<Dimension Bound for Doubly Badly Approximable Affine Forms>

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Homogeneous DA:

Approximate $\alpha \notin \mathbb{Q}$ by $\frac{p}{q}$, $p, q \in \mathbb{Z}$

<u>Dirichlet thm</u> $\forall N \in \mathbb{N}$, $\exists\, p, q \in \mathbb{Z}$, $0 < q < N$

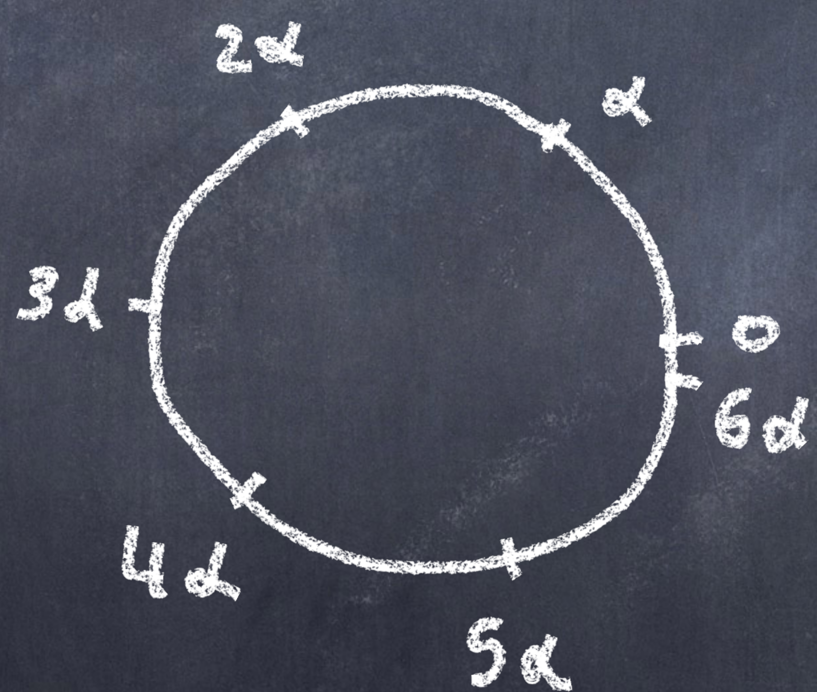
s.t. $\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN}$.

$|q\alpha - p| < \frac{1}{N}$

$\|q\alpha\| = d(q\alpha, \mathbb{Z}) < \frac{1}{N} < \frac{1}{q}$

dist to $\mathbb{Z}$

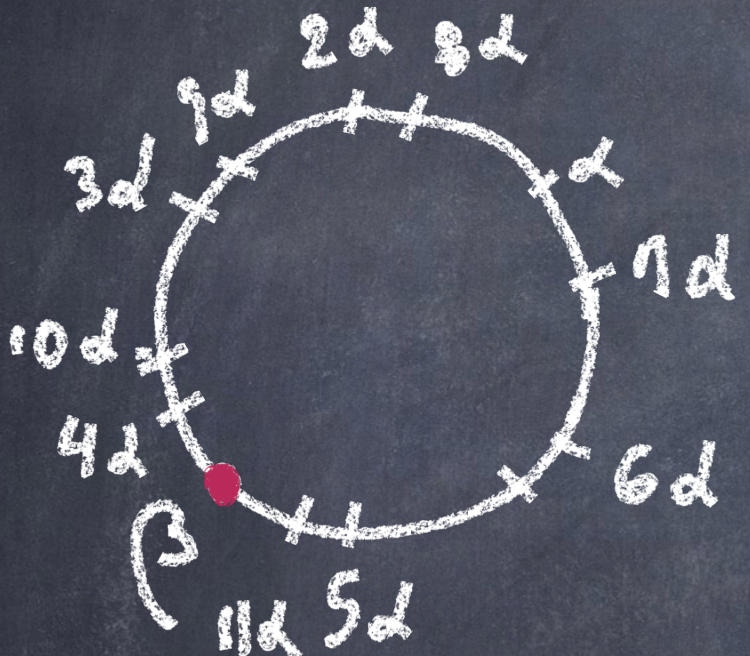
$\mathbb{R}/\mathbb{Z}$



"distribution $q\alpha$ mod $\mathbb{Z}$"

Inhomogeneous D.A. $\alpha \notin \mathbb{Q}$

Distribution of $q\alpha$ mod $\mathbb{Z}$ near β:



$\forall \beta, \quad \|q\alpha - \beta\| < \frac{1}{N} < \frac{1}{q}$

$\Rightarrow q\|q\alpha - \beta\| < 1.$ (Minkowski: $q\|q\alpha-\beta\| < \frac{1}{4}$)

In fact, $\liminf_{q\to\infty} q\,\|q\alpha - \beta\| = 0$

a.e. α, β.

