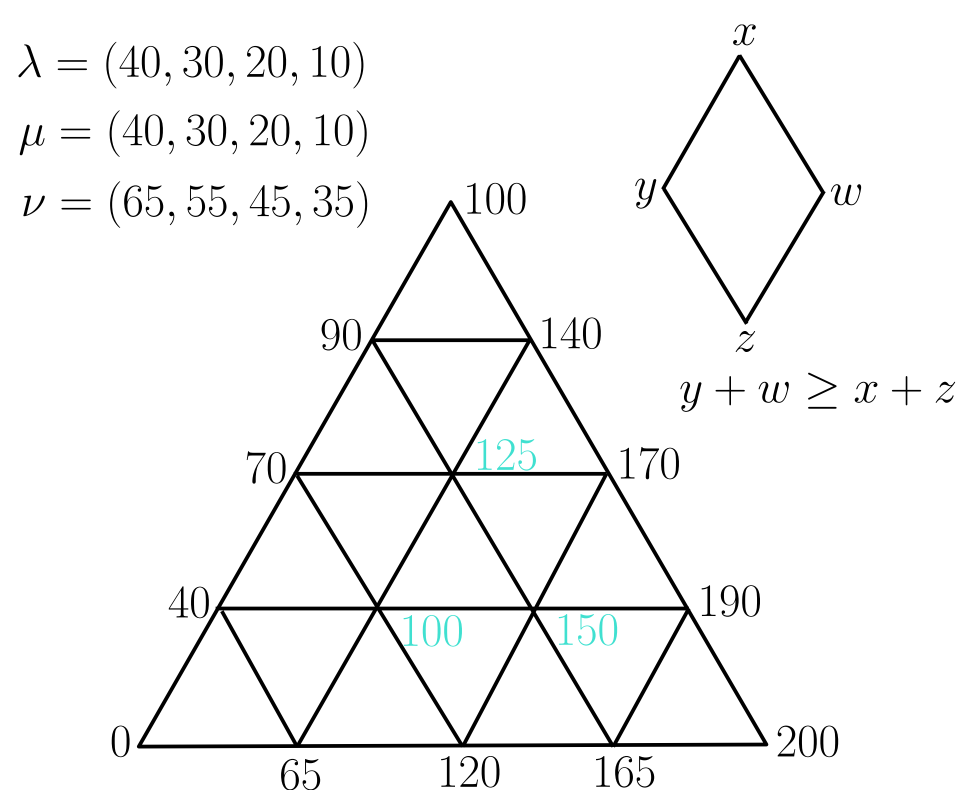
Overall intuition

- Why do we expect a random concave function with a periodic hessian having average s to be close to quadratic?
- Suppose a function f has a hessian that deviates in a systematic way from s over large regions. Then, the resulting large scale variation in the hessian, together with something like the strict concavity of entropy, ought to make the measure of an ϵ neighbourhood of f small.

Questions for the future

- What about periodic boundary conditions with other average hessians?

- What is the shape of a random hive chosen uniformly from $P_{\lambda\mu}^{\nu}$, when the mesh of the equilateral lattice tends to zero, where λ, μ, ν are equally spaced samples from some one dimensional concave functions.



Thank you!