Lyapunov exponents of probability distributions with non-compact support

Adriana Sánchez Chavarría

September, 2019

Products of random matrices

- $M = SL(2, \mathbb{R})^{\mathbb{N}}$,
- $f : M \to M$ the shift map given by

$$f((\alpha_n)_n) = (\alpha_{n+1})_n.$$

- The product of random matrices is the function given by

$$A((\alpha_n)_n) = \alpha_0,$$

Its iterates are dynamically defined by

$$A^k((\alpha_n)_n) = \alpha_{k-1} \cdots \alpha_0.$$

Lyapunov Exponents

We call **Lyapunov exponents** the limits

$$\lambda_+(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|,$$
$$\lambda_-(x) = \lim_{n\to\infty} \frac{1}{n} \log \|(A^n(x))^{-1}\|^{-1},$$

which exist for μ-almost every $x \in M$ if $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$.

Lyapunov Exponents

If μ is ergodic the Lyapunov exponents are constant for μ-almost every point x. In this case we write

$$\lambda_+(x) = \lambda_+(\mu) = \inf_{n \geq 1} \frac{1}{n} \int \log \|A^n\| d\mu,$$
$$\lambda_-(x) = \lambda_-(\mu) = \sup_{n \geq 1} \frac{1}{n} \int \log \|(A^n)^{-1}\|^{-1} d\mu.$$

Weak* topology

We say that μ_n converges to μ in the weak* topology if

$$\left| \int \psi d\mu_n - \int \psi \mu \right| \to 0,$$

for all bounded continuous functions.

Semicontinuity

If μ has compact support then, λ_+ is upper semi-continuous and λ_- is lower semi-continuous.

Bocker-Viana

Let $\mathcal{G}(2)$ be the space of compactly supported probability measures on $GL(2, \mathbb{R})$, equipped with the weakest topology, $\mathcal{T}$, such that

- $p \mapsto \int \varphi dp$ is continuous for every continuous function $\varphi : G \to \mathbb{R}$ and
- $p \mapsto \operatorname{supp} p$ is continuous relative to the Hausdorff topology.

Then the extremal Lyapunov exponents $\mu \mapsto \lambda_{\pm}(\mu)$ vary continuously on $\mathcal{G}(2)$.

Measures with non compact support

The function $\mu \mapsto \lambda_+(\mu)$ is not upper semi-continuous relative to the weak* topology. The same remains valid for $\mu \mapsto \lambda_-(\mu)$ with lower semi-continuity.

Support

Let $0 < r < 1/2 < s < 1$, $l = s/r > 1$ and

$$\alpha(2k-1) = \alpha(2k)^{-1} = \begin{pmatrix} e^{l^k} & 0 \\ 0 & e^{-l^k} \end{pmatrix}$$

Measure q

Consider the weights

$$p_{2k} = p_{2k-1} = r^k \text{ for } k \geq 2,$$
$$p_1 = p_2 = \frac{1}{2}\left(1 - 2\frac{r^2}{r-1}\right).$$

Define $q = \sum p_k \delta_{\alpha(k)}$.

Measure q

Consider the weights

$$p_{2k} = p_{2k-1} = r^k \text{ for } k \geq 2,$$
$$p_1 = p_2 = \frac{1}{2}\left(1 - 2\frac{r^2}{r-1}\right).$$

Define $q = \sum p_k \delta_{\alpha(k)}$. Then

$$\lambda_+(q) = 0.$$

Measure q_n

Change the weights

$$p^n_{2n} = r^n + l^{-n},$$
$$p^n_2 = \frac{1}{2}\left(1 - 2\frac{r^2}{r-1}\right) - l^{-n}.$$

Define $q_n = \sum p^n_k \delta_{\alpha(k)}$.

Measure q_n

Change the weights

$$p^n_{2n} = r^n + l^{-n},$$
$$p^n_2 = \frac{1}{2}\left(1 - 2\frac{r^2}{r-1}\right) - l^{-n}.$$

Define $q_n = \sum p^n_k \delta_{\alpha(k)}$. Then

$$\lambda_+(q_n) = 1 + l^{-n-1} \geq 1.$$

Wasserstein space

The **Wasserstein space** is the space

$$P_1(M) := \left\{ \mu \in P(M) : \int_M d(x_0, x) d\mu(x) < +\infty \right\},$$

where $x_0 \in M$ is arbitrary.