Similarly, a.e. $A \in M_{m\times n}(\mathbb{R})$, $b \in \mathbb{R}^m$,

$$\liminf_{\substack{|q|\to\infty \\ q\in\mathbb{Z}^n}} |q|^{n/m} \|Aq - b\| = 0$$

dist to $\mathbb{Z}^m$

Bugeaud, Harrap, Kristensen, Velani (2010) $\forall A \in M_{m\times n}(\mathbb{R})$

$$\dim_H \{ b \in \mathbb{R}^m : \liminf_{|q|\to\infty} |q|^{\frac{n}{m}} \|Aq - b\| > 0 \} = m.$$

Question (Uri) Fix $\varepsilon > 0$. For which A

$$\dim_H \{ b \in \mathbb{R}^m : \liminf_{|q|\to\infty} |q|^{\frac{n}{m}} \|Aq - b\| > \varepsilon \} < m \,?$$

"$=: \mathrm{Bad}_A(\varepsilon)$"

Thm (L-de Saxcé-Shapira) For a.e. A,

$\forall \varepsilon > 0$, $\dim_H \mathrm{Bad}_A(\varepsilon) < m$.

Rmk A singular $\Rightarrow \bigcap_{\varepsilon > 0} \mathrm{Bad}_A(\varepsilon) = \mathrm{Bad}_A(\infty)$ full dim

Einsiedler-Tseng

$$\left(\begin{array}{c|c|c} e^{nt} I_m & & \\ \hline & e^{-mt} I_n & \\ \hline & & 1 \end{array}\right)\left(\begin{array}{c|c|c} I_m & A & -b \\ \hline & I_n & 0 \\ \hline & & 1 \end{array}\right)\left(\begin{array}{c} p \\ \hline q \\ \hline 1 \end{array}\right) = \begin{pmatrix} e^{nt}(Aq+p-b) \\ e^{-mt} q \\ 1 \end{pmatrix}$$

Suppose it is in $B_\varepsilon(0) \subset \mathbb{R}^d$ for $\exists p$.

$$\left.\begin{array}{l} e^{nt}\,\|Aq-b\| < \varepsilon \\ e^{-mt}\,|q| < \varepsilon \end{array}\right\} \Rightarrow \underset{e^{nt}\varepsilon^{\frac{n}{m}}}{\underset{\wedge}{|q|^{\frac{n}{m}}}}\ \underset{e^{-nt}\varepsilon}{\underset{\wedge}{\|Aq-b\|}} < \varepsilon^{\frac{n}{m}+1}$$

for $t \nearrow \infty \Rightarrow \liminf\limits_{|q|\to\infty} |q|^{n/m}\,\|Aq-b\| < \varepsilon^{\frac{n+m}{n}}$.

$$\begin{pmatrix} e^{nt} I_m & & \\ & e^{-mt} I_n & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_m & A & -b \\ & I_n & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} p \\ q \\ 1 \end{pmatrix} = \begin{pmatrix} e^{nt}(Aq+p-b) \\ e^{-mt} q \\ 1 \end{pmatrix}$$

$$\begin{array}{ccc} \| & \| & \rotatebox{90}{$\in$} \\ a_t & u_{A,b} & \mathbb{Z}^{m+n+1} \end{array}$$

$d = m+n$

$$a_t x_{A,b} := a_t u_{A,b} \mathbb{Z}^{m+n} \in Y = SL_d(\mathbb{R}) \ltimes \mathbb{R}^d \big/ SL_d(\mathbb{Z}) \ltimes \mathbb{Z}^d$$

grid : translated lattice

"space of unimodular grids in $\mathbb{Z}^d$"

$$\begin{array}{l} Y = ASL_d(\mathbb{R}) / ASL_d(\mathbb{Z}) \\ \qquad\quad x \quad + b \\ \big\downarrow \pi \\ X = SL_d(\mathbb{R}) / SL_d(\mathbb{Z}) \\ \qquad\quad x \end{array}$$

