INVARIANT MEASURES FOR NON-AUTONOMOUS SYSTEMS AND ERGODIC INVERSE SHADOWING

S. Kryzhevich[a], S. Pilyugin
Saint-Petersburg State University, Russia

E-mail[a]: kryzhevicz@gmail.com

source: **arXiv:1907.06792**

Smooth and Homogeneous Dynamics – 2019,
International Centre for Theoretical Sciences,
Bengaluru, India, September, 23, 2019

Shadowing and Inverse Shadowing

Problem.

We work with discrete dynamical systems given by a C^1 diffeomorphisms $f : M \to M$ of a compact C^1 - smooth Riemannian manifold M with metrics dist. Orbits of the map:

$$O_f(x_0) = \{x_{k+1} = f(x_k) : k \in \mathbb{Z}\},$$

or, equivalently, $x_k = f^k(x_0)$. There is a numerical method, represented by a sequence of homeomorphisms $g_k : M \to M, k \in \mathbb{Z}$ (that is a non-autonomous difference equation). Orbits:

$$O_g(x_0) = \{y_{k+1} = g_k(y_k) : k \in \mathbb{Z}\}.$$

Question (classical form).

Are orbits of f and g pointwise close?

Shadowing and Inverse Shadowing in a nutshell

Shadowing.

Given an 'approximate solution' $\{y_k\}$ could we expect, there is a true solution $\{x_k\}$, ε – close to it? In other words: does the approximate dynamics correspond to anything in the modelled system?

Inverse shadowing.

Given an true solution $\{x_k\}$ could we expect, there is a approximate solution $\{y_k\}$, generated by the given method that is ε – close to it? In other words: does every exact dynamics correspond to anything we get by the given numerical method?

Shadowing and Inverse Shadowing: precise definitions

Shadowing.

We say that a diffeomorphism f has the *shadowing property* if for any $\varepsilon > 0$ there exists a $d = d(\varepsilon) > 0$ such that for any sequence y_k: $\text{dist}(y_{k+1}, f(y_k)) \leq d$ for any k (the so-called *d-pseudotrajectory*) there exists a point x_0 such that if $x_k = f^k(x_0)$ then

$$\text{dist}(y_k, x_k) \leq \varepsilon, \quad k \in \mathbb{Z}. \tag{1}$$

Inverse shadowing.

We say that a diffeomorphism f has the *inverse shadowing property* if for any $\varepsilon > 0$ there exists a $d = d(\varepsilon) > 0$ such that for any trajectory $\{x_k = f^k(x) : k \in \mathbb{Z}\}$ of f and for any d-method $g = \{g_k\}$ for f there exists a trajectory $\{y_k\}$ of g such that Eq. (1.1) takes place.

Literature

Shadowing

- S. Yu. Pilyugin. *Shadowing in dynamical systems*, Lect. Notes Math, 1706. Berlin, Heidelberg, New York, Springer-Verlag, 1999.
- K. Palmer. *Shadowing in dynamical systems. Theory and applications*, Dordrecht, Kluwer, 2000.
- S.Yu. Pilyugin and K. Sakai. *Shadowing and hyperbolicity*, Lect. Notes Math, 2193. Cham, Springer-Verlag, 2017.

Literature

Inverse Shadowing

- R.M. Corless and S.Yu. Pilyugin. *Approximate and real trajectories for generic dynamical systems.* J. Math. Anal. Appl. 189, 409-423, 1995.
- A. Al-Nayef, P. Diamond, P. Kloeden, V. Kozyakin, and A. Pokrovskii. *Bi-Shadowing and Delay Equations,* Dynamics and Stability of Systems 11 (2), 121–134, 1996. 24, 1996.
- K. Lee. *Continuous inverse shadowing and hyperbolicity*, Bull. Austral. Math. Soc. 67, 15-26. 2003.
- S.Yu. Pilyugin. *Inverse shadowing by continuous methods.* Discrete Cont. Dyn. Syst. 8, 29–38. 2002.
- S.Yu. Pilyugin. *Spaces of dynamical systems*, Berlin, De Gruyter. 2012.

Literature

Metric analogs for Shadowing

- C.A. Morales. *Shadowable points*, Dyn. Syst. 31, 347-356. 2016.
- K. Moriyasu, K. Sakai, and N. Sumi. *Diffeomorphisms with shadowable measures*, Axioms. 2018.

Some basic definitions from topologic dynamics.

Minimal points

A point x is *minimal* or quasiperiodic for f if $\overline{O_f(y)} = \overline{O_f(x)}$ for any point $y \in \overline{O_f(x)}$.

Recurrent (Poisson stable) points

A point x is *recurrent* if it is both positive and negative for $O_f(x)$ (in other words it is both α and ω limit for itself).

Nonwandering points

A point x is *nonwandering* if for any neighbourhood $U \ni x$ there exist points $y, z \in U$ and $n \in \mathbb{N}$ such that $f^n(y) = z$.

$$\overline{Per(f)} \subset \overline{R(f)} \subset \Omega(f) = \overline{\Omega(f)}$$

Some basic definitions from Ergodic Theory.

Invariant measures

A Borel probability measure μ is *invariant* with respect to the map f if $\mu(f^{-1}(A)) = \mu(A)$ for any measurable set A.

The set $\mathcal{M}(f)$ of all invariant measures is non-empty (Krylov-Bogolubov Theorem), convex and compact in the $*$-weak topology,

For any invariant measure all points of its support are recurrent (follows from Poincaré Recurrence Theorem).

Spaces of sets of invariant measures.

