Weak hyperbolicity for singular flows

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Introduction

In this work, we present some weak hyperbolic structures and one obtain characterizations (or some conditions to get) of these properties.

Introduction

From now on,
M is a closed n-dimensional manifold, $n \geq 3$.

$\mathfrak{X}^1(M)$ is the space of C^1 vector fields X on M,

$X_t : M \to M, t \in \mathbb{R}$ is the C^1 flow generated by $X \in \mathfrak{X}^1(M)$.

$\Lambda \subseteq M$ is a compact invariant set for X_t such that all singularities are hyperbolic, if any.

Uniform Hyperbolicity

Definition

The set Λ is said to be **hyperbolic** for a flow $X_t : M \to M$ if there are a continuous invariant splitting $T_xM = E^s_x \oplus E^X \oplus E^u_x$ and constants $C, \lambda > 0$ such that, $\forall x \in \Lambda$,

$$\|DX_t|_{E^s_x}\| \leq ce^{-\lambda t}, \ \|DX_{-t}|_{E^u_{(X_t(x))}}\| \leq ce^{-\lambda t}.$$

Dominated splittings

A **dominated splitting** over Λ is a continuous DX_t-invariant splitting $T_\Lambda M = E \oplus F$ with $E_x \neq \{0\}$, $F_x \neq \{0\}$ for every $x \in \Lambda$ and such that there are positive constants K, λ satisfying

$$\|DX_t|_{E_x}\| \cdot \|DX_{-t}|_{F_{X_t(x)}}\| < Ke^{-\lambda t},$$
$$\text{for all } x \in \Lambda, \text{ and all } t > 0.$$

Partial hyperbolicity

Definition

The set Λ is said to be **partially hyperbolic** for X if there is a dominated splitting $T_xM = E_x \oplus F_x$ such that E_x is uniformly contracted (or F_x is uniformly expanded) by DX_t.

Singular hyperbolicity

Given E a vector space, we denote by $\wedge^p E$ the exterior power of order p of E, defined as follows. If $v_1, \dots, v_n$ is a basis of E then $\wedge^p E$ is generated by

$$\left\{v_{i_1} \wedge \dots \wedge v_{i_p}\right\}_{1 \leq i \leq n, i_j \neq i_k, j \neq k}.$$

Singular hyperbolicity

Any linear transformation $A : E \to F$ induces a transformation $\wedge^p A : \wedge^p E \to \wedge^p F$.
We can see a p-plane as $\widetilde{v} \in \wedge^p(E_x) \setminus \{0\}$ of norm one.

Singular hyperbolicity

Hence, p-sectional expansion is given by the inequality

$$\| \wedge^p DX_t(x).\widetilde{v}\| > Ce^{\lambda t},$$

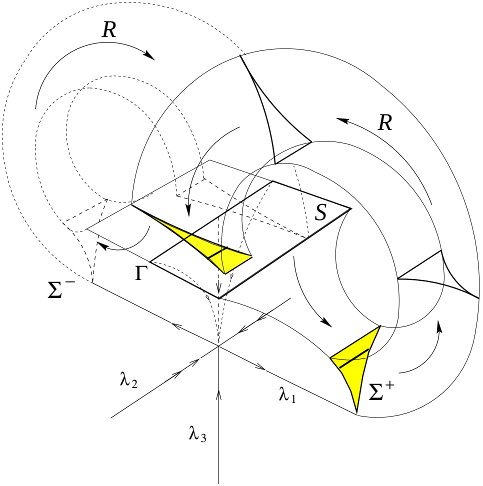
for some $\lambda > 0$ and every $t > 0$.

Singular/sectional hyperbolicity

The set Λ is said to be **p-sectionally hyperbolic** (or **singular hyperbolic of order p**) for a smooth flow X if:

1. there exists a partially hyperbolic splitting $T_\Lambda M = E \oplus F$;
2. E is uniformly contracting and the central subbundle F is p-sectionally expanding, with $2 \leq p \leq \dim(F)$.

Geometric Lorenz attractor



Turaev-Shil'nikov attractor

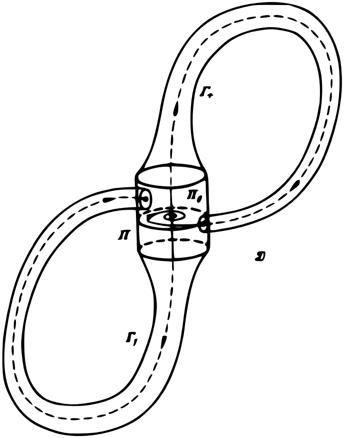


Figure: An attractor Γ of a n-dimensional vector field, $n \geq 4$, with 3-sectional expansion which is not (2-)sectionally expanding.