$\mathbb{R}^d/\mathbb{Z}^d$

fiber: $=$ $\{b\}$

$\mathbb{R}^d/x$

x: $\exists$ small vector

(Mahler)

$$a_t = \begin{pmatrix} e^{nt} I_m & \\ & e^{-mt} I_n \end{pmatrix} \begin{array}{l} \curvearrowright X \\ \curvearrowright Y \end{array}$$

both linearly on $\left\{ \begin{array}{l} \text{lattices} \\ \text{grids} \end{array} \right.$

x <u>convergent on average</u> if $\forall \delta > 0$,
(no escape of mass on average)

$\exists K$ cpt, $\liminf_{T\to\infty} \frac{1}{T} \left| \{ 0 < t < T : a_t x \notin K \} \right| < \delta$.

<u>Thm1</u> (LSS) x_A convergent on average

$\Rightarrow \forall \varepsilon > 0$, $\dim_H \mathrm{Bad}_A(\varepsilon) < m$.

<u>Thm2</u> (LSS) $\exists A$ nonsingular (x_A non divergent)

s.t. $\exists \varepsilon > 0$, $\dim_H \mathrm{Bad}_A(\varepsilon) = m$.

<u>Q</u> What if x has <u>δ-escape of mass on average</u>

i.e. $\forall K$ cpt, $\liminf_{T\to\infty} \frac{1}{T} \left| \{ 0 < t < T : a_t x \notin K \} \right| \geq \delta$?

($\delta < 1$)

$\underline{n = m = 1}$:

1. $\exists$ small vector $\Rightarrow$ the other basis vector is large.

2. We can use continued fraction $A = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$.

best approx: $\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}} = \dfrac{p_n}{q_n}$

<u>Thm</u> 1 (Bugeaud - D. Kim - L. - Rams) For $n = m = 1$, TFAE:

i) $\forall \varepsilon > 0$, $\dim_H \mathrm{Bad}_A(\varepsilon) < n$.

ii) $\lim\limits_{k \to \infty} q_k^{1/k} < \infty$.

iii) x_A not divergent on average ($\delta < 1$).

Thm 1 (Wooyeon Kim - L.) A : δ-escape of mass on average

$0 < \delta < 1$ but not for $\delta = 1$.

$\Rightarrow \forall \varepsilon > 0, \ \dim_H \mathrm{Bad}_A(\varepsilon) < m.$

A : ε-bad for b $\iff$ $\liminf_{|q| \to \infty} |q|^{\frac{n}{m}} \|Aq - b\| > \varepsilon.$

$\mathrm{Bad}^b(\varepsilon) = \{ A \in M_{m \times n}(\mathbb{R}) : A : \varepsilon\text{-bad for } b \}$

$\mathrm{Bad}(\varepsilon) = \{ (A, b) \in M_{m \times n}(\mathbb{R}) \times \mathbb{R}^m :$ " $\}$

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<u>Thm 2</u> (W.K.-L.) $\forall \varepsilon > 0$, $\dim_H \bigcup_{b \in \mathbb{R}^m} \mathrm{Bad}^b(\varepsilon) < mn$.

$\therefore \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\dim_H \mathrm{Bad}^b(\varepsilon) < mn - \delta$, $\forall b \in \mathbb{R}^m$.

<u>Coro</u> (W.K.-L.) $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\dim_H \mathrm{Bad}(\varepsilon) < mn + m - \delta.$$

"doubly metric case"

Idea of proof (LSS)

large $\dim_H$ $\rightsquigarrow$ measure of large entropy relative to X

$\pi^{-1}(B_x)$

S_k

e^{-n_k}-sep set

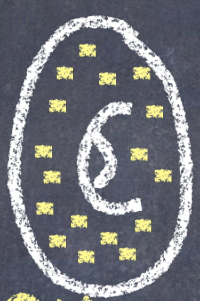
$$\frac{1}{n_k}\sum_{n=0}^{n_k-1}\frac{1}{|S_k|}\sum 1_{S_k} \xrightarrow{w^*} \mu : \text{prob meas}$$

x : convergent on average : $\exists\, n_k \nearrow \infty$ s.t.

$$\frac{1}{n_k}\sum_{n=0}^{n_k-1}\delta_{a^n x} \xrightarrow{w^*} \mu^0 : \text{prob measure}$$

Idea of proof (LSS)

large $\dim_H$ $\rightsquigarrow$ measure of large entropy relative to X



$S_K \subset S$ unstable leaf in $\mathrm{Bad}_{x_A}(\varepsilon)$.

max e^{-n_K}-sep set

$$\frac{1}{n_K}\sum_{n=0}^{n_K-1}\frac{1}{|S_K|}\sum 1_{S_K} \xrightarrow{w^*} \mu : \text{prob meas}$$

$$\dim_H S \leq \liminf_{\delta\to 0}\frac{\log N_d(S,\delta)}{\log\frac{1}{\delta}} \nwarrow \frac{\log|S_K|}{n_K}$$

lower box

Choose $K_i \nearrow Y$.