Remark

The Lyapunov exponents exist for every measure p in the Wasserstein space $P_1(SL(2, \mathbb{R}))$.

Remark

The Lyapunov exponents exist for every measure p in the Wasserstein space $P_1(SL(2,\mathbb{R}))$. If $\mu = p^{\mathbb{N}}$ then

$$\int \log \|A(x)\| d\mu = \int \log \|\alpha\| dp \leq \int d(\alpha, \mathrm{id}) dp < \infty.$$

Wasserstein topology

We say that μ_k converge in the Wasserstein topology to μ, if

- $\mu_k \xrightarrow{*} \mu$,
- $\int d(x_0, x) d\mu_k(x) \to \int d(x_0, x) d\mu(x)$.

Wasserstein topology

We say that μ_k converge in the Wasserstein topology to μ, if

- $\mu_k \xrightarrow{*} \mu$,
- $\int d(x_0, x) d\mu_k(x) \to \int d(x_0, x) d\mu(x)$.

This convergence is metrized by the Wasserstein distance

$$W_1(\mu, \nu) = \sup \left\{ \int_M \psi d\mu - \int_M \psi d\nu \right\},$$

where the supremum on the right is over all 1-Lipschitz functions ψ.

Wasserstein topology

The following properties are equivalent to the convergence in the Wasserstein topology:

1. $\mu_k \xrightarrow{*} \mu$ and

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int_{d(x_0,x)\geq R} d(x_0,x)d\mu_k(x) = 0;$$

2. For all continuous functions φ with $|\varphi(x)| \leq C(1 + d(x_0,x))$, $C \in \mathbb{R}$, one has

$$\int \varphi(x)d\mu_k(x) \to \int \varphi(x)d\mu(x).$$

Semicontinuity with Wasserstein topology

Theorem (Viana, S)

The function $\lambda_+ : P_1(SL(2,\mathbb{R})) \to \mathbb{R}$ is upper semi-continuous relative to the Wasserstein topology. The same remains valid for the function λ_- with lower semi-continuity.

Stationary measures

If η is a probability measure such that for every measurable set $E \subset \mathbb{P}^1$

$$\eta(E) = \int \eta(\alpha^{-1}E)dp(\alpha),$$

then we have the following identity

$$\lambda_+(p) = \max\left\{\int \Phi dp \times \eta : \eta \;\; p - \text{stationary}\right\},$$

where $\Phi : SL(2,\mathbb{R}) \times \mathbb{P}^1 \to \mathbb{R}$ is given by

$$\Phi(\alpha, [v]) = \log \frac{\|\alpha v\|}{\|v\|}.$$

Discontinuity example

Let $p_k = e^{-\sqrt{k}}$,

$$\alpha(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Discontinuity example

Let $p_k = e^{-\sqrt{k}}$,

$$\alpha(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- $q = \sum p_k \delta_{\alpha(k)}$ and $\lambda_+(q) = \sum e^{-\sqrt{k}} \log k > 0$.

Discontinuity example

Let $p_k = e^{-\sqrt{k}}$,

$$\alpha(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- $q = \sum p_k \delta_{\alpha(k)}$ and $\lambda_+(q) = \sum e^{-\sqrt{k}} \log k > 0$.
- $q_n = \sum_{k \neq n} p_k \delta_{\alpha(k)} + p_n \delta_B$ and $\lambda_+(q_n) = 0$.

Discontinuity example

Let $p_k = e^{-\sqrt{k}}$,

$$\alpha(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- $q = \sum p_k \delta_{\alpha(k)}$ and $\lambda_+(q) = \sum e^{-\sqrt{k}} \log k > 0$.
- $q_n = \sum_{k \neq n} p_k \delta_{\alpha(k)} + p_n \delta_B$ and $\lambda_+(q_n) = 0$.
- $W(q_n, q) \leq p_n d(\alpha(n), B) \sim n e^{-\sqrt{n}} \to 0$.

Thank You!!!