Characterization of domination property

Theorem (S.-2018)

A continuous DX_t-invariant splitting $T_\Lambda M = E \oplus F$ is dominated if, and only if, exists $\eta < 0$ for which

$$\liminf_{t\to+\infty} \frac{1}{t} \log |DX_t|_{E_x}| - \limsup_{t\to+\infty} \frac{1}{t} \log m(DX_t|_{F_x}) < \eta,$$

in a total probability set of Λ.

Characterization of partial hyperbolicity

Theorem (S.-2018)

The set Λ is a partially hyperbolic for X, if and only if, there exists a continuous invariant splitting of the tangent bundle, $T_\Lambda M = E \oplus F$, of Λ sucht that:

1. *the Lyapunov exponents on E are negative (or positive on F), and*
2. $\liminf_{t\to+\infty} \frac{1}{t} \log |DX_t|_{E_x}| - \limsup_{t\to+\infty} \frac{1}{t} \log m(DX_t|_{F_x}) < 0,$

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p-sectional Lyapunov exponents

Definition

The *p-sectional Lyapunov exponents* (or *Lyapunov exponents of order p*) of x along F are the limits

$$\lim_{t \to +\infty} \frac{1}{t} \log \| \wedge^p DX_t(x).\widetilde{v} \|$$

whenever they exists, where $\widetilde{v} \in \wedge^p F_x - \{0\}$.

Characterization of p-sectional hyperbolicity

Theorem (S.-2018)

A continuous invariant splitting $T_\Lambda M = E \oplus F$ is p-sectional hyperbolic for X if, and only if, on a set of total probability in Λ,

1. $\liminf_{t\to+\infty} \frac{1}{t} \log |DX_t|_{E_x}| - \limsup_{t\to+\infty} \frac{1}{t} \log m(DX_t|_{F_x}) < 0,$
2. *the Lyapunov exponents in the E direction are negative and*
3. *the p-sectional Lyapunov exponents in the F direction are positive .*

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Hyperbolicity via quadratic forms

Let f be a diffeomorphism on a closed manifold M and B a quadratic form on TM.
Consider the quadratic form
$f^\sharp B_x(v) = B_{f(x)}(Df(x)v), x \in M, v \in T_xM.$

Theorem (Lewowicz-1980): $f \in \mathrm{Diff}^r(M), r \geq 1$ is Anosov (hyperbolic over whole M) iff there exists a continuous non-degenerate quadratic form $B : TM \to \mathbb{R}$ such that $f^\sharp B - B > 0$.

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$\mathcal{J}$-separation and monotonicity

Let $\mathcal{J}$ be an indefinite non-degenerate quadratic form on TM.
Denote by $C_{\pm}(x) = \{v \in T_xM; \pm\mathcal{J}_x(v) > 0\} \cup \{0\}$ the open cones of positive and negative vectors, and $C_0(x) = \{v \in T_xM; \mathcal{J}_x(v) = 0\}$ their common boundary.

$\mathcal{J}$-separation and monotonicity

Given a continuous field of non-degenerate quadratic forms $\mathcal{J}$, we say that a cocycle $A_t(x)$ over X is

- *strictly $\mathcal{J}$-separated* if $A_t(x)(C_+(x) \cup C_0(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$;
- *strictly $\mathcal{J}$-monotone* if $\partial_t \big(\mathcal{J}_{X_t(x)}(A_t(x)v) \big) \mid_{t=0} > 0$, for all $v \in T_xM \setminus \{0\}$, $t > 0$ and $x \in U$;

Strict $\mathcal{J}$ separation

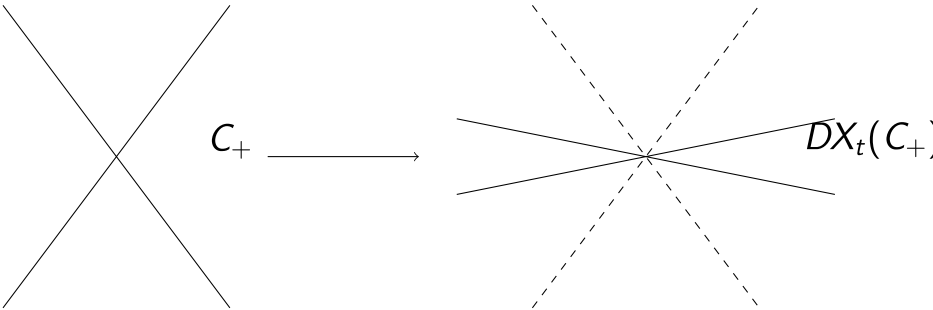

Figure: strict $\mathcal{J}$-separation.