$\exists\, \rho$ partition s.t. $\frac{1}{q}H_\mu(\rho^{(q)} \mid X) \geq \dim_H S - C\eta_i$

$\eta_i \to 0$.

Idea of proof (LSS)

large $\dim_H$ $\rightsquigarrow$ measure of large entropy relative to X

$\overset{[EL]}{\rightsquigarrow}$ invariant under U unstable horospherical

$h_\mu(a|X) \leq m$

"=" $\Rightarrow$ μ : U-invariant

inv. translation by b

μ : a_t-inv, $\underbrace{U\text{-inv.}}_{\text{contains full fiber}}$ $\Rightarrow\Leftarrow$ "small translates" cannot avoid a ball.

Idea of pf (WKim-L) for $\mathrm{Bad}^b(\varepsilon)$ and $\mathrm{Bad}(\varepsilon)$.

$$\mathcal{L}_\varepsilon = \{ \text{grids } \Lambda_y : \Lambda_y \cap B^{\mathbb{R}^d}_{\varepsilon^{\frac{m}{m+n}}}(0) = \phi \} \subset Y$$

non-cpt closed

Observation 1 $(A, b) \in \mathrm{Bad}(\varepsilon)$

$\Rightarrow$ either $\exists q \in \mathbb{Z}^n$ s.t. $\| Aq - b \| = 0$ —— $(*)$

or

$$a_t x_{A,b} \in \mathcal{L}_\varepsilon \quad \forall t \gg 0.$$

Observation 2 $\mathrm{Bad}^b_{(*)}(\varepsilon) \subset \mathrm{Bad}^0(\varepsilon)$

$\uparrow$

$\dim_H < mn$ (Broderick-Kleinbock 13)

$\rightsquigarrow$ consider $a_t x_{A,b} \in \mathcal{L}_\varepsilon$ eventually.

Work on $\overline{Y} = ASL_d(\mathbb{R}) / ASL_d(\mathbb{Z}) \cup \{\infty\}$.

$$W = \left\{ \begin{pmatrix} I_m & A & 0 \\ & I_n & 0 \\ & & 1 \end{pmatrix} \right\} \text{ instead of } U = \left\{ \begin{pmatrix} I_m & 0 & b \\ & I_n & 0 \\ & & 1 \end{pmatrix} \right\}.$$

unstable in $M_{m,n}(\mathbb{R}^d)$

$\forall b$, $\exists \mu_b$ with

i) $\operatorname{supp} \mu_b \subset \mathcal{L}_\varepsilon = \{$grids avoiding $\varepsilon^{\frac{m}{m+n}}$-ball$\}$

ii) $\mu_b(\overline{Y} \setminus Y) \leq \eta = \frac{mn}{m+n}(mn - \dim_H \mathrm{Bad}^b(\varepsilon)) > 0$

iii) K: cpt $\nearrow Y$.

$$\Rightarrow h_{\mu_b}(\overline{D}^{(q)} \mid \mathcal{B}^u_{\overline{Y}}) \geq (m+n) \dim_H^b(\varepsilon)$$

max entropy if full-dim

Kadyrov - Kleinbock - Lindenstrauss - Margulis

Das - Fishman - Simmons - Urbanski

<u>Rmks</u> 1. Still open for some A : div. on average.

$\exists\, y_k$: vector of best approximation $Y_k = |y_k|$.

<u>Thm 2</u> (BKLR) $A \in M_{m,n}$ s.t. ${}^tA\mathbb{Z}^m + \mathbb{Z}^n$ full rk. If $Y_k^{\frac{1}{k}} \to \infty$,

then $\exists\, \varepsilon > 0$ s.t. $\dim_H \mathrm{Bad}_A(\varepsilon) = n$.

