Cut-and-Project Quasicrystals: Patch Frequency and Moduli Spaces

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ICTS Smooth and Homogeneous Dynamics 2019

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 754475).

Outline

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- Λ is uniformly discrete
  $\exists r > 0$ such that $B_r(x) \cap \Lambda = \{x\}$ for all $x \in \Lambda$
- Λ is relatively dense
  $\exists R > 0$ such that $B_R(0) + \Lambda$ covers X.

Take Λ the pointset of vertices of the tiling with two rhombuses.

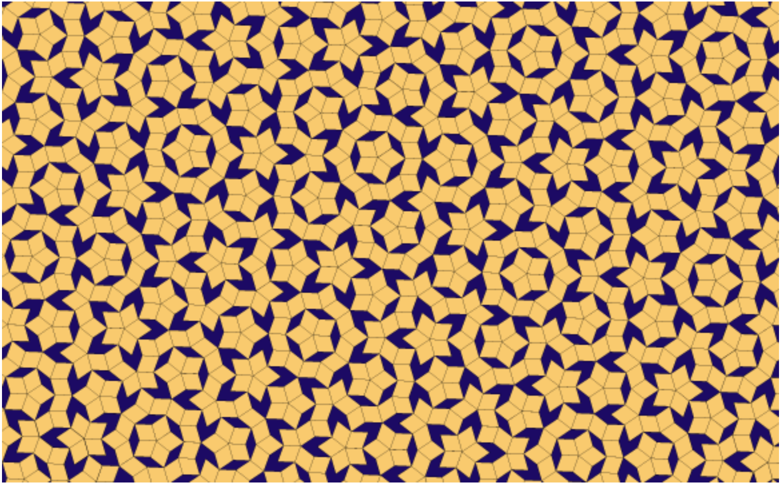

FIGURE: Penrose Rhomb '74. Source: Tilings Encyclopedia.
https://tilings.math.uni-bielefeld.de/

A local criterion for regularity of a system of points.

Definition: T-patch

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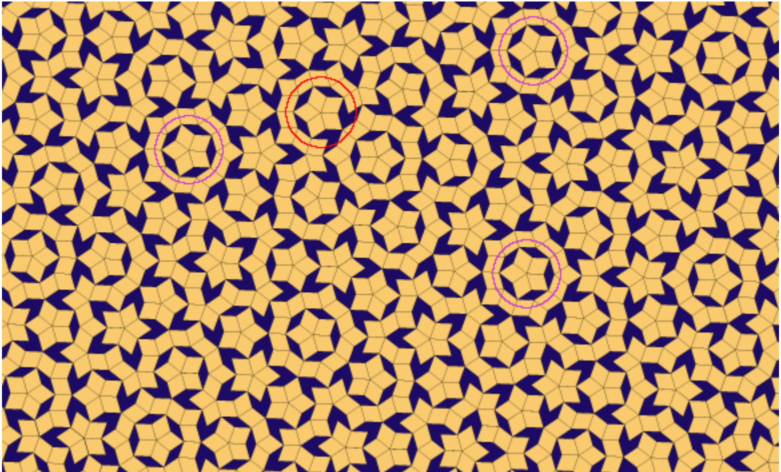

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Thm(Delone,Dolbilin,Shtogrin,Galiulin;'76)

Let $\Lambda \subset \mathbb{R}^d$ be Delone. There exists $T_0 > 0$ such that if all T_0-patches are equivalent up to isometry then there exists a lattice $\Gamma < \mathrm{Isom}(\mathbb{R}^d)$ and $x \in \Lambda$ s.t.

$$\Lambda = \Gamma x.$$

Same paper: $\mathbb{S}^n$, $\mathbb{H}^n$.

Non-examples: discrete orbits of $\mathrm{SL}_2(\mathbb{R})$

Question of Barak Weiss: For which translation surface are the set of holonomy vectors Delone?
Answer: No lattice surface is.

Theorem (Wu)

Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ a lattice and $x \in \mathbb{R}^2$ with $\Lambda = \Gamma x$ discrete.

- If Γ is arithmetic then Λ is not relatively dense
- If Γ is non-arithmetic then Λ is not uniformly discrete.