Kantorovich - Wasserstein distance

Let $\mathcal{P}(\mathcal{M})$ be the set of all Borel probability measures on M.

$$W_1(\mu_1, \mu_2) = \sup_{\varphi \in \mathrm{Lip}_1} \left| \int_K \varphi \, d\mu_1 - \int_K \varphi \, d\mu_2 \right|$$

It defines $*$-weak topology.

Metric space of sets of invariant measures.

Sets of f-invariant measures are always compact. So, we may introduce the Hausdorff distance, corresponding to W_1.

$\mathcal{K}(M)$ – set of all compact subsets of $\mathcal{P}(\mathcal{M})$. $\mathcal{M} : f \to \mathcal{M}(f) \in \mathcal{K}(M)$.

Some basic definitions from Hyperbolic Theory.

Hyperbolic invariant sets

A compact f - invariant set K is hyperbolic if $Df|_K$ can be represented as $E^s \oplus E^u$ such that $DfE^{s,u} = E^{s,u}$ and there exist $C > 0$, $\lambda \in (0,1)$ such that $\|Df^n|_{E^s}\| \le C\lambda^n$ and $\|Df^{-n}|_{E^u}\| \le C\lambda^n$ for any $n \in \mathbb{N}$.

Axiom A

Definition

A diffeo f satisfies Axiom A if periodic points are dense in $\Omega(f)$ and the latter set is hyperbolic.

Smale's Decomposition Theorem

Axiom A implies $\Omega(f) = \Omega_1 \bigcup \ldots \bigcup \Omega_n$ (sets are disjoint, invariant, compact and transitive).

Structural stability and Ω – stability

Arrows

$\Omega_i \to \Omega_j$ if $i \neq j$ and $W^s(\Omega_j) \bigcap W^u(\Omega_i) \neq \emptyset$.

Structural stability

Equivalent to Axiom A and Strong Transversality Condition.

Ω-stability

Equivalent to Axiom A and no-cycle condition (weaker than Structural Stability).

Shadowing and Structural Stability

Interiors of (Inverse) Shadowing

$$\mathrm{Int}^1\mathsf{S}(M) = \mathrm{Int}^1\mathsf{IS}(M) = \mathsf{StS}(M).$$

S.Yu. Pilyugin, K. Sakai, K. Lee.

Main problem

Which maps are continuity points for the map $\mathcal{M}$? Could we use numerical methods to model invariant measures?

Invariant measures for methods

Let $g = \{g_k : k \in \mathbb{Z}_+\}$ be a method; denote $g_0^k = g_k \circ \ldots \circ g_0$. For a point $x \in K$, we define the set

$$\mathcal{M}_0(g, x) = \bigcap_{n=1}^{\infty} \overline{\left\{ \frac{1}{N} \sum_{k=0}^{N-1} \delta(g_0^k(x)) \ : \ N \geq n \right\}}.$$

In other words, this is the set of limit points for the sequence

$$\left\{ \frac{1}{N} \sum_{k=0}^{N-1} \delta(g_0^k(x)) : N \in \mathbb{N} \right\}.$$

Note that that all the sets $\mathcal{M}_0(g, x)$ are nonempty as intersections of nested nonempty compact sets. Let

$$\mathcal{M}_0(g) := \bigcup_{x \in K} \mathcal{M}_0(g, x) \text{ and } \mathcal{M}(g) := \overline{\operatorname{conv} \mathcal{M}_0(g)}.$$

We call any measure of the set $\mathcal{M}(g)$ invariant with respect to the method g.

Invariant measures for methods-2

Theorem 1.

If a method g is generated by iterations of a single map g_0, the set $\mathcal{M}(g)$ coincides with $\mathcal{M}(g_0)$ in its classical sense.

Upper semicontinuity of the set of invariant measures

Theorem 2.

Let methods $g_k = \{g_{km}\}$ be such that $g_{km} \rightrightarrows f$ as $k \to \infty$. Then for any > 0 there exists $K \in \mathbb{N}$ such that for any $k \geq K$ we have

$$\mathcal{M}(g_k) \subset U_\varepsilon(\mathcal{M}(f))$$

(in Hausdorff metrics) .

Remark.

By Takens' Tolerance stability results continuity of invariant measures is typical in C^k topology for any k.

Ergodic Inverse Shadowing

Definition

We say that a mapping f has the *ergodic inverse shadowing property* ($f \in$ EIS) if for any $\varepsilon > 0$ there exists a $d = d(\varepsilon) > 0$ such that for any trajectory $\{x_k = f^k(x) : \; k \in \mathbb{Z}_+\}$ of f and for any d-method g for f there exists a trajectory $\{y_k : \; k \in \mathbb{Z}_+\}$ of g such that

$$\liminf \frac{\#\{1 \le k \le n : (x_k, y_k) \le \varepsilon\}}{n} \ge 1 - \varepsilon. \qquad (2)$$

Evidently IS $\subset$ EIS.

Lower semicontinuity of measures and Ergodic Inverse Shadowing.

Theorem 3.

Let methods $g_k = \{g_{km}\}$ be such that $g_{km} \rightrightarrows f \in \mathrm{EIS}$ as $k \to \infty$. Then $\mathcal{M}(g_k) \to \mathcal{M}(f)$ in Hausdorff metrics.

Corollary.

Let $f \in \mathrm{EIS}$ and x be a minimal point of f then for any $\varepsilon > 0$ there is $\delta > 0$ the ε – neighbourhood of x contains recurrent points for all maps g that are δ close to f in C^0.

Ergodic Inverse Shadowing and hyperboicity of $\Omega(f)$.

Theorem 4.

If the set $\Omega(f)$ is hyperbolic, $f \in \mathrm{EIS}$.

Corollary.

$\mathrm{Int}^1\mathrm{EIS} \supset \Omega S$.

Corollary.

$\mathrm{IS} \subsetneq \mathrm{EIS}$.

Thank you!