(Generalized to weighted case by

Chow - Ghosh - Guan - Marnet - Simmons)

Best approximation:

$$M(y) = \inf_{p \in \mathbb{Z}^m} |Ay - p|$$

$\exists\, Y_i = \|y_i\|$ and $M_i = M(y_i) = \|Ay_i\| \overset{\text{set}}{=} |AY_i + x_i|$

i) $Y_1 < Y_2 < \dots$ and $M_1 > M_2 > \dots$

ii) $M(y) \geq M_i \quad \forall y$ s.t. $\|y\| < Y_{i+1}$.

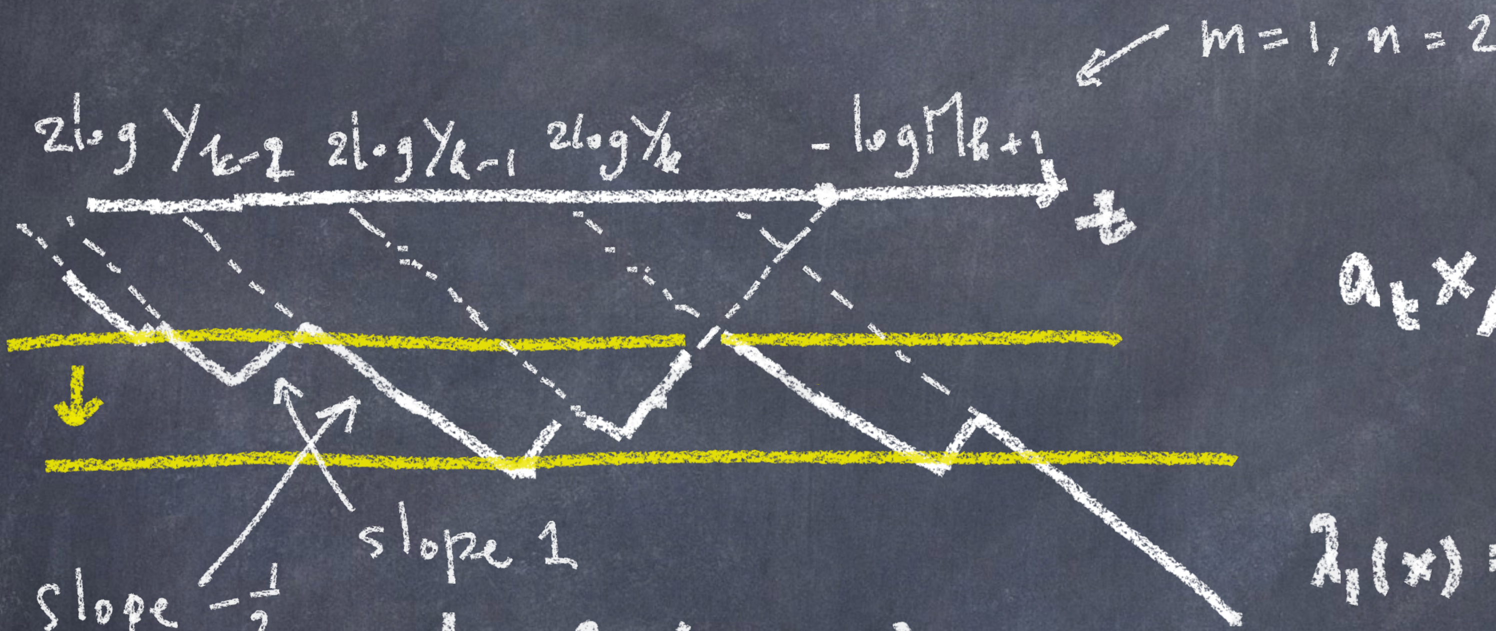
$$a_t u_A \mathbb{Z}^n \ni \begin{pmatrix} e^{t/m} (Aq + p) \\ e^{-t/n} \quad q \end{pmatrix}$$