Primitive Lattice points (Baake-Moody-Pleasants)

Meyer sets

Meyer's definition of a quasicrystal

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This generalizes lattices:

- If E is trivial, Λ is a lattice.
- Λ is Meyer if and only if Λ and $\Lambda - \Lambda$ are Delone. (Lagarias)

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Embedding Theorem (Meyer '72)

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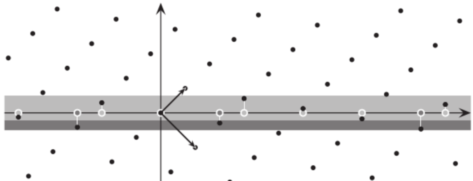

Silver mean chain. Source: APERIODIC ORDER Vol. I: A Mathematical Invitation. Baake-Grimm '13

Definition: cut-and-project scheme

The data $(\mathcal{L}, \mathbb{R}^d, \mathcal{A})$ defines a cut-and-project scheme if

$\mathcal{A}$ is a locally compact Abelian group,

$\mathcal{L}$ is a lattice in the group $\mathbb{R}^d \times \mathcal{A}$

$\pi = \pi_{\mathbb{R}^d}, \pi_{\text{int}} = \pi_{\mathcal{A}}$ natural projections satisfy

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathcal{A}$.

$$
\begin{array}{ccccccc}
\mathbb{R}^d & \xleftarrow{\ \pi\ } & \mathbb{R}^d \times \mathcal{A} & \xrightarrow{\ \pi_{\text{int}}\ } & \mathcal{A} & \overset{\textit{window}}{\supset} & \mathcal{W} \\
\cup & & \bigvee \ \text{lattice} & & \cup \ \text{dense} & & \\
\pi(\mathcal{L}) & \overset{\text{inj}}{\hookleftarrow} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) & &
\end{array}
$$

cut-and-project sets

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Definition: Window and cut-and-project sets

Window $\mathcal{W} \subset \mathcal{A}$ bounded and define the cut-and-project set

$$\Lambda(\mathcal{W}, \mathcal{L}) = \pi\left(\mathcal{L} \cap \left(\mathbb{R}^d \times \mathcal{W}\right)\right)$$

If $\mathcal{W}$ has non-empty interior, call it cut-and-project quasicrystal

Minkowski Space

Meyer '72: Λ Meyer and $\theta\Lambda \subset \Lambda$ for real algebraic θ then θ is Pisot or Salem
de Bruijn '81, Pleasants '01: Penrose tilings are a cut-and-project quasicrystal

S-adics

- $\mathbb{K}$ is a number field of degree N. $\mathfrak{o}$ ring of integers.
- field embeddings $\sigma_i : \mathbb{K} \to \overline{\mathbb{K}}$ with completions $\mathbb{K}_\nu$, $\nu \in S$
- $\mathbb{K}_S = \prod_{\nu \in S} \mathbb{K}_\nu$
- $\mathcal{L} = \{(\sigma_1(v), \ldots, \sigma_N(v)) : v \in \mathfrak{o}^k\} < \mathbb{K}_S^k$ lattice

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$\mathbb{R}^d \hookrightarrow \mathbb{K}_{\nu_1}^k$ for some ν_1, $\mathbb{K}_S^k = \mathbb{R}^d \oplus \mathbb{R}^m = \mathbb{R}^n$, $\mathcal{W} \subset \mathbb{R}^m$

$$\mathbb{R}^d \xleftarrow{\pi} \mathbb{K}_S^k = \mathbb{R}^n \xrightarrow{\pi_{\text{int}}} \mathbb{R}^m$$

$$\Lambda = \pi(\mathcal{L} \cap \mathbb{R}^d \times \mathcal{W})$$

Related projection method

- Non-commutative cut-and-project sets (Amos' 1st lecture, Björklund-Hartnick-Pogorzelski '18)
- Realize integer points of varities by lifting to S-adics
  E.g. Lubotzky-Phillips-Sarnak '86, Ellenberg-Venkatesh '06, Ellenberg-Michel-Venkatesh '10, Aka-Einsiedler-Shapira '14,15

