On the representation of measures

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Representation of measures

Consider a family of Borel probability measures $\{\mu_x\}_{x\in X}$ on a Riemannian manifold M, parametrised by a Riemannian manifold X. A representation of $\{\mu_x\}_{x\in X}$ is a mapping $T : (x, \omega) \in X \times \Omega \mapsto T(x, \omega) \in M$ such that, for each x,

$$\mu_x = T(x, \cdot)_* \mathbb{P},$$

where $(\Omega, \mathbb{P})$ is an auxiliary probability space.

Problem:

Which conditions assure a representation s.t. $T(\cdot, \omega)$ are C^k-regular uniformly in ω?

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Markov chain

Let $f : M \to M$ be C^r for $r \geq 0$ and a small perturbation parameter $\varepsilon > 0$.
The Markov chain: a family $\{p_\varepsilon(\cdot\,|x)\}$ of Borel probability measures.

- Every $p_\varepsilon(\cdot\,|x)$ is supported inside an ε-neighbourhood of $f(x)$.
- Random orbit: $\{x_j\}$ where each x_{j+1} has distribution $p_\varepsilon(\cdot\,|x_j)$.
- Jumps $x_j \mapsto f(x_j)$ and disperses with distribution $p_\varepsilon(\cdot\,|x_j)$.

Iteration of random maps

We consider $f : M \to M$ to be C^r for $r \geq 0$ and a small perturbation parameter $\varepsilon > 0$. The random iteration of maps is given by

- Assuming the existence of a family of probability distributions $\{\nu_\varepsilon\}$ on the space of C^r-maps.
- Support of ν_ε is in a ε-neighbourhood of $f(x)$.
- Random orbit: $x_j = f_{\omega_j} \circ \cdots \circ f_{\omega_1}(x_0)$, where f_{ω_j} are i.i.d. random variables with distribution ν_ε.
- The random orbits generated by the random maps indeed give rise to a discrete time Markov chain.

Stochastic stability

A system (f, μ) is stochastically stable under the perturbation scheme $\{p_\varepsilon(\cdot\,|x)\}$ or $\{\nu_\varepsilon : \varepsilon > 0\}$ if

$$\lim_{\varepsilon\to 0} \int \varphi d\mu_\varepsilon = \int \varphi d\mu \quad \text{for every continuous } \varphi : U \to \mathbb{R}.$$

- Several contributions proving stochastic stability of different systems: Sinai, Kifer, L.-S. Young, Keller, Araújo, Alves, Benedicks, Viana, etc.
- Arguments: assume existence of probability on the space of maps, $\nu_\varepsilon \xrightarrow[\varepsilon\to 0]{} \delta_f$, control of distortion, hyperbolic times, thermodynamics formalism, etc.
- Questions: dependence on the probability distributions of the Markov chains, relation with structural properties, shadowing, etc.

Representation of Markov chains

The sequence of random maps is a representation of the Markov chain if for any Borel U

$$p_\varepsilon(U|x) = \nu_\varepsilon(\{f_\omega : f_\omega(x) \in U\}).$$

- Some contributions: Blumenthal and Corso '70, Kifer '86, Quas '91, Araújo '00, Benedicks and Viana '06, ...
- We tackled the general case in terms of a transportation problem and the Moser trick.

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Main result

Theorem (Jost, Matveev, Portegies, R.)

We show that a family $\{\mu_x\}_{x\in X}$ is C^k-representable in case M is a compact, oriented, and connected Riemannian manifold of class C^{k+2} and X is a compact Riemannian manifold of class C^k, and the measures $\{\mu_x\}_{x\in X}$ have positive densities of class C^k on the interior of M (and have suitable decay towards the boundary of M).

Jost, Matveev, Portegies, R. - To appear in *Communications in Analysis and Geometry (2019)*.

Further results: Jost, Kell, R., *Calculus of Variations and PDE's* 2015

On optimal transport

- Basic problem (G. Monge, 1781): moving a distribution like a pile of sand from a place to another at minimum cost.
- Modern context: given μ and ν probability measures, find a measurable map T s.t.

$$\min_{S_*\mu=\nu}\left\{\int_X c(x,S(x))d\mu(x)\right\},$$

where $c : X \times Y \to [0,\infty]$ is some given *cost* function, and the minimum is taken over all measurable maps $S : X \to Y$, such that $S_*\mu = \nu$.

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The Monge-Ampère equation

Alternatively: weak solutions (Kantorovich): γ on $X \times Y$, with $\pi_{\mathcal{P}(X)*}\gamma = \mu$ and $\pi_{\mathcal{P}(Y)*}\gamma = \nu$,
Minimisation problem:

$$C(\mu, \nu) = \inf_{\gamma \in \mathcal{P}(X \times Y)} \int_{X \times Y} c(x, y) d\gamma(x, y),$$

$c : X \times Y \to [0, +\infty]$.

- Duality $\to$ existence of optimal transport maps of form: $\sim \nabla u$
- Jacobian equations: $S_* \mu = \nu \to$ Monge-Ampère equation.
- Regularity of solutions $u \Rightarrow$ regularity of transport maps.

Price: positivity of densities, and convexity on/of the support.

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Using transportation

1. The Moser trick: oriented, connected, compact manifolds.
    - Volume forms are isomorphic.
    - There exists a flow Φ_t that connects μ_0 and μ_t along a deformation.
    - Φ_1 provides the solution of Monge-Kantorovich problem.
2. Find regular family of transport maps.
    - $T(x, \cdot) : M \to M$ that pushs a reference measure μ to μ_x.
3. Show regular dependence of $T(x, \cdot)$ on x.
    - The map $T : X \times M \to M$ is the representation, for which $T(\cdot, \omega)$ is regular.

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Thanks for your attention!