: norm $= \max\left(e^{t/m}\|Aq\|, e^{-t/n}|q|\right)$

min achieved by

(x_i, y_i) at $\left(e^{t/m} M_i, e^{-t/n} Y_i\right)$

$Y_k = |y_k|$ $\quad y_k$: vector of best approximation



$$y = \log \lambda_1(a_t x_A)$$

$$a_t x_A = a_t u_A \mathbb{Z}^n \subset \mathbb{R}^n$$

lattice

$\lambda_1(x)$ = 1st minimum of lattice x

= min length of a vector in x

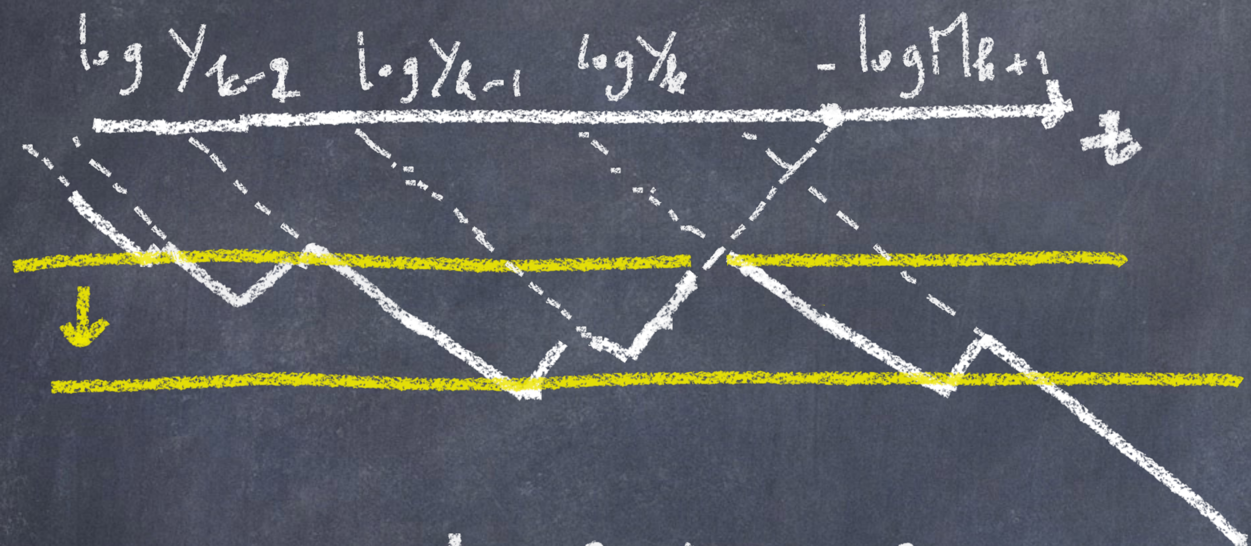
min achieved by

$$(x_i, y_i) \text{ at } \left(e^{t/m} M_i,\ e^{-t/n} Y_i\right)$$

$\cap$ x-axis :

$$\max\left(e^{t/m} M_i,\ e^{-t/n} Y_i\right) = 1$$

$$\therefore t = -m \log M_i \text{ or } n \log Y_i$$



$y = \log \lambda_1 (a_t x_A)$

If $Y_k^{\frac{1}{k}} \to \infty$, then

[BKLR, CGGMS] shows (*).

More generally, if

$\left\{ \begin{array}{l} Y_{n(k)}^{1/n(k+1)} \longrightarrow \infty \\ M_{n(k)} Y_{n(k+1)}^{2} \le 1 \end{array} \right\}$ (*) is true.

(*) $\mathrm{Bad}^{\alpha}_{\{y_i\}} = \{\theta \in \mathbb{R}^n : |\theta - y_i| \ge \alpha, \forall i\} \subset \mathrm{Bad}_A(\varepsilon)$ $\left(\varepsilon \approx \frac{1}{R}\left(\frac{\alpha^2}{4mn}\right)^{1/\delta}\right)$

and $\dim_H(\mathrm{Bad}^{\alpha}_{\{y_i\}}) \ge n - C \limsup_{k \to \infty} \frac{k}{\log Y_k}$ $(R \to \infty : n)$

Rmk 2. If $\lim_{\text{stat}} Y_{k+1} M_k \to 0$, then $\dim_H \mathrm{Bad}_\Lambda(\varepsilon) = n$.

(Joint w/ Taehyung Kim)

$$\left(\underline{\text{Def}} \quad \lim_{\text{stat}} a_k \to \xi \iff \forall \varepsilon, \ \lim_{N\to\infty} \frac{\{n : |a_n - \xi| > \varepsilon\} \cap [1, N]}{N} = 0 \right)$$

A singular on average $\Rightarrow$ $Y_k^{*} \to \infty$ or

$$\liminf_{N\to\infty} \frac{\{ i \le N : Y_{i+1} M_i > \varepsilon \} \cap [1, N]}{N} = 0$$