Primitive integer points on large spheres

$$\mathrm{SO}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{Z}) \longleftarrow \mathrm{SO}_d(\mathbb{R}\times\mathbb{Q}_p)/\mathrm{SO}_d(\mathbb{Z}[\tfrac{1}{p}])$$

"Window" given by $\mathrm{SO}_d(\mathbb{Z}_p)\mathrm{SO}_d(\mathbb{Z}[\frac{1}{p}])$ and project an orbit $\mathbb{H}_v(\mathbb{R}\times\mathbb{Q}_p)\mathrm{SO}_d(\mathbb{Z}[\frac{1}{p}])$ where $\mathbb{H}_v = \mathrm{Stab}_v(\mathrm{SO}_d)$ for v primitive. Produces friends Λ_v of v.
Equidistributes when projecting on sphere as $v \to \infty$ ($\|v\|^2$ coprime to p).

Window with empty interior

Adeles

Consider the internal space $\mathcal{A} = \mathbb{A}_f^d = \prod_p \mathbb{Q}_p^d$ and $\mathcal{L} = \mathbb{Q}^d$.

$$\mathbb{R}^d \xleftarrow{\pi} \mathbb{A}_{\mathbb{Q}}^d = \mathbb{R}^d \times \mathbb{A}_f^d \xrightarrow{\pi_{\text{int}}} \mathbb{A}_f^d$$

$$\Lambda = \pi\left(\mathbb{Q}^d \cap \mathbb{R}^d \times \mathcal{W}\right)$$

- primitive lattice points of $\mathbb{Z}^d$. Take $\mathcal{W} = \prod_p \mathbb{Z}_p^d \setminus \prod_p p\mathbb{Z}_p^d$.
- square-free numbers in $\mathbb{Z}$. Take $\mathcal{W} = \prod_p \mathbb{Z}_p \setminus \prod_p p^2\mathbb{Z}_p$.

Patch Counting

Sometimes assume (see Moody, Pleasants)

1. Λ is *regular* if the boundary of $\mathcal{W}$ has zero measure
2. Λ is *generic* if $\pi_{\text{int}}(\mathcal{L})$ doesn't intersect the boundary of $\mathcal{W}$.

For any T-patch $\mathcal{P} = \mathcal{P}(y, T) = \Lambda \cap B_T(y)$ define

$$\text{freq}_\Lambda(\mathcal{P}) = \lim_{t \to \infty} \frac{\#\{x \in \Lambda \cap B_t(0) : \mathcal{P}(x, T) \sim_{\mathbb{R}^d} \mathcal{P}\}}{t^d}.$$

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Theorem on Patch Frequency Asymptotics

Fix $d + m = n$. There exists $\kappa > 0$ such that for a **random (of rational type)** cut-and-project quasicrystal Λ of $(\mathcal{L}, \mathbb{R}^d, \mathbb{R}^m)$ we have

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Patches $\mathcal{P}$ of $\Lambda(\mathcal{L}, \mathcal{W})$ correspond to points in $\Lambda(\mathcal{L}, \mathcal{W}_\mathcal{P})$ for some $\mathcal{W}_\mathcal{P} \subset \mathcal{W}$. Reduces to counting $\mathcal{L} \cap B_t(0) \times \mathcal{W}_\mathcal{P}$.

Spaces of quasicrystals

Construction of Marklof and Strömbergsson

- $\Lambda = \Lambda(\mathcal{L}, \mathcal{W})$ cut-and-project quasicrystal of $(\mathcal{L}, \mathbb{R}^d, \mathbb{R}^m)$
- Suppose $\mathcal{L} = g\mathbb{Z}^n \in X_n$ for $g \in \mathrm{SL}_n(\mathbb{R})$ and consider $\mathrm{SL}_d(\mathbb{R}) < \mathrm{SL}_n(\mathbb{R})$ top-left block.
- $\overline{\mathrm{SL}_d(\mathbb{R})\mathcal{L}} = L'\mathcal{L}$ for some $L' < \mathrm{SL}_n(\mathbb{R})$ (Ratner)

The space of quasicrystals associated to $\mathcal{L}$

$$\mathfrak{Q} = \{\Lambda(y, \mathcal{W}) : y \in L'\mathcal{L}\}$$

with probability measure $\mu_{\mathfrak{Q}}$ (push forward of $m_{L'\mathcal{L}}(y)$ under $y \mapsto \Lambda(y, \mathcal{W})$).