Partially hyperbolic case

A field of quadratic forms $\mathcal{J}_0$ on U is equivalent with $\mathcal{J}$ if there is $C > 1$ such that

$$\frac{1}{C}\mathcal{J}_0 \leq \mathcal{J} \leq C\mathcal{J}_0. \tag{1}$$

Consider the operator $\tilde{J}_{0,x} := J_0 \cdot DX(x) + DX(x)^* \cdot J_0$, where $DX(x)^*$ is the adjoint of $DX(x)$ with respect to the adapted inner product.

The Partially hyperbolic case

Theorem (Araujo-S. 2013)

Let X be a $\mathfrak{J}$-non-negative vector field on U, with $\mathfrak{J}$ has constant index I_U. Then, Λ is partially hyperbolic of index I_U for X_t if, and only if, there are a equivalent $\mathfrak{J}_0$ and a function $\delta : U \to \mathbb{R}$ such that

$$\tilde{J}_{0,x} - \delta(x) J_0 \quad \text{is positive definite,} \quad x \in U.$$

Hyperbolicity via Lyapunov functions

Define

$$\Delta(x,t) := \int_0^t \delta(X_s(x))\, ds. \tag{2}$$

The area under the function δ allows us to detect different dominated splittings with respect to linear multiplicative cocycles on vector bundles.

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3-dimensional singular hyperbolicity

If $\wedge^2 DX_t$ is strictly separated with respect to some family $\mathcal{J}$ of quadratic forms, then there exists the function δ_2 with respect to the cocyle $\wedge^2 DX_t$.

We set

$$\widetilde{\Delta}_a^b(x) := \int_a^b \delta_2(X_s(x))\, ds$$

the area under the function $\delta_2 : U \to \mathbb{R}$ with respect to $\wedge^2 DX_t$ and its infinitesimal generator.

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3-dimensional singular hyperbolicity

Theorem (Araujo-S. 2015)

Suppose that X is three-dimensional vector field on M which is non-negative strictly $\mathcal{J}$-separated over a non-trivial subset Λ, where $\mathcal{J}$ has index 1. Then

1. $\wedge^2 DX_t$ *is strictly* $(-\mathcal{J})$*-separated;*
2. Λ *is a singular hyperbolic set if either one of the following properties is true*
   1. $\widetilde{\Delta}_0^t(x) \xrightarrow[t\to+\infty]{} -\infty$ *for all* $x \in \Lambda$.
   2. $\tilde{\mathcal{J}} - 2\,\mathrm{tr}(DX)\mathcal{J} > 0$ *on* Λ.

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The Lorenz conjecture and the 14th Smale's problem

In 1963, E. Lorenz introduced the following system of nonlinear differential equations:

$$\dot{x}_1 = -\sigma x_1 + \sigma x_2,$$
$$\dot{x}_2 = \varrho x_1 - x_2 + x_1 x_3,$$
$$\dot{x}_3 = -\beta x_3 + x_1 x_3.$$

The Lorenz conjecture and the 14th Smale's problem

Theorem (Tucker 2002)

For the classical parameter values $(\sigma = 10, \beta = \frac{8}{3}, \varrho = 28)$*, the Lorenz equations support a robust strange attractor* $\mathcal{A}$*. Furthermore, the flow admits a unique SRB measure* μ *with* $\operatorname{supp}(\mu) = \mathcal{A}$*.*

Indeed, he proves that the attracting set is a *Singular Hyperbolic Attractor*!!

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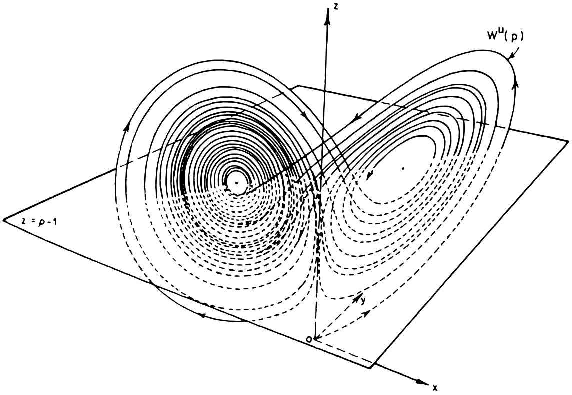

Figure: The Lorenz flow.

Acknowledgments

Thanks!