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MS actually consider $\mathrm{ASL}_d(\mathbb{R})$-orbit closure.
For $m_{L'\mathcal{L}}$-a.e. y defines a cut-and-project scheme i.e. it holds

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$.

What are the spaces of quasicrystals?

Identify X_n with $\mathrm{SL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{Z})$.

Classification Theorem I

Write $Y = \overline{\mathrm{SL}_d(\mathbb{R}) g\, \mathrm{SL}_n(\mathbb{Z})} = L'\mathcal{L} = gL\,\mathrm{SL}_n(\mathbb{Z})$. Then L is almost $\mathbb{Q}$-simple.

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Classification Theorem II

By the classification of semi-simple groups it follows that $\widetilde{L} \simeq \mathbb{G}(\mathbb{K}_S)$ where

- $\mathbb{G} = \mathrm{SL}_k$ for $k \geq d$ or Sp_{2k} (if $d = 2$)
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Generalizes [MS '13] $Y = X_n$ for the case $d > m$.
Unpublished work of O. Sargent.
Private communication with M. Einsiedler.

Lemma about rational invariant subspaces

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$.

(I) and (D) implies (L) Lemma

A vector space $V < \mathbb{R}^n$ is called $\mathcal{L}$-rational if $V \cap \mathcal{L}$ is a lattice in V.

The following implications hold.

a. **(D)** $\Rightarrow$ $\mathbb{R}^d$ is not contained in a proper $\mathcal{L}$-rational subspace.

b. **(I)** $\Rightarrow$ $\mathbb{R}^m$ contains no non-trivial $\mathcal{L}$-rational subspace.

Define

(L) There exists no proper $\mathcal{L}$-rational subspace of $\mathbb{R}^n$ that is L'-invariant w.r.t. matrix multiplication.

c. **(I)** and **(D)** $\Rightarrow$ **(L)**.

(L) implies simple almost $\mathbb{Q}$-simple

Theorem

Write $Y = \overline{\mathrm{SL}_d(\mathbb{R}) g \, \mathrm{SL}_n(\mathbb{Z})} = L' \mathcal{L} = gL \, \mathrm{SL}_n(\mathbb{Z})$.
Then L *is almost* $\mathbb{Q}$*-simple algebraic group.* $\tilde{L} \simeq \mathbb{G}(\mathbb{K}_S)$ *where*

- $\mathbb{G} = \mathrm{SL}_k$ *for* $k \geq d$ *or* Sp_k *(if* $d = 2$*)*
- $\mathbb{K}$ *real number field of degree* N *such that* $kN = n$

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Proof.

- Shah '91: L minimal $\mathbb{Q}$-group generated by unipotents containing $H' = g^{-1}\,\mathsf{SL}_d(\mathbb{R})g$
- L is semi-simple: Let U denote the unipotent radical of L. V^U is a rational subspace. Cannot be by Lemma.
- H' must be contained in an almost direct factor (over $\mathbb{R}$). It follows that L is almost $\mathbb{Q}$-simple
- Use classification (see Tits '66, Morris-Witte '14)

□

Theorem (in Tits '66)

If L is almost $\mathbb{Q}$-simple and simply connected, there exists a field $\mathbb{K}$ and an absolutely almost simple simply connected group G defined over $\mathbb{K}$ such that $L \simeq_{\mathbb{Q}} Res_{\mathbb{K}/\mathbb{Q}}(G)$.

Theorem (Morris-Witte in Appendix of Solomon-Weiss '14)

If L as above contains a conjugate of the top left $\mathrm{SL}_d(\mathbb{R})$ then G is either of type A_k or C_k.

Patch frequency

If $\mathbb{K} = \mathbb{Q}$, we know that $\mathsf{SL}_k(\mathbb{R})$ and $\mathsf{Sp}_{2k}(\mathbb{R})$ are groups of Rogers type. (Rogers/Schmidt, Kelmer-Yu).
Apply Schmidt's counting theorem to count patches.

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Thank you